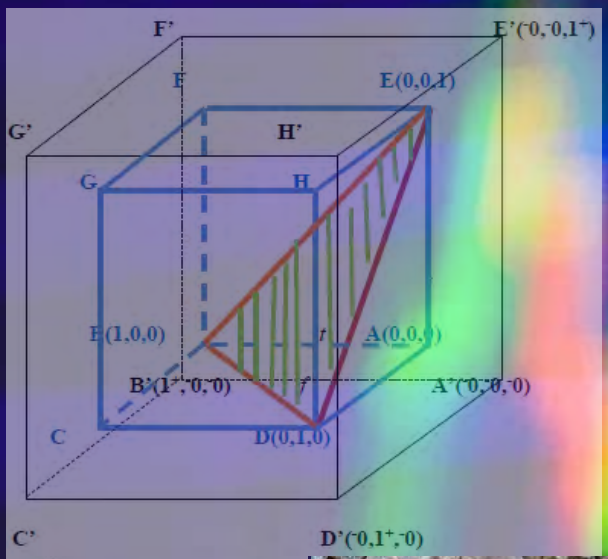
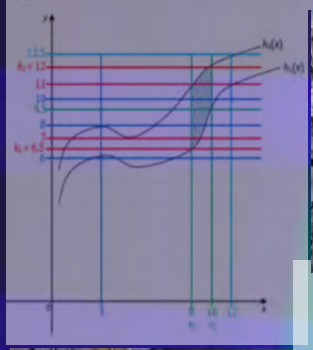
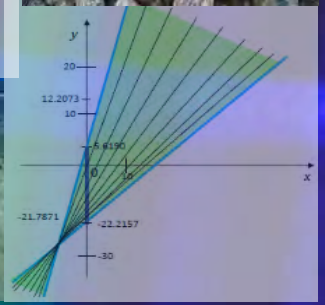
Volume 78,2025

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering







Editors-in-Chief
Florentin Smarandache
Mohamed Abdel-Baset
Maikel Leyva-Vázquez

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

<A> <neutA> <antiA>

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Copyright Notice

Copyright @ Neutrosophics Sets and Systems

All rights reserved. The authors of the articles do hereby grant Neutrosophic Sets and Systems non-exclusive, worldwide, royalty-free license to publish and distribute the articles in accordance with the Budapest Open Initiative: this means that electronic copying, distribution and printing of both full-size version of the journal and the individual papers published therein for non-commercial, academic or individual use can be made by any user without permission or charge. The authors of the articles published in Neutrosophic Sets and Systems retain their rights to use this journal as a whole or any part of it in any other publications and in any way they see fit. Any part of Neutrosophic Sets and Systems howsoever used in other publications must include an appropriate citation of this journal.

Information for Authors and Subscribers

*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:

http://fs.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science:

http://fs.unm.edu/ScienceLibrary.htm.

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

Information about the neutrosophics you get from the UNM website:

http://fs.unm.edu/neutrosophy.htm. The

home page of the journal is accessed on

http://fs.unm.edu/NSS.

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

**** NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, the NSS articles are indexed in Scopus.**

NSS ABSTRACTED/INDEXED IN
SCOPUS,
Google Scholar,
Google Plus,
Google Books,
EBSCO,
Cengage Thompson Gale (USA),
Cengage Learning (USA),
ProQuest (USA),
Amazon Kindle (USA),
University Grants Commission (UGC) - India,
DOAJ (Sweden),
International Society for Research Activity (ISRA),
Scientific Index Services (SIS),
Academic Research Index (ResearchBib),
Index Copernicus (European Union),
CNKI (Tongfang Knowledge Network Technology Co.,
Beijing, China),
Baidu Scholar (China),
Redalyc - Universidad Autonoma del Estado de Mexico (IberoAmerica),
Publons,
Scimago, etc.

Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster
Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free
Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

Copyright © Neutrosophic Sets and Systems, 2025

ISSN 2331-6055 (print) NSS **Editorial Board** *ISSN 2331-608X (online)*

University of New Mexico

Editors-in-Chief

Prof. Emeritus Florentin Smarandache, PhD, Postdoc, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: smarand@unm.edu.

Dr. Mohamed Abdel-Baset, Head of Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamedbasset@ieee.org.

Prof. Dr. Maikel Leyva-Vázquez, Instituto Superior Tecnológico Bolivariano de Tecnología, Ecuador: maikel.leyvav@ug.edu.ec.

Associate Editors

Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

Assoc. Prof. Alok Dhital, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: adhital@unm.edu.

Dr. S. A. Edalatpanah, Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran, Email: saedalatpanah@gmail.com.

Charles Ashbacker, Charles Ashbacher Technologies, Box 294, 118 Chaffee Drive, Hiawatha, IA 52233, United States, Email: cashbacher@prodigy.net.

Prof. Dr. Xiaohong Zhang, Department of Mathematics, Shaanxi University of Science &Technology, Xian 710021, China, Email: zhangxh@shmtu.edu.cn.

Prof. Dr. W. B. Vasantha Kandasamy, School of Computer Science and Engineering, VIT, Vellore 632014, India, Email: vasantha.wb@vit.ac.in.

Editors

Yanhui Guo, University of Illinois at Springfield, One University Plaza, Springfield, IL 62703, United States,Email: yguo56@uis.edu.

Giorgio Nordo, MIFT - Department of Mathematical and Computer Science, Physical Sciences and Earth Sciences, Messina University, Italy,Email: giorgio.nordo@unime.it.

Mohamed Elhoseny, American University in the Emirates,Dubai,UAE,
Email: mohamed.elhoseny@aue.ae.

Le Hoang Son, VNU Univ. of Science, Vietnam National Univ. Hanoi, Vietnam, Email: sonlh@vnu.edu.vn.

Huda E. Khalid, Head of Scientific Affairs and Cultural Relations Department, Nineveh Province, Telafer University, Iraq, Email: dr.huda-ismael@uotelafer.edu.iq.

A. A. Salama, Dean of the Higher Institute of Business and Computer Sciences, Arish, Egypt, Email: ahmed_salama_2000@sci.psu.edu.eg.

Young Bae Jun, Gyeongsang National University, South Korea, Email: skywine@gmail.com.

Yo-Ping Huang, Department of Computer Science and Information, Engineering National Taipei University, New Taipei City, Taiwan, Email: yphuang@ntut.edu.tw.

Tarek Zayed, Department of Building and Real Estate, The Hong Kong Polytechnic University, Hung Hom, 8 Kowloon, Hong Kong, China, Email: tarek.zayed@polyu.edu.hk.

Vakkas Ulucay, Kilis 7 Aralık University, Turkey, Email: vulucay27@gmail.com.

Peide Liu, Shandong University of Finance and Economics, China, Email: peide.liu@gmail.com.

Jun Ye, Ningbo University, School of Civil and Environmental Engineering, 818 Fenghua Road, Jiangbei District, Ningbo City, Zhejiang Province, People's Republic of China, Email: yejun1@nbu.edu.cn.

Memet Şahin, Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey, Email: mesahin@gantep.edu.tr.

Muhammad Aslam & Mohammed Alshumrani, King Abdulaziz Univ., Jeddah, Saudi Arabia, Emails magmuhammad@kau.edu.sa, maalshmrani @kau.edu.sa.

Mutaz Mohammad, Department of Mathematics, Zayed University, Abu Dhabi 144534, United Arab Emirates. Email: Mutaz.Mohammad@zu.ac.ae.

Abdullahi Mohamud Sharif, Department of Computer Science, University of Somalia, Makka Al-mukarrama Road, Mogadishu, Somalia, Email: abdullahi.shariif@uniso.edu.so.

.

Copyright © Neutrosophic Sets and Systems, 2025

Katy D. Ahmad, Islamic University of Gaza, Palestine,Email: katyon765@gmail.com.
NoohBany Muhammad, American University of Kuwait,Kuwait,Email: noohmuhammad12@gmail .com.
Soheyb Milles, Laboratory of Pure and Applied Mathematics, University of Msila, Algeria, Email: soheyb.milles@univ-msila.dz.
Pattathal Vijayakumar Arun, College of Science and Technology,Phuentsholing,Bhutan,
Email: arunpv2601@gmail.com.
Endalkachew Teshome Ayele, Department of Mathematics, Arbaminch University, Arbaminch, Ethiopia,Email: endalkachewteshome83@yahoo. com.
A. Al-Kababji, College of Engineering, Qatar University,Doha,Qatar,
Email: ayman.alkababji@ieee.org.
Xindong Peng, School of Information Science and Engineering, Shaoguan University, Shaoguan 512005, China, Email: 952518336@qq.com.
Xiao-Zhi Gao, School of Computing, University of Eastern Finland, FI-70211 Kuopio, Finland, xiao-zhi.gao@uef.fi.
Madad Khan, Comsats Institute of Information Technology,Abbottabad,Pakistan,
Email: madadmath@yahoo.com.
G. Srinivasa Rao, Department of Statistics, The University of Dodoma, Dodoma, PO. Box: 259, Tanzania,Email: gaddesrao@gmail.com.
Ibrahim El-henawy, Faculty of Computers and Informatics,ZagazigUniversity,Egypt,
Email: henawy2000@yahoo.com.
Muhammad Saeed, Department of Mathematics, University of Management and Technology, Lahore, Pakistan,
Email: muhammad.saeed@umt.edu.pk.
A. A. A. Agboola, Federal University of Agriculture, Abeokuta, Nigeria,
Email: agboolaaaa@funaab.edu.ng.
Abduallah Gamal, Faculty of Computers and Informatics, Zagazig University, Egypt,
Email: abduallahgamal@zu.edu.eg.
Ebenezer Bonyah, Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi 00233, Ghana,
Email: ebbonya@gmail.com.
Roan Thi Ngan, Hanoi University of Natural Resources and Environment, Hanoi, Vietnam,
Email: rtngan@hunre.edu.vn.
Sol David Lopezdomínguez Rivas, Universidad Nacional de Cuyo, Argentina.
Email: sol.lopezdominguez@fce.uncu.edu.ar.
Maikel Yelandi Leyva Vázquez, Universidad Regional Autónoma de los Andes (UNIANDES), Avenida Jorge Villegas, Babahoyo, Los Ríos, Ecuador,
Email: ub.c.investigacion@uniandes.edu.ec.
Arlen Martín Rabelo, Exxis, Avda. Aviadores del Chaco N° 1669 c/ San Martin, Edif. Aymac I, 4to. piso, Asunción, Paraguay,
Email: arlen.martin@exxis-group.com.
Carlos Granados, Estudiante de Doctorado en Matematicas, Universidad del Antioquia, Medellın, Colombia,
Email: carlosgranadosortiz@outlook.es.
Tula Carola Sanchez Garcia, Facultad de Educacion de la Universidad Nacional Mayor de San Marcos, Lima, Peru,
Email: tula.sanchez1@unmsm.edu.pe.
Carlos Javier Lizcano Chapeta, Profesor - Investigador de pregrado y postgrado de la Universidad de Los Andes, Mérida 5101, Venezuela, Email: lizcha_4@hotmail.com.
Noel Moreno Lemus, Procter & Gamble International Operations S.A., Panamá,
Email: nmlemus@gmail.com.
Asnioby Hernandez Lopez, Mercado Libre, Montevideo, Uruguay.
Email: asnioby.hernandez@mercadolibre.com.
Muhammad Akram, University of the Punjab, New Campus, Lahore, Pakistan,
Email: m.akram@pucit.edu.pk.
Tatiana Andrea Castillo Jaimes, Universidad de Chile, Departamento de Industria, Doctorado en Sistemas de Ingeniería, Santiago de Chile, Chile,
Email: tatiana.a.castillo@gmail.com.
Irfan Deli, Muallim Rifat Faculty of Education, Kilis 7 Aralik University, Turkey,
Email: irfandeli@kilis.edu.tr.
Ridvan Sahin, Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey, Email: mat.ridone@gmail.com.
Ibrahim M. Hezam, Department of computer, Faculty of Education, Ibb University, Ibb City,

Yemen, Email: ibrahizam.math@gmail.com.
Moddassir khan Nayeem, Department of Industrial and Production Engineering, American International University-Bangladesh, Bangladesh; nayeem@aiub.edu.
Aiyared Iampan, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, Email: aiyared.ia@up.ac.th.
Ameirys Betancourt-Vázquez, 1 Instituto Superior Politécnico de Tecnologias e Ciências (ISPTEC), Luanda, Angola, Email: ameirysbv@gmail.com.
H. E. Ramaroson, University of Antananarivo, Madagascar, Email: erichansise@gmail.com.
G. Srinivasa Rao, Department of Mathematics and Statistics, The University of Dodoma, Dodoma PO. Box: 259, Tanzania.
Onesfole Kuramaa, Department of Mathematics, College of Natural Sciences, Makerere University, P.O Box 7062, Kampala, Uganda, Email: onesfole.kurama@mak.ac.ug.
Karina Pérez-Teruel, Universidad Abierta para Adultos (UAPA), Santiago de los Caballeros, República Dominicana, Email: karinaperez@uapa.edu.do.
Neilys González Benítez, Centro Meteorológico Pinar del Río, Cuba, Email: neilys71@nauta.cu.
Jesus Estupinan Ricardo, Centro de Estudios para la Calidad Educativa y la Investigation Cinetifica, Toluca,Mexico,Email: jestupinan2728@gmail.com
Victor Christianto, Malang Institute of Agriculture (IPM), Malang, Indonesia, Email: victorchristianto@gmail.com.
Wadei Al-Omeri, Department of Mathematics, Al-Balqa Applied University, Salt 19117, Jordan, Email: wadeialomeri@bau.edu.jo.
Ganeshsree Selvachandran, UCSI University, Jalan Menara Gading, Kuala Lumpur, Malaysia, Email: Ganeshsree@ucsiuniversity.edu.my.
Ilanthenral Kandasamy, School of Computer Science and Engineering (SCOPE), Vellore Institute of Technology (VIT), Vellore 632014, India, Email: ilanthenral.k@vit.ac.in
Kul Hur, Wonkwang University, Iksan, Jeollabukdo, South Korea, Email: kulhur@wonkwang.ac.kr.
Kemale Veliyeva & Sadi Bayramov, Department of Algebra and Geometry, Baku State University, 23 Z. Khalilov Str., AZ1148, Baku, Azerbaijan, Email: kemale2607@mail.ru,
Email: baysadi@gmail.com.
Irma Makharadze & Tariel Khvedelidze, Ivane Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences, Tbilisi, Georgia.
Inayatur Rehman, College of Arts and Applied Sciences, Dhofar University Salalah, Oman, Email: irehman@du.edu.om.
Mansour Lotayif, College of Administrative Sciences, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain.
Riad K. Al-Hamido, Math Department, College of Science, Al-Baath University, Homs, Syria, Email: riad-hamido1983@hotmail.com.
Saeed Gul, Faculty of Economics, Kardan University, Parwan-e- Du Square, Kabil, Afghanistan, Email: s.gul@kardan.edu.af.
Faruk Karaaslan, Çankırı Karatekin University, Çankırı, Turkey, Email: fkaraaslan@karatekin.edu.tr.
Morrisson Kaunda Mutuku, School of Business, Kenyatta University, Kenya
Surapati Pramanik, Department of Mathematics, Nandalal Ghosh B T College, India, Email: drspramanik@isns.org.in.
Suriana Alias, Universiti Teknologi MARA (UiTM) Kelantan, Campus Machang, 18500 Machang, Kelantan, Malaysia, Email: suria588@kelantan.uitm.edu.my.
Arsham Borumand Saeid, Dept. of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran, Email: arsham@uk.ac.ir.
Ahmed Abdel-Monem, Department of Decision support, Zagazig University, Egypt, Email: aabdelmounem@zu.edu.eg.
Çağlar Karamasa, Anadolu University, Faculty of Business,
Turkey, Email: ckaramasa@anadolu.edu.tr.
Mohamed Talea, Laboratory of Information Processing, Faculty of Science Ben M'Sik, Morocco, Email: taleamohamed@yahoo.fr.
Assia Bakali, Ecole Royale Navale, Casablanca, Morocco, Email: assiabakali@yahoo.fr.
V.V. Starovoytov, The State Scientific Institution «The United Institute of Informatics Problems of the National Academy of Sciences of Belarus»,

Copyright © Neutrosophic Sets and Systems, 2025

Minsk, Belarus, Email: ValeryS@newman.bas-net.by.
E.E. Eldarova, L.N. Gumilyov Eurasian National University, Nur-Sultan, Republic of Kazakhstan, Email: Doctorphd_eldarova@mail.ru. Mukhamed iyeva Dilnoz Tulkunovna & Egamberdiev Nodir Abdunazarovich, Science and innovation center for information and communication technologies, Tashkent University of Information Technologies (named after Muhammad Al-Khwarizmi), Uzbekistan.
Mohammad Hamidi, Department of Mathematics, Payame Noor University (PNU), Tehran, Iran. Email: m.hamidi@pnu.ac.ir.
Lemnaouar Zedam, Department of Mathematics, Faculty of Mathematics and Informatics, University Mohamed Boudiaf, M'sila, Algeria, Email: l.zedam@gmail.com.
M. Al Tahan, Department of Mathematics, Lebanese International University, Bekaa, Lebanon, Email: madeline.tahan@liu.edu.lb.
Mohammad Abobala, Tishreen University, Faculty of Science, Department of Mathematics, Lattakia, Syria,Email" mohammad.abobala@tishreen.edu.sy
Rafif Alhabib, AL-Baath University, College of Science, Mathematical Statistics Department, Homs, Syria, Email: ralhabib@albaath-univ.edu.sy.
R. A. Borzooei, Department of Mathematics, Shahid Beheshti University, Tehran, Iran, borzooei@hatef.ac.ir.
Selcuk Topal, Mathematics Department, Bitlis Eren University, Turkey, Email: s.topal@beu.edu.tr.
Qin Xin, Faculty of Science and Technology, University of the Faroe Islands, Tórshavn, 100, Faroe Islands.
Sudan Jha, Pokhara University,Kathmandu, Nepal, Email: jhasudan@hotmail.com.
Mimosette Makem and Alain Tiedeu, Signal, Image and Systems Laboratory, Dept. of Medical and Biomedical Engineering, Higher Technical Teachers' Training College of EBOLOWA, PO Box 886, University of Yaoundé, Cameroon, E-mail: alain_tiedeu@yahoo.fr.
Mujahid Abbas, Department of Mathematics and Applied Mathematics, University of Pretoria Hatfield 002, Pretoria, South Africa, Email: mujahid.abbas@up.ac.za.
Željko Stević, Faculty of Transport and Traffic Engineering Doboj, University of East Sarajevo, Lukavica, East Sarajevo, Bosnia and Herzegovina, Email: zeljko.stevic@sf.ues.rs.ba.
Michael Gr. Voskoglou, Mathematical Sciences School of Technological Applications, Graduate Technological Educational Institute of Western Greece, Patras, Greece, Email: voskoglou@teiwest.gr.
Saeid Jafari, College of Vestsjaelland South, Slagelse, Denmark, Email: sj@vucklar.dk.
Angelo de Oliveira, Ciencia da Computacao, Universidade Federal de Rondonia, Porto Velho - Rondonia, Brazil, Email: angelo@unir.br.
Valeri Kroumov, Okayama University of Science, Okayama, Japan, Email: val@ee.ous.ac.jp.
Rafael Rojas, Universidad Industrial de Santander, Bucaramanga, Colombia, Email: rafael2188797@correo.uis.edu.co.
Walid Abdelfattah, Faculty of Law, Economics and Management, Jendouba, Tunisia, Email: abdelfattah.walid@yahoo.com.
Akbar Rezaei, Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran, Email: rezaei@pnu.ac.ir.
John Frederick D. Tapia, Chemical Engineering Department, De La Salle University - Manila, 2401 Taft Avenue, Malate, Manila, Philippines, Email: john.frederick.tapia@dlsu.edu.ph.
Darren Chong, independent researcher, Singapore, Email: darrenchong2001@yahoo.com.sg.
Galina Ilieva, Paisii Hilendarski, University of Plovdiv, 4000 Plovdiv, Bulgaria, Email: galili@uni-plovdiv.bg.
Paweł Pławiak, Institute of Teleinformatics, Cracow University of Technology, Warszawska 24 st., F-5, 31-155 Krakow, Poland, Email: plawiak@pk.edu.pl.
E. K. Zavadskas, Vilnius Gediminas Technical University, Vilnius, Lithuania, Email: edmundas.zavadskas@vgtu.lt.
Darjan Karabasevic, University Business Academy, Novi Sad, Serbia, Email: darjan.karabasevic@mef.edu.rs.
Dragisa Stanujkic, Technical Faculty in Bor, University of Belgrade, Bor, Serbia, Email: dstanujkic@tfbor.bg.ac.rs.
Katarina Rogulj, Faculty of Civil Engineering,

Architecture and Geodesy, University of Split, Matice Hrvatske 15, 21000 Split, Croatia; Email: katarina.rogulj@gradst.hr.
Luige Vladareanu, Romanian Academy, Bucharest, Romania, Email: luigiv@arexim.ro.
Hashem Bordbar, Center for Information Technologies and Applied Mathematics, University of Nova Gorica, Slovenia, Email: Hashem.Bordbar@ung.si.
N. Smidova, Technical University of Kosice, SK 88902, Slovakia, Email: nsmidova@yahoo.com.
Quang-Thinh Bui, Faculty of Electrical Engineering and Computer Science, VŠB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic, Email: qthinhbui@gmail.com.
Mihaela Colhon & Stefan Vladutescu, University of Craiova, Computer Science Department, Craiova, Romania,Emails: colhon.mihaela@ucv.ro, vladutescu.stefan@ucv.ro.
Philippe Schweizer, Independent Researcher, Av. de Lonay 11, 1110 Morges, Switzerland, Email: flippe2@gmail.com.
Madjid Tavanab, Business Information Systems Department, Faculty of Business Administration and Economics University of Paderborn, D-33098 Paderborn, Germany, Email: tavana@lasalle.edu.
Rasmus Rempling, Chalmers University of Technology, Civil and Environmental Engineering, Structural Engineering, Gothenburg, Sweden.
Fernando A. F. Ferreira, ISCTE Business School, BRU-IUL, University Institute of Lisbon, Avenida das Forças Armadas, 1649-026 Lisbon, Portugal, Email: fernando.alberto.ferreira@iscte-iul.pt.
Julio J. Valdés, National Research Council Canada, M-50, 1200 Montreal Road, Ottawa, Ontario K1A 0R6, Canada, Email: julio.valdes@nrc-cnrc.gc.ca.
Tieta Putri, College of Engineering Department of Computer Science and Software Engineering, University of Canterbury, Christchurch, New Zeeland.
Phillip Smith, School of Earth and Environmental Sciences, University of Queensland, Brisbane, Australia, phillip.smith@uq.edu.au.
Sergey Gorbachev, National Research Tomsk State University, 634050 Tomsk, Russia, Email: gsv@mail.tsu.ru.
Sabin Tabirca, School of Computer Science, University College Cork, Cork, Ireland, Email: tabirca@neptune.ucc.ie.
Umit Cali, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway, Email: umit.cali@ntnu.no.
Willem K. M. Brauers, Faculty of Applied Economics, University of Antwerp, Antwerp, Belgium, Email: willem.brauers@uantwerpen.be.
M. Ganster, Graz University of Technology, Graz, Austria, Email: ganster@weyl.math.tu-graz.ac.at.
Ignacio J. Navarro, Department of Construction Engineering, Universitat Politècnica de València, 46022 València, Spain, Email: ignamar1@cam.upv.es.
Francisco Chiclana, School of Computer Science and Informatics, De Montfort University, The Gateway, Leicester, LE1 9BH, United Kingdom, Email: chiclana@dmu.ac.uk.
Jean Dezert, ONERA, Chemin de la Huniere, 91120 Palaiseau, France, Email: jean.dezert@onera.fr.

.

Copyright © Neutrosophic Sets and Systems, 2025

ISSN 2331-6055 (print) NSS ISSN 2331-608X (online)
University of New Mexico

Contents

Takaaki Fujita, Florentin Smarandache, Neutrosophic Circular-arc Graphs and Proper circular-arc Graphs……1-30
Lunsen Zou, Zhiqiang Wang, Jin Wang, Linwei Wang, Liangrui Zhou, Modified WASPAS Technique for Triangular Fuzzy Neutrosophic Number Multiple-Attribute Decision-Making to Safe Operation Risk Assessment of Batteries in Energy Storage Power Stations……..……31-46
Ahmed Salem Heilat, An Approach to Numerical Solutions for Refined Neutrosophic Differential Problems of High-Orders……..…….47-59
Qiwei Liu, Jiayin Li, Chang Liu, Lagrange-Integrated GRA Model for Urban Landscape Sculpture Site Selection with Neutrosophic Decision-Making……60-73
Mohamed eassa, M. Myvizhi, Mohamed Elkholy, Ahmed Abdelhafeez, Hussam Elbehiery, Enhanced MADM Strategy with Heptapartitioned Neutrosophic Distance Metrics……..……74-96
Florentin Smarandache, Neutrosophic Perspectives on the Body-Mind-Soul-Spirit Fluidity……..…97-104
A.Meenakshi, Shivangi Mishra J, Correlation and Regression Coefficients of Neutrosophic Graph Products using Minimum Spanning Tree for Enhancing Healthcare Strategies……..……..105-126
Xiangke Zhang, Shoukui He, Xu Chen, A Cross-Entropy-Based Framework Using HyperSoft Type-2 Neutrosophic Numbers for Innovation Capability Evaluation in Central Cities of Emerging Economies……..…..127-148
Mona Mohamed, Ahmed M. AbdelMouty, Khalid Mohamed, Florentin Smarandache, SuperHyperSoft-Driven Evaluation of Smart Transportation in Centroidous-Moosra: Real-World Insights for the UAV Era……...149-163
Aiping Cui, Zhiyao Liu, Utilizing CoCoSo and Interval Neutrosophic MAGDM with SuperHyperSoft for Competitiveness Assessment in Rural Tourism of Surrounding Cities……....164-180
Zahid Khan and Katrina Lane Krebs, Enhancing Neutrosophic Data Analysis: A Review of Neutrosophic Measures and Applications with Neutrostat……..……...181-190
Abdullah Ali Salamai, A Robust Neutrosophic Model for Risk Management in Livestock Supply Chains: A Case Study Towards Sustainable Practices ……..…….191-205
Heng He, A Novel Approach to Assessing Art Education Teaching Quality in Vocational Colleges Based on Double-Valued Neutrosophic Numbers and Multi-Attribute Decision-Making with Tree Soft Sets……206-218
Seher BODUR, Serpil KAYA and Selçuk TOPAL, A Novel Analysis on the Perception of Physical Environmental Quality: An In-Depth Study Comparing the Neutrosophic Likert Scale with Score Function and the Classic Likert Scale - Statistical Analysis of Data from a Restaurant Survey……219-234
Peng Tong, Bin Xie, A Refined Dual-Valued Neutrosophic COPRAS-Based Approach for Holistic Evaluation of English Teaching Standards……..……..235-251
Giorgio Nordo, Saeid Jafari and Maikel Yelandi Leyva V azquez, A Python Framework Enhancement for Neutrosophic Topologies……252-273
Yi Zheng, Jun Chen, A Triangular Fuzzy Neutrosophic Model for Assessing and Optimizing Educational Quality in University Badminton Initiatives: A comprehensive study……..274-291
K. Radhika, T. Harikrishnan, R. Ambrose Prabhu, P.Tharaniya, M.John peter, M.Anandhkumar, On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices……292-322
Changshun Bai, Pute Lan, DVNN-CoCoSo: A Hybrid Neutrosophic Model for Evaluating Social and Economic Impacts of Sports Events……..…323-340
Bhimraj Basumatary, Jeevan Krishna Khaklary, A Study on Continuity functions in Neutro-Topological Spaces……..……..341-352
Prasanta Kumar Raut, Surapati Pramanik, Deepak Kumar Mohapatra, Srikanta Kumar Sahoo, Solving the shortest path based on the traveling salesman problem with a genetic algorithm in a Fermatean neutrosophic environment……..……353-366
Haihua Ding, Haifeng Chen, Quality Assessment in Higher Education Management using the Modified MARCOS Method with Double-Valued Neutrosophic Numbers: A Case Study ……..……..367-385
Yifang Zhao, Elevating Service Quality in Mass Sports Events Through Trapezoidal Neutrosophic Weighted Product Models……..……386-401
Ibrahim Yasser, Aya A. Abd El-Khalek, Huda E. Khalid, A. A. Salama, Ahmed K. Essa, Cloud Computing and Neutrosophic Logic: Navigating Uncertainty in Educational Administration……..……..402-451
Rong Yan, Yongguang Han, Liqing Hao, An Efficient Neutrosophic Weighted Sum Approach with Insights into Engineering Project Management Performance Assessment……..……..452-469
Seher BODUR, Selçuk TOPAL, and Hacı GÜRKAN, A Statistical Comparison Study of a Real-Life Survey Data on Procedural Justice via Neutrosophic Likert Scale with Score Function and Classical Likert Scale……..……..470-489
Dongdong Liu, REGIME Framework for Performance Ability Evaluation of Excellent Football Players with Probabilistic Simplified Neutrosophic Sets……490-505
K. Radhika, S. Senthil, N. Kavitha, R.Jegan, M.Anandhkumar, A. Bobin, Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making……506-539
Ying Du, A Methodological Approach to Decision-Making Using Interval-Valued Neutrosophic Framework for Enhancing and Evaluating English Blended Teaching Quality ……540-552
A.N. Revathi and S. Mohanaselvi, A Neutrosophic Fixed Charge Multi-Objective 4-Dimensional Multi-item Transportation Problem with Carbon Emission and Budget Constraints……553-576
Raed Hatamleh, On A Novel Topological Space Based on Partially Ordered Ring Of Weak Fuzzy Complex Numbers and Its Relation With The Partially Ordered Neutrosophic Ring of Real Numbers……..……577-590

Copyright © Neutrosophic Sets and Systems, 2025

NSS

Neutrosophic Sets and Systems, Vol.78, 2025

University of New Mexico

Neutrosophic Circular-arc Graphs and Proper circular-arc graphs

Takaaki Fujita[1]∗, Florentin Smarandache[2],

[1]∗ Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp

[2] University of New Mexico, Gallup Campus, NM 87301, USA. fsmarandache@gmail.com :

Abstract. Graph theory is a fundamental branch of mathematics that studies networks made up of nodes (vertices) and connections (edges). A key concept in graph theory is the intersection graph, where vertices represent sets, and edges are drawn between vertices if their corresponding sets intersect. A circular-arc graph specifically models the intersections of arcs on a circle, with vertices corresponding to the arcs and edges existing between intersecting arcs. This paper delves into the study of circular-arc graphs within the frameworks of fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam Neutrosophic graphs, all of which incorporate uncertainty into graph structures. Additionally, we examine the concept of proper circular-arc graphs.

Keywords: Neutrosophic graph, Fuzzy graph, Intersection graph, Circular-arc Graph

1. Introduction

1.1. *Circular arc graphs*

Graph theory is a fundamental branch of mathematics that studies networks consisting of nodes (vertices) and connections (edges), which are crucial for analyzing the structure, paths, and properties of these networks [16]. One important example in graph theory is the intersection graph, where vertices correspond to sets, and edges are drawn between vertices if their corresponding sets intersect [32,61,73]. Many related graph classes have been extensively researched, such as interval graphs [28,33], mixed interval graphs [36–38].

In this paper, we focus on circular-arc graphs [31, 35, 84], a specific type of intersection graph that has garnered significant attention due to its practical applications and importance in the study of various graph classes. A circular-arc graph represents the intersections of arcs on a circle, where vertices correspond to the arcs, and edges exist between arcs that intersect [31,35,84]. Circular-arc graphs are known for their linear-time recognition algorithms [44, 46, 48].

Additionally, several related graph classes have been studied, including: Circular Arc Bigraphs [7], Proper Circular-Arc Bigraphs [69], Chordal Proper Circular Arc Graphs [5], Unit Circular-Arc Graphs [18,47,56], Proper Circular-Arc Graphs [1,8,20,81,86], Helly Circular-Arc Graphs [9, 34, 54, 68], Proper Helly Circular-Arc Graphs [55], Weighted Circular-Arc Graphs [53, 59], Circular-Arc Overlap Graphs [49, 82, 83], Bipartite Co-Circular-Arc Graphs [19], and Circular-Arc Product Graphs [39].

1.2. *Fuzzy Graphs and Neutrosophic Graphs*

In this paper, we explore Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. These graph concepts were developed to handle uncertainty in practical applications.

A fuzzy graph assigns a membership value between 0 and 1 to each vertex and edge, indicating the degree of uncertainty or imprecision associated with them [67]. Fuzzy graphs essentially represent fuzzy sets [17, 52, 88, 89] and are widely used in fields such as social networks, decision-making, and transportation systems, where relationships can be uncertain or ambiguous [62, 67].

Intuitionistic Fuzzy Graphs extend fuzzy graphs by introducing both membership and non-membership degrees for vertices and edges, allowing for a more nuanced representation of uncertainty in relationships [30, 63, 64].

Neutrosophic Graphs [12,22,23,25–27,76–79], derived from neutrosophic set theory [4,75,80], add three components—truth, indeterminacy, and falsity—to classical and fuzzy logic. This approach offers greater flexibility in representing uncertainty .

Turiyam Neutrosophic Graphs, an extension of neutrosophic and fuzzy graphs, assign four attributes—truth, indeterminacy, falsity, and a liberal state—to each vertex and edge [29]. Note that Turiyam Neutrosophic Graphs are a subset of quadripartitioned neutrosophic graphs (cf. [74]).

In addition, various types of intersection graphs have been studied within these frameworks. Research has focused on fuzzy intersection graphs [14, 60, 65, 72], neutrosophic intersection graphs [11], fuzzy permutation graphs [66], and fuzzy interval graphs [14]. Many studies have also explored related topics [24].

1.3. *Our Contribution*

Our motivation and contribution are as follows. As previously mentioned, extensive research has been conducted on Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Likewise, significant work has been done on Circular-arc graphs in classical graph theory. These studies hold both mathematical and practical

significance due to their wide range of applications. However, while intersection graphs have been explored within the frameworks of fuzzy and neutrosophic theories, their full properties and characteristics remain largely unexplored.

In this paper, we define Circular-arc graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs, and examine their properties and interrelationships. Through these studies, we aim to contribute to the advancement of graph theory, particularly in the areas of intersection graphs, Fuzzy Graphs, and Neutrosophic Graphs.

2. **Preliminaries and definitions**

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, and neutrosophic graphs.

2.1. *Basic Graph Concepts*

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [15, 16, 87].

Definition 2.1 (Graph)**.** [16] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.2 (Degree)**.** [16] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 2.3 (Subgraph)**.** [16] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Definition 2.4 (Induced subgraph)**.** [42, 51] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset $V' \subseteq V$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set consists of all edges from E that have both endpoints in V'. Formally, the induced subgraph $G[V'] = (V', E')$ is defined as follows:

$$E' = \{(u, v) \in E \mid u \in V', v \in V'\}.$$

In other words, $G[V']$ is the subgraph of G that contains all vertices in V' and all edges from G whose endpoints are both in V'.

Definition 2.5 (Complete Graph)**.** A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

2.2. *Intersection graph*

Intersection graphs have been extensively studied. The definition is provided below [32, 61, 73].

Definition 2.6 (Intersection graph)**.** [32,61,73] A *intersection graph* is a graph that represents the intersection relationships between sets. Formally, let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a collection of sets. The *intersection graph* $G = (V, E)$ associated with $\mathcal{S}$ is a graph where:

- The vertex set V corresponds to the sets in $\mathcal{S}$, i.e., $V = \{v_1, v_2, \ldots, v_n\}$, where each vertex v_i represents the set $S_i \in \mathcal{S}$.
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

2.3. *Fuzzy, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs*

In this subsection, we examine Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs.

Fuzzy graphs are frequently discussed in comparison with crisp graphs, which represent the classical form of graphs [67].

Definition 2.7. (cf. [50, 67]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Taking the above into consideration, we define Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs as follows. Please note that the definitions have been consolidated for simplicity.

Definition 2.8 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs)**.** (cf. [23]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

(1) *Fuzzy Graph [67]:*
- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v.

(2) *Intuitionistic Fuzzy Graph (IFG) [2]:*
- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $v_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + v_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $v_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + v_B(u, v) \leq 1$.

(3) *Neutrosophic Graph [3, 43, 45, 70]:*
- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
  - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
  - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
  - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
  - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.

(4) *Turiyam Neutrosophic Graph [29]:*
- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
  - $t(v) \in [0, 1]$ is the truth value,
  - $iv(v) \in [0, 1]$ is the indeterminacy value,
  - $fv(v) \in [0, 1]$ is the falsity value,
  - $lv(v) \in [0, 1]$ is the liberal state value,
  - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.

- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

3. Result: Circular-arc graph

In this section, we extend the concept of Circular-arc graphs to Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definition of a Circular-arc graph is provided below. As mentioned in the introduction, Circular-arc graphs are a well-studied topic in graph theory, with numerous ongoing research efforts [10, 35, 57, 71, 81, 85].

Definition 3.1 (Circular-arc graph)**.** [85] A *circular-arc graph* is the intersection graph of a set of arcs on a circle. Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc of the circle. The corresponding circular-arc graph $G = (V, E)$ is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$ consists of one vertex for each arc in the set.
- The edge set E is defined such that there is an edge between two vertices I_α and I_β if and only if their corresponding arcs I_α and I_β intersect, i.e.,

$$\{I_\alpha, I_\beta\} \in E \iff I_\alpha \cap I_\beta \neq \emptyset.$$

A family of arcs corresponding to the graph G is referred to as an *arc model* of the circular-arc graph.

The following theorem is clearly valid.

Theorem 3.2. *A Circular-Arc Graph satisfies the definition of an Intersection Graph.*

Proof. In a circular-arc graph $G = (V, E)$, the vertex set $V = \{I_1, I_2, \ldots, I_n\}$ corresponds to a collection of arcs $\{I_1, I_2, \ldots, I_n\}$ on a circle C_1. Each vertex v_i in the graph represents an arc I_i on the circle. This directly satisfies the definition of an intersection graph, where each vertex corresponds to a set (in this case, an arc).

And in a circular-arc graph, there is an edge between two vertices I_α and I_β if and only if the corresponding arcs I_α and I_β intersect, i.e., $I_\alpha \cap I_\beta \neq \emptyset$. This is consistent with the edge definition of an intersection graph, where an edge exists between two vertices if their corresponding sets (arcs) have a non-empty intersection.

By these definitions, we can see that the structure of a circular-arc graph satisfies the criteria of an intersection graph. Each vertex in a circular-arc graph represents a set (an arc), and there is an edge between two vertices if and only if their corresponding arcs (sets) intersect. Therefore, a *Circular-Arc Graph* is an instance of an *Intersection Graph.* □

We extend the above concept to Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definitions are provided below.

Definition 3.3 (Fuzzy Circular-Arc Graph)**.** (cf. [41, 65]) Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. A *Fuzzy Circular-Arc Graph* is a fuzzy graph $\psi = (V, \sigma, \mu)$ defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$, where each vertex corresponds to an arc I_i.
- The vertex membership function $\sigma : V \to [0, 1]$ assigns to each vertex I_i a membership degree $\sigma(I_i)$, representing the degree of membership of the arc I_i in the fuzzy graph.
- The edge membership function $\mu : V \times V \to [0, 1]$ is defined by:

$$\mu(I_\alpha, I_\beta) = \begin{cases} \min\{\sigma(I_\alpha), \sigma(I_\beta)\}, & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

These functions satisfy the following conditions:

(1) $\mu(I_\alpha, I_\beta) \leq \min\{\sigma(I_\alpha), \sigma(I_\beta)\}$ for all $I_\alpha, I_\beta \in V$,
(2) $\mu(I_\alpha, I_\beta) = \mu(I_\beta, I_\alpha)$ (symmetry),
(3) $\mu(I_\alpha, I_\alpha) = 0$ (no self-loops).

The resulting fuzzy graph represents the degrees of connection between arcs based on their intersections and membership degrees.

Definition 3.4 (Intuitionistic Fuzzy Circular-Arc Graph)**.** Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. An *Intuitionistic Fuzzy Circular-Arc Graph* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The intuitionistic fuzzy vertex set $A = \{\langle I_i, \mu_A(I_i), \nu_A(I_i)\rangle : I_i \in V\}$, where:
  - $\mu_A(I_i) \in [0, 1]$ is the degree of membership of I_i,
  - $\nu_A(I_i) \in [0, 1]$ is the degree of non-membership of I_i,
  - $\mu_A(I_i) + \nu_A(I_i) \leq 1$.
- The intuitionistic fuzzy edge set $B = \{\langle (I_\alpha, I_\beta), \mu_B(I_\alpha, I_\beta), \nu_B(I_\alpha, I_\beta)\rangle : I_\alpha, I_\beta \in V\}$, where:
  - $\mu_B(I_\alpha, I_\beta) = \begin{cases} \min\{\mu_A(I_\alpha), \mu_A(I_\beta)\}, & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$
  - $\nu_B(I_\alpha, I_\beta) = \begin{cases} \max\{\nu_A(I_\alpha), \nu_A(I_\beta)\}, & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$

Definition 3.5 (Neutrosophic Circular-Arc Graph)**.** Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. A *Neutrosophic Circular-Arc Graph* is a neutrosophic graph $G = (V, E, \sigma, \mu)$ defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The neutrosophic vertex membership function $\sigma : V \to [0,1]^3$, assigning to each vertex I_i a triple $\sigma(I_i) = (\sigma_T(I_i), \sigma_I(I_i), \sigma_F(I_i))$, where:
  - $\sigma_T(I_i)$ is the truth-membership degree,
  - $\sigma_I(I_i)$ is the indeterminacy-membership degree,
  - $\sigma_F(I_i)$ is the falsity-membership degree,
  - $0 \leq \sigma_T(I_i) + \sigma_I(I_i) + \sigma_F(I_i) \leq 3$.
- The edge set $E = \{(I_\alpha, I_\beta) : I_\alpha, I_\beta \in V, I_\alpha \cap I_\beta \neq \emptyset\}$.
- The neutrosophic edge membership function $\mu : E \to [0,1]^3$, assigning to each edge (I_α, I_β) a triple $\mu(I_\alpha, I_\beta) = (\mu_T(I_\alpha, I_\beta), \mu_I(I_\alpha, I_\beta), \mu_F(I_\alpha, I_\beta))$, where:
  - $\mu_T(I_\alpha, I_\beta) = \min\{\sigma_T(I_\alpha), \sigma_T(I_\beta)\}$,
  - $\mu_I(I_\alpha, I_\beta) = \max\{\sigma_I(I_\alpha), \sigma_I(I_\beta)\}$,
  - $\mu_F(I_\alpha, I_\beta) = \max\{\sigma_F(I_\alpha), \sigma_F(I_\beta)\}$,
  - $0 \leq \mu_T(I_\alpha, I_\beta) + \mu_I(I_\alpha, I_\beta) + \mu_F(I_\alpha, I_\beta) \leq 3$.

Definition 3.6 (Turiyam Neutrosophic Circular-Arc Graph)**.** Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. A *Turiyam Neutrosophic Circular-Arc Graph* is a Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ defined as follows:

- The vertex set $V^T = \{I_1, I_2, \ldots, I_n\}$.
- For each vertex $I_i \in V^T$, the Turiyam Neutrosophic membership functions assign:
  - Truth value $t(I_i) \in [0,1]$,
  - Indeterminacy value $iv(I_i) \in [0,1]$,
  - Falsity value $fv(I_i) \in [0,1]$,
  - Liberal value $lv(I_i) \in [0,1]$,
  - Constraint: $0 \leq t(I_i) + iv(I_i) + fv(I_i) + lv(I_i) \leq 4$.
- The edge set $E^T = \{(I_\alpha, I_\beta) : I_\alpha, I_\beta \in V^T, I_\alpha \cap I_\beta \neq \emptyset\}$.
- For each edge $(I_\alpha, I_\beta) \in E^T$, the Turiyam Neutrosophic membership functions assign:
  - Truth value $t(I_\alpha, I_\beta) = \min\{t(I_\alpha), t(I_\beta)\}$,
  - Indeterminacy value $iv(I_\alpha, I_\beta) = \max\{iv(I_\alpha), iv(I_\beta)\}$,
  - Falsity value $fv(I_\alpha, I_\beta) = \max\{fv(I_\alpha), fv(I_\beta)\}$,
  - Liberal value $lv(I_\alpha, I_\beta) = \max\{lv(I_\alpha), lv(I_\beta)\}$,
  - Constraint: $0 \leq t(I_\alpha, I_\beta) + iv(I_\alpha, I_\beta) + fv(I_\alpha, I_\beta) + lv(I_\alpha, I_\beta) \leq 4$.

We now examine the properties of the graph mentioned above. The following theorem holds.

Theorem 3.7. *A Neutrosophic circular-arc graph can be transformed into a classic circular-arc graph.*

Proof. Obviously holds. □

Corollary 3.8. *A Fuzzy Circular-Arc Graph, Intuitionistic Fuzzy Circular-Arc Graph, or Turiyam Neutrosophic Circular-Arc Graph can be transformed into a classic Circular-Arc Graph.*

Proof. Obviously holds. □

Theorem 3.9. *Neutrosophic Graph can be represented as a Neutrosophic Circular-Arc Graph.*

Proof. We represent the vertices of the neutrosophic graph as arcs on a circle. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices.
- $E \subseteq V \times V$ is the set of edges between the vertices.
- $\sigma(v_i) = (\sigma_T(v_i), \sigma_I(v_i), \sigma_F(v_i))$ assigns truth, indeterminacy, and falsity membership degrees to each vertex $v_i \in V$.
- $\mu(v_i, v_j) = (\mu_T(v_i, v_j), \mu_I(v_i, v_j), \mu_F(v_i, v_j))$ assigns truth, indeterminacy, and falsity membership degrees to each edge $(v_i, v_j) \in E$.

We now need to represent G as a Neutrosophic Circular-Arc Graph. To do this, map each vertex v_i to an arc I_i on a circle C_1.

We define the vertex set of the Neutrosophic Circular-Arc Graph. In the Neutrosophic Circular-Arc Graph, each vertex $v_i \in V$ of the neutrosophic graph is represented by an arc $I_i \subseteq C_1$. Therefore, the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ in the neutrosophic graph corresponds to the set of arcs $\{I_1, I_2, \ldots, I_n\}$ on the circle C_1. Each vertex I_i in the Neutrosophic Circular-Arc Graph is assigned the same neutrosophic membership degree as the corresponding vertex v_i in the neutrosophic graph:

$$\sigma(I_i) = (\sigma_T(I_i), \sigma_I(I_i), \sigma_F(I_i)) = \sigma(v_i).$$

We define the edge set of the Neutrosophic Circular-Arc Graph. In the Neutrosophic Circular-Arc Graph, an edge exists between two arcs I_α and I_β if and only if the arcs intersect, i.e., $I_\alpha \cap I_\beta \neq \emptyset$. This corresponds to the edges in the neutrosophic graph. That is, there is an edge between v_α and $v_\beta \in V$ in the neutrosophic graph if and only if there is an edge between the corresponding arcs I_α and I_β in the Neutrosophic Circular-Arc Graph.

For the Neutrosophic Circular-Arc Graph, the neutrosophic membership degrees for each edge (I_α, I_β) are defined based on the membership degrees of the corresponding vertices:

$$\mu_T(I_\alpha, I_\beta) = \min\{\sigma_T(I_\alpha), \sigma_T(I_\beta)\},$$

$$\mu_I(I_\alpha, I_\beta) = \max\{\sigma_I(I_\alpha), \sigma_I(I_\beta)\},$$

$$\mu_F(I_\alpha, I_\beta) = \max\{\sigma_F(I_\alpha), \sigma_F(I_\beta)\}.$$

This assignment of membership degrees ensures that the edge relationships in the neutrosophic graph are preserved in the Neutrosophic Circular-Arc Graph.

We have successfully mapped each vertex of the neutrosophic graph G to an arc on the circle and preserved the edge relationships by ensuring that arcs intersect if and only if their corresponding vertices are connected by an edge in G. Additionally, the neutrosophic membership degrees for vertices and edges have been preserved. □

Theorem 3.10. *Every subgraph of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph where each vertex represents an arc on a circle, and let $G' = (V', E', \sigma', \mu')$ be a subgraph of G with $V' \subseteq V$ and $E' \subseteq E$. The vertices in V' correspond to a subset of the arcs in G, and the edges in E' represent the intersections between these arcs, following the same intersection rules as in G.

Since the vertices and edges of G' inherit their structure from G, including the neutrosophic membership functions σ' and μ', the arcs in V' still form a valid arc model on the circle. Therefore, the subgraph G' is also a Neutrosophic Circular-Arc Graph.

Thus, every subgraph of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph. □

Theorem 3.11. *The disjoint union of two Neutrosophic Circular-Arc Graphs is also a Neutrosophic Circular-Arc Graph.*

Proof. Let $G_1 = (V_1, E_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, E_2, \sigma_2, \mu_2)$ be two Neutrosophic Circular-Arc Graphs. The disjoint union $G = G_1 \sqcup G_2$ is defined by taking the union of the vertex sets and edge sets, i.e., $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, with no edges between V_1 and V_2.

Since the arcs corresponding to vertices in V_1 and V_2 form valid arc models independently on separate parts of the circle, their disjoint union also forms a valid arc model. The neutrosophic membership functions σ and μ are preserved for each respective graph, ensuring that G satisfies the conditions for being a Neutrosophic Circular-Arc Graph.

Thus, the disjoint union of two Neutrosophic Circular-Arc Graphs is also a Neutrosophic Circular-Arc Graph. □

Theorem 3.12. *The complement of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph, where V represents a set of arcs on a circle, and E represents the edges between intersecting arcs. The complement graph $\overline{G} = (V, \overline{E}, \overline{\sigma}, \overline{\mu})$ has the same vertex set, but the edge set $\overline{E}$ consists of all pairs (u, v) such that $(u, v) \notin E$.

In the context of circular-arc graphs, an edge $(u, v) \in E$ exists if and only if the corresponding arcs I_u and I_v intersect. For the complement graph $\overline{G}$, the edge $(u, v) \in \overline{E}$ exists if and only if I_u and I_v do not intersect. Since non-intersecting arcs also form valid configurations on a circle, the complement graph retains the structure of a circular-arc graph.

The neutrosophic membership functions $\overline{\sigma}(v)$ and $\overline{\mu}(u, v)$ are defined as follows:

$$\overline{\sigma}(v) = \sigma(v),$$

$$\overline{\mu}(u, v) = \begin{cases} \min(\sigma_T(u), \sigma_T(v)) & \text{if } I_u \cap I_v = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Since the complement preserves the structure and neutrosophic properties, the complement of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph. □

Theorem 3.13. *If the union of all arcs in a Neutrosophic Circular-Arc Graph c overs the entire circle, then the graph is connected.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph, where each vertex $v \in V$ corresponds to an arc I_v on a circle C_1. Assume that the union of all arcs covers the entire circle:

$$\bigcup_{v \in V} I_v = C_1.$$

We aim to show that G is connected; that is, for any two vertices $u, w \in V$, there exists a path in G connecting u and w.

Since the arcs $\{I_v\}_{v \in V}$ cover C_1 completely, we can traverse the circle from any point in I_u to any point in I_w without leaving the union of the arcs. Along this traversal, we pass through a sequence of arcs corresponding to vertices in V. Specifically, there exists a sequence of vertices $v_0, v_1, \ldots, v_k \in V$ such that:

- $v_0 = u$ and $v_k = w$,
- For each $i = 0, 1, \ldots, k-1$, the arcs I_{v_i} and $I_{v_{i+1}}$ intersect: $I_{v_i} \cap I_{v_{i+1}} \neq \emptyset$.

Because arcs I_{v_i} and $I_{v_{i+1}}$ intersect, there is an edge $(v_i, v_{i+1}) \in E$. Therefore, the sequence $v_0, v_1, \ldots, v_k$ forms a path in G connecting u and w.

Since u and w were arbitrary, G is connected. □

Theorem 3.14. *Deleting a vertex (and its incident edges) from a Neutrosophic Circular-Arc Graph results in another Neutrosophic Circular-Arc Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph with an arc model $\{I_v \mid v \in V\}$ on the circle C_1. Let $v_0 \in V$ be the vertex to be deleted. Define the new graph $G' = (V', E', \sigma', \mu')$ as follows:

- $V' = V \setminus \{v_0\}$,
- $E' = \{(u, v) \in E \mid u, v \in V'\}$,
- σ' and μ' are the restrictions of σ and μ to V' and E', respectively.

The arcs corresponding to G' are $\{I_v \mid v \in V'\}$, which are the same as in G except that I_{v_0} is removed. These arcs still lie on the circle C_1.

The intersection relationships among the remaining arcs are preserved. For any $u, w \in V'$, the arcs I_u and I_w intersect if and only if $(u, w) \in E'$. The neutrosophic membership functions σ' and μ' retain the same values as in G for the vertices and edges of G'.

Therefore, G' satisfies all the conditions of a Neutrosophic Circular-Arc Graph. Thus, deleting a vertex and its incident edges from G results in another Neutrosophic Circular-Arc Graph. □

We examine the relationship between Neutrosophic Circular-Arc Graphs and Neutrosophic Interval Graphs. In interval graphs [13, 33, 40], each vertex represents an interval on the real line, and edges exist between vertices whose intervals overlap. A Neutrosophic Interval Graph extends this concept by incorporating neutrosophic membership degrees, allowing for the representation of uncertainty in both vertices and edges. The formal definition of a Neutrosophic Interval Graph is provided below.

Definition 3.15 (Neutrosophic Interval Graph)**.** [21] Let V be a finite set of vertices, and let

$$\mathcal{N} = \{(\mu_1, \tau_1, \zeta_1), (\mu_2, \tau_2, \zeta_2), \ldots, (\mu_n, \tau_n, \zeta_n)\}$$

be a finite family of neutrosophic intervals on the real line $\mathbb{R}$. Each triple (μ_i, τ_i, ζ_i) represents the truth-membership function $\mu_i : \mathbb{R} \to [0, 1]$, the indeterminacy-membership function $\tau_i : \mathbb{R} \to [0, 1]$, and the falsity-membership function $\zeta_i : \mathbb{R} \to [0, 1]$ such that:

$$\mu_i(x) + \tau_i(x) + \zeta_i(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

The *Neutrosophic Interval Graph* $G = (V, \mu_V, \tau_V, \zeta_V, \rho_\mu, \rho_\tau, \rho_\zeta)$ is defined as follows:

- The vertex membership functions $\mu_V : V \to [0, 1]$, $\tau_V : V \to [0, 1]$, and $\zeta_V : V \to [0, 1]$ are given by:

$$\mu_V(v_i) = \sup_{x \in \mathbb{R}} \mu_i(x), \quad \tau_V(v_i) = \sup_{x \in \mathbb{R}} \tau_i(x), \quad \zeta_V(v_i) = \sup_{x \in \mathbb{R}} \zeta_i(x).$$

- The neutrosophic adjacency relations $\rho_\mu : V \times V \to [0,1]$, $\rho_\tau : V \times V \to [0,1]$, and $\rho_\zeta : V \times V \to [0,1]$ are defined by:

$$\rho_\mu(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_\tau(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\tau_i(x), \tau_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_\zeta(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\zeta_i(x), \zeta_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

Theorem 3.16. *[21] a Neutrosophic Interval Graph can be transformed into a classic Interval Graph.*

Proof. Refer to the literature [21]. □

Theorem 3.17. *Every Neutrosophic Interval Graph can be represented as a Neutrosophic Circular-Arc Graph.*

Proof. Consider a *Neutrosophic Interval Graph* $G = (V, E, \sigma, \mu)$, where each vertex corresponds to an interval on the real line, and edges exist between vertices whose intervals overlap. The neutrosophic membership functions assign truth, indeterminacy, and falsity degrees to each vertex and edge. We will prove that this interval graph can be transformed into a neutrosophic circular-arc graph.

We begin by mapping the intervals on the real line to arcs on a circle.

- Consider the real line as a straight line segment where each vertex in the interval graph represents an interval.
- Map this straight line onto a circle C_1, ensuring that no intervals wrap around the circle. In this process, each interval on the real line is transformed into an arc on the circle.
- Thus, each vertex $v \in V$ that corresponds to an interval on the real line will now correspond to an arc on the circle C_1.

Next, we verify that the intersection relationships between intervals are preserved during this transformation.

- Two intervals on the real line intersect if and only if their corresponding arcs on the circle intersect, provided there is no overlap or wrapping.

- Since the mapping from intervals to arcs does not introduce any overlap or wrapping, the intersection relationships between intervals are exactly maintained in the corresponding arcs on C_1.
- Therefore, for each pair of intervals that overlap on the real line, their corresponding arcs will intersect on the circle.

Now, we transfer the neutrosophic membership functions from the interval graph to the corresponding circular-arc graph.

- For each vertex $v \in V$ in the interval graph, the neutrosophic membership function $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, which assigns truth, indeterminacy, and falsity degrees, remains unchanged when transferred to the corresponding arc in the circular-arc graph.
- Similarly, for each edge $e \in E$, which corresponds to the overlap of intervals, the neutrosophic membership function $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ is preserved when transferred to the corresponding edge in the circular-arc graph.
- Thus, the neutrosophic membership degrees of both vertices and edges are maintained in the transformation from the interval graph to the circular-arc graph.

By mapping the intervals to arcs, preserving the intersection relationships, and maintaining the neutrosophic membership functions, we have shown that every Neutrosophic Interval Graph can be represented as a Neutrosophic Circular-Arc Graph. □

4. **Result: Proper-Circular-Arc Graph**

In this section, we extend the definition of Proper Circular-Arc Graphs to Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs. These graph models incorporate uncertainty and assign degrees of membership, indeterminacy, and falsity, depending on the specific type of graph. A Proper Circular-Arc Graph is essentially a more constrained version of a Circular-Arc Graph, where no arc is properly contained within another.

Below, we provide the classical definition of a Proper Circular-Arc Graph. As mentioned in the introduction, Proper Circular-Arc Graphs, like Circular-Arc Graphs, have been the subject of various studies [1, 8, 20, 86].

Definition 4.1 (Proper Circular-Arc Graph)**.** A *proper circular-arc graph* is a circular-arc graph with an intersection model in which no arc properly contains another arc.

Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc of the circle. The corresponding circular-arc graph $G = (V, E)$ is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$ consists of one vertex for each arc in the set.

- The edge set E is defined such that there is an edge between two vertices I_α and I_β if and only if their corresponding arcs I_α and I_β intersect, i.e.,

$$\{I_\alpha, I_\beta\} \in E \iff I_\alpha \cap I_\beta \neq \emptyset.$$

A circular-arc graph is said to be *proper* if there exists an arc model in which no arc is properly contained within another, i.e., for all $I_i, I_j \in \{I_1, I_2, \ldots, I_n\}$, it holds that:

$$I_i \subset I_j \quad \text{or} \quad I_j \subset I_i \quad \text{does not occur.}$$

The generalized definitions for each type of graph are provided below.

Definition 4.2 (Fuzzy Proper Circular-Arc Graph)**.** A *Fuzzy Proper Circular-Arc Graph* is a fuzzy graph $G = (V, \sigma, \mu)$ with an arc model in which no arc properly contains another arc. Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc. The graph G is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The membership function $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex I_i.
- The edge membership function $\mu : V \times V \to [0, 1]$ is given by:

$$\mu(I_\alpha, I_\beta) = \begin{cases} \min(\sigma(I_\alpha), \sigma(I_\beta)) & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

- The arcs satisfy the property that no arc properly contains another, i.e., $I_i \subset I_j$ for $i \neq j$ does not hold.

Definition 4.3 (Intuitionistic Fuzzy Proper Circular-Arc Graph)**.** An *Intuitionistic Fuzzy Proper Circular-Arc Graph* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ with a proper arc model. Let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on the circle C_1. The graph G_{IF} is defined as follows:

- The vertex set $A = \{(I_i, \mu_A(I_i), \nu_A(I_i)) : I_i \in V\}$, where $\mu_A(I_i)$ is the membership degree and $\nu_A(I_i)$ is the non-membership degree.
- The edge set $B = \{((I_\alpha, I_\beta), \mu_B(I_\alpha, I_\beta), \nu_B(I_\alpha, I_\beta))\}$, where:

$$\mu_B(I_\alpha, I_\beta) = \min(\mu_A(I_\alpha), \mu_A(I_\beta)),$$

$$\nu_B(I_\alpha, I_\beta) = \max(\nu_A(I_\alpha), \nu_A(I_\beta)).$$

- No arc properly contains another, i.e., $I_i \subset I_j$ does not hold.

Definition 4.4 (Neutrosophic Proper Circular-Arc Graph)**.** A *Neutrosophic Proper Circular-Arc Graph* is a neutrosophic graph $G = (V, E, \sigma, \mu)$ defined with a proper arc model on a circle. Let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a set of arcs, with the following structure:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The neutrosophic membership function $\sigma : V \to [0,1]^3$ assigns a triple $(\sigma_T(I_i), \sigma_I(I_i), \sigma_F(I_i))$ to each arc I_i.
- The edge set $E = \{(I_\alpha, I_\beta)\}$, where the neutrosophic edge membership function μ is given by:

$$\mu_T(I_\alpha, I_\beta) = \min(\sigma_T(I_\alpha), \sigma_T(I_\beta)),$$
$$\mu_I(I_\alpha, I_\beta) = \max(\sigma_I(I_\alpha), \sigma_I(I_\beta)),$$
$$\mu_F(I_\alpha, I_\beta) = \max(\sigma_F(I_\alpha), \sigma_F(I_\beta)).$$

- No arc properly contains another, i.e., $I_i \subset I_j$ does not hold.

Definition 4.5 (Turiyam Neutrosophic Proper Circular-Arc Graph)**.** A *Turiyam Neutrosophic Proper Circular-Arc Graph* is a Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ where no arc properly contains another. Let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1. The graph G^T is defined as follows:

- The vertex set $V^T = \{I_1, I_2, \ldots, I_n\}$.
- Each vertex $I_i \in V^T$ is assigned a quadruple of membership degrees: $(t(I_i), iv(I_i), fv(I_i), lv(I_i))$, representing truth, indeterminacy, falsity, and liberal values.
- The edge set $E^T = \{(I_\alpha, I_\beta)\}$, where:

$$t(I_\alpha, I_\beta) = \min(t(I_\alpha), t(I_\beta)),$$
$$iv(I_\alpha, I_\beta) = \max(iv(I_\alpha), iv(I_\beta)),$$
$$fv(I_\alpha, I_\beta) = \max(fv(I_\alpha), fv(I_\beta)),$$
$$lv(I_\alpha, I_\beta) = \max(lv(I_\alpha), lv(I_\beta)).$$

- No arc properly contains another, i.e., $I_i \subset I_j$ does not hold.

Theorem 4.6. *A Neutrosophic Proper Circular-Arc Graph can be transformed into a Classic Proper Circular-Arc Graph by assigning specific values to the truth, indeterminacy, and falsity membership functions.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Proper Circular-Arc Graph. We aim to transform G into a *Classic Proper Circular-Arc Graph* by assigning specific values to the truth, indeterminacy, and falsity membership functions.

We Set the truth membership for vertices. In the Neutrosophic Graph, each vertex I_i has associated truth $\sigma_T(I_i)$, indeterminacy $\sigma_I(I_i)$, and falsity $\sigma_F(I_i)$. To transform this graph into a classic one, we must ensure that the presence of vertices is certain and that there is no indeterminacy or falsity.

To achieve this, we assign the following values to all vertices:

$$\sigma_T(I_i) = 1, \quad \sigma_I(I_i) = 0, \quad \sigma_F(I_i) = 0 \quad \forall I_i \in V.$$

This guarantees that each vertex is fully present in the graph, with no uncertainty or falsity.

We Set the truth membership for edges. For each edge $(I_\alpha, I_\beta) \in E$, the neutrosophic membership function assigns values to the truth, indeterminacy, and falsity of the edges. To create a Classic Proper Circular-Arc Graph, we set the truth membership for all edges to 1 and indeterminacy and falsity degrees to 0:

$$\mu_T(I_\alpha, I_\beta) = 1, \quad \mu_I(I_\alpha, I_\beta) = 0, \quad \mu_F(I_\alpha, I_\beta) = 0 \quad \forall (I_\alpha, I_\beta) \in E.$$

This ensures that edges definitively exist between intersecting arcs, without uncertainty or falsity.

We explain about preserving the properness condition. A key property of a *Proper Circular-Arc Graph* is that no arc is properly contained within another. In the Neutrosophic context, this property can be maintained by ensuring that no vertex has indeterminacy or falsity values that could imply partial or uncertain containment.

By setting $\sigma_I(I_i) = 0$ and $\sigma_F(I_i) = 0$ for all vertices, we ensure that no arc is improperly nested within another. This guarantees that the arcs in the resulting Classic Proper Circular-Arc Graph remain distinct and non-nested. □

Corollary 4.7. *A Fuzzy Proper Circular-Arc Graph, Intuitionistic Fuzzy Proper Circular-Arc Graph, or Turiyam Neutrosophic Proper Circular-Arc Graph can be transformed into a classic Proper Circular-Arc Graph.*

Proof. It can be proven in the same way as above. □

Theorem 4.8. *Neutrosophic Proper Circular-Arc Graph is special case of Neutrosophic Circular-Arc Graph.*

Proof. This follows directly from the definitions. □

Corollary 4.9. *A Fuzzy Proper Circular-Arc Graph is a special case of a Fuzzy Circular-Arc Graph, an Intuitionistic Fuzzy Proper Circular-Arc Graph is a special case of an Intuitionistic Fuzzy Circular-Arc Graph, and a Turiyam Neutrosophic Proper Circular-Arc Graph is a special case of a Turiyam Neutrosophic Circular-Arc Graph.*

Proof. This follows directly from the definitions. In each case, the "proper" version of the graph imposes an additional constraint that no arc is properly contained within another, while retaining the underlying structure and properties of the more general graph. Therefore, the proper versions are specific instances of their respective general forms. □

Theorem 4.10. *A Neutrosophic Proper Circular-Arc Graph can be transformed into a Fuzzy Proper Circular-Arc Graph, Intuitionistic Fuzzy Proper Circular-Arc Graph, or Turiyam Neutrosophic Proper Circular-Arc Graph by appropriately adjusting the truth, indeterminacy, and falsity membership functions.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a *Neutrosophic Proper Circular-Arc Graph*. We aim to show how G can be transformed into either a *Fuzzy Proper Circular-Arc Graph*, an *Intuitionistic Fuzzy Proper Circular-Arc Graph*, or a *Turiyam Neutrosophic Proper Circular-Arc Graph*.

First, we consider about transformation to Fuzzy Proper Circular-Arc Graph. To transform G into a Fuzzy Proper Circular-Arc Graph, we simplify the neutrosophic membership functions by collapsing the three degrees (truth, indeterminacy, falsity) into a single fuzzy membership value. We do this by taking only the truth-membership degree:

$$\sigma_{\text{fuzzy}}(I_i) = \sigma_T(I_i), \quad \mu_{\text{fuzzy}}(I_\alpha, I_\beta) = \mu_T(I_\alpha, I_\beta).$$

This transformation preserves the structure of G while discarding the indeterminacy and falsity components, yielding a proper fuzzy circular-arc graph.

We consider about transformation to Intuitionistic Fuzzy Proper Circular-Arc Graph. To transform G into an Intuitionistic Fuzzy Proper Circular-Arc Graph, we use the truth-membership degree $\sigma_T(I_i)$ as the membership degree and the falsity-membership degree $\sigma_F(I_i)$ as the non-membership degree:

$$\mu_A(I_i) = \sigma_T(I_i), \quad \nu_A(I_i) = \sigma_F(I_i),$$

$$\mu_B(I_\alpha, I_\beta) = \mu_T(I_\alpha, I_\beta), \quad \nu_B(I_\alpha, I_\beta) = \mu_F(I_\alpha, I_\beta).$$

The indeterminacy component $\sigma_I(I_i)$ is implicitly accounted for by the condition that:

$$\mu_A(I_i) + \nu_A(I_i) \leq 1.$$

Thus, G is transformed into an Intuitionistic Fuzzy Proper Circular-Arc Graph.

We consider about transformation to Turiyam Neutrosophic Proper Circular-Arc Graph. To transform G into a Turiyam Neutrosophic Proper Circular-Arc Graph, we assign the following values:

$$t(I_i) = \sigma_T(I_i), \quad iv(I_i) = \sigma_I(I_i), \quad fv(I_i) = \sigma_F(I_i), \quad lv(I_i) = 0.$$

Similarly, for edges:

$$t(I_\alpha, I_\beta) = \mu_T(I_\alpha, I_\beta), \quad iv(I_\alpha, I_\beta) = \mu_I(I_\alpha, I_\beta), \quad fv(I_\alpha, I_\beta) = \mu_F(I_\alpha, I_\beta), \quad lv(I_\alpha, I_\beta) = 0.$$

The liberal value $lv(I_i) = 0$ is set to zero since it is not present in the original neutrosophic graph. This transforms G into a Turiyam Neutrosophic Proper Circular-Arc Graph while maintaining the original structure of the arcs and intersections. □

Theorem 4.11. *In a Neutrosophic Proper Circular-Arc Graph, no neutrosophic Circular-Arc is properly contained within another with respect to the truth-membership function μ_T.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Proper Circular-Arc Graph.

By definition, a Proper Circular-Arc Graph does not allow any arc I_i to be properly contained within another arc I_j. Formally, for all $I_i, I_j \in V$, we have:

$$I_i \subset I_j \quad \text{or} \quad I_j \subset I_i \quad \text{does not occur.}$$

In a Neutrosophic Proper Circular-Arc Graph, the truth-membership function μ_T respects this containment restriction. If $I_i \subset I_j$ were to hold, we would expect $\mu_T(I_i) \leq \mu_T(I_j)$, but since proper containment does not occur, this situation is impossible.

Therefore, no neutrosophic arc is properly contained within another with respect to μ_T, and the theorem follows. □

Next, we define p-Proper Circular-Arc Graphs and q-Improper Circular-Arc Graphs, and extend these concepts to neutrosophic graphs by introducing Neutrosophic p-Proper Circular-Arc Graphs and Neutrosophic q-Improper Circular-Arc Graphs. The generalized concepts of p-Proper Circular-Arc Graphs and q-Improper Circular-Arc Graphs are natural extensions of the classical circular-arc graph. Below, we provide the formal definitions.

Definition 4.12 (p-Proper Circular-Arc Graph)**.** A graph $G = (V, E)$ is called a *p-proper circular-arc graph* if there exists a circular-arc representation of G such that no arc in the representation is properly contained within more than p other arcs.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G. The graph G is p-proper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_v \subset A_u$ is at most p:

$$\forall u \in V, \quad |\{v \in V \mid A_v \subset A_u\}| \leq p.$$

Definition 4.13 (q-Improper Circular-Arc Graph)**.** A graph $G = (V, E)$ is called a *q-improper circular-arc graph* if there exists a circular-arc representation of G such that no arc in the representation properly contains more than q other arcs.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G. The graph G is q-improper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_u \subset A_v$ is at most q:

$$\forall u \in V, \quad |\{v \in V \mid A_u \subset A_v\}| \leq q.$$

Definition 4.14 (Neutrosophic p-Proper Circular-Arc Graph)**.** A graph $G = (V, E, \sigma, \mu)$ is called a *Neutrosophic p-Proper Circular-Arc Graph* if there exists a circular-arc representation of G such that no arc in the representation is properly contained within more than p other arcs with respect to the truth-membership function μ_T.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G, where each arc A_v is assigned a truth-membership degree $\mu_T(v)$. The graph G is p-proper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_v \subset A_u$ with respect to $\mu_T(v)$ is at most p:

$$\forall u \in V, \quad |\{v \in V \mid A_v \subset A_u \text{ and } \mu_T(A_v) \leq \mu_T(A_u)\}| \leq p.$$

Definition 4.15 (Neutrosophic q-Improper Circular-Arc Graph)**.** A graph $G = (V, E, \sigma, \mu)$ is called a *Neutrosophic q-Improper Circular-Arc Graph* if there exists a circular-arc representation of G such that no arc in the representation properly contains more than q other arcs with respect to the truth-membership function μ_T.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G, where each arc A_v is assigned a truth-membership degree $\mu_T(v)$. The graph G is q-improper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_u \subset A_v$ with respect to $\mu_T(u)$ is at most q:

$$\forall u \in V, \quad |\{v \in V \mid A_u \subset A_v \text{ and } \mu_T(A_u) \leq \mu_T(A_v)\}| \leq q.$$

Theorem 4.16. *Neutrosophic p-Proper circular-arc graph is a Neutrosophic graph. Also Neutrosophic q-Improper circular-arc graph is a Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 4.17. *A Neutrosophic p-Proper circular-arc graph can be transformed into a classic p-Proper circular-arc graph.*

Proof. Obviously holds. □

Theorem 4.18. *A Neutrosophic q-Improper circular-arc graph can be transformed into a classic q-Improper circular-arc graph.*

Proof. Obviously holds. □

Theorem 4.19. *A Neutrosophic* 0*-Proper Circular-Arc Graph is exactly a Neutrosophic Proper Circular-Arc Graph.*

Proof. By definition, a Neutrosophic p-Proper Circular-Arc Graph is a graph in which no arc is properly contained within more than p other arcs with respect to the truth-membership function μ_T.

For the case $p = 0$, this condition simplifies to no arc being properly contained within any other arc. Formally, for each arc $A_u \in \mathcal{A}$, the number of arcs $A_v \in \mathcal{A}$ such that $A_v \subset A_u$ is zero:

$$\forall u \in V, \quad |\{v \in V \mid A_v \subset A_u \text{ and } \mu_T(A_v) \leq \mu_T(A_u)\}| = 0.$$

This directly matches the definition of a Neutrosophic Proper Circular-Arc Graph, which states that no arc is properly contained within another. Therefore, a Neutrosophic 0-Proper Circular-Arc Graph is precisely a Neutrosophic Proper Circular-Arc Graph. □

Theorem 4.20. *A Neutrosophic* 0*-Improper Circular-Arc Graph is a Neutrosophic Proper Circular-Arc Graph.*

Proof. By definition, a Neutrosophic q-Improper Circular-Arc Graph is a graph in which no arc properly contains more than q other arcs with respect to the truth-membership function μ_T.

For the case $q = 0$, this condition simplifies to no arc properly containing any other arc. Formally, for each arc $A_u \in \mathcal{A}$, the number of arcs $A_v \in \mathcal{A}$ such that $A_u \subset A_v$ is zero:

$$\forall u \in V, \quad |\{v \in V \mid A_u \subset A_v \text{ and } \mu_T(A_u) \leq \mu_T(A_v)\}| = 0.$$

This condition ensures that no arc properly contains any other arc, which is exactly the definition of a Neutrosophic Proper Circular-Arc Graph. Thus, a Neutrosophic 0-Improper Circular-Arc Graph is a Neutrosophic Proper Circular-Arc Graph. □

Theorem 4.21. *A Neutrosophic* 1*-Proper Circular-Arc Graph allows each arc to be properly contained within at most one other arc with respect to the truth-membership function* μ_T*.*

Proof. Obviously holds. □

5. Future tasks: Graph Power

A Power Graph is a concept that examines the powers of a graph. The general definition is provided below [6, 58].

Definition 5.1 (Graph Power)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. The k-th power of G, denoted G^k, is defined as a graph with the same vertex set V, but with an edge between two vertices $u, v \in V$ if and only if the distance between u and v in G is at most k. Formally, the edge set E^k of G^k is given by:

$$E^k = \{(u, v) \in V \times V : d_G(u, v) \leq k\},$$

where $d_G(u, v)$ denotes the shortest path distance between u and v in the original graph G.

For example, G^2 is called the square of G, and G^3 is called the cube of G.

At the conceptual stage, we outline the definitions for Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs below. Moving forward, we plan to investigate the relationships between these concepts and various types of graphs, such as Circular-Arc Graphs.

Definition 5.2 (Fuzzy Graph Power)**.** Let $G = (V, E, \sigma, \mu)$ be a Fuzzy Graph, where:

- $\sigma : V \to [0, 1]$ assigns membership degrees to vertices.
- $\mu : E \to [0, 1]$ assigns membership degrees to edges.

The k-th power of G, denoted $G^k = (V, E^k, \sigma^k, \mu^k)$, is defined as:

(1) $\sigma^k(v) = \sigma(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degree $\mu^k(u, v)$ is defined as:

$$\mu^k(u, v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\min_{e \in P_{uv}} \mu(e) \right),$$

where:

- P_{uv} is a path from u to v in G,
- $\ell(P_{uv})$ is the length of the path P_{uv} (number of edges),
- The maximum is taken over all paths of length at most k connecting u and v,
- The minimum is taken over the membership degrees of edges along each path.

Definition 5.3 (Intuitionistic Fuzzy Graph Power)**.** Let $G = (V, E, \mu_A, v_A, \mu_B, v_B)$ be an Intuitionistic Fuzzy Graph, where:

- $\mu_A : V \to [0, 1]$ and $v_A : V \to [0, 1]$ assign degrees of membership and non-membership to vertices, respectively.
- $\mu_B : E \to [0, 1]$ and $v_B : E \to [0, 1]$ assign degrees of membership and non-membership to edges, respectively.

The k-th power of G, denoted $G^k = (V, E^k, \mu_A^k, v_A^k, \mu_B^k, v_B^k)$, is defined as:

(1) $\mu_A^k(v) = \mu_A(v), \quad v_A^k(v) = v_A(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degrees are defined as:

$$\mu_B^k(u,v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\min_{e \in P_{uv}} \mu_B(e) \right),$$

$$v_B^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\max_{e \in P_{uv}} v_B(e) \right).$$

Definition 5.4 (Neutrosophic Graph Power)**.** Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Graph, where:

- $\sigma_T, \sigma_I, \sigma_F : V \to [0,1]$ assign truth, indeterminacy, and falsity membership degrees to vertices.
- $\mu_T, \mu_I, \mu_F : E \to [0,1]$ assign truth, indeterminacy, and falsity membership degrees to edges.

The k-th power of G, denoted $G^k = (V, E^k, \sigma^k, \mu^k)$, is defined as:

(1) $\sigma^k(v) = \sigma(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degrees are defined as:

$$\mu_T^k(u,v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\min_{e \in P_{uv}} \mu_T(e) \right),$$

$$\mu_I^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\max_{e \in P_{uv}} \mu_I(e) \right),$$

$$\mu_F^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\max_{e \in P_{uv}} \mu_F(e) \right).$$

Definition 5.5 (Turiyam Neutrosophic Graph Power)**.** Let $G = (V, E, \sigma = (t, iv, fv, lv), \mu = (t_e, iv_e, fv_e, lv_e))$ be a Turiyam Neutrosophic Graph, where:

- $t, iv, fv, lv : V \to [0,1]$ assign truth, indeterminacy, falsity, and liberal state values to vertices.
- $t_e, iv_e, fv_e, lv_e : E \to [0,1]$ assign truth, indeterminacy, falsity, and liberal state values to edges.

The k-th power of G, denoted $G^k = (V, E^k, \sigma^k, \mu^k)$, is defined as:

(1) $\sigma^k(v) = \sigma(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degrees are defined as:

$$t^k(u,v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\min_{e \in P_{uv}} t_e(e) \right),$$

$$iv^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \le k}} \left(\max_{e \in P_{uv}} iv_e(e) \right),$$

$$fv^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} fv_e(e) \right),$$

$$lv^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} lv_e(e) \right).$$

Theorem 5.6. *The Turiyam Neutrosophic Graph Power reduces to the Neutrosophic Graph Power when the liberal state functions are set to zero.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Turiyam Neutrosophic Graph. Define a Neutrosophic Graph $G' = (V, E, \sigma', \mu')$ by setting:

$$\sigma'(v) = (t(v), iv(v), fv(v)), \quad \forall v \in V,$$
$$\mu'(e) = (t_e(e), iv_e(e), fv_e(e)), \quad \forall e \in E.$$

Similarly, the k-th power G^k transforms to $(G')^k$ with edge membership degrees:

$$t'^k(u,v) = t^k(u,v),$$
$$iv'^k(u,v) = iv^k(u,v),$$
$$fv'^k(u,v) = fv^k(u,v),$$

for all $(u,v) \in E^k$.

Since the liberal state functions $lv(v)$ and $lv_e(e)$ are set to zero and omitted, the Turiyam Neutrosophic Graph Power reduces to the Neutrosophic Graph Power. □

Theorem 5.7. *The Neutrosophic Graph Power reduces to the Intuitionistic Fuzzy Graph Power when the falsity-membership functions are set to one minus the truth-membership functions, and indeterminacy-membership functions are set to zero.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Graph. Define an Intuitionistic Fuzzy Graph $G'' = (V, E, \mu_A, v_A, \mu_B, v_B)$ by setting:

$$\mu_A(v) = t(v), \quad v_A(v) = fv(v), \quad \forall v \in V,$$
$$\mu_B(e) = t_e(e), \quad v_B(e) = fv_e(e), \quad \forall e \in E.$$

Assuming $iv(v) = 0$ and $iv_e(e) = 0$ for all $v \in V$ and $e \in E$, the indeterminacy is removed. The k-th power $(G')^k$ transforms to $(G'')^k$ with edge membership degrees:

$$\mu_B^k(u,v) = t'^k(u,v),$$
$$v_B^k(u,v) = fv'^k(u,v),$$

for all $(u,v) \in E^k$.

Thus, the Neutrosophic Graph Power reduces to the Intuitionistic Fuzzy Graph Power under these conditions. □

Theorem 5.8. *The Intuitionistic Fuzzy Graph Power reduces to the Fuzzy Graph Power when the non-membership functions are set to one minus the membership functions.*

Proof. Let $G'' = (V, E, \mu_A, v_A, \mu_B, v_B)$ be an Intuitionistic Fuzzy Graph. Define a Fuzzy Graph $G''' = (V, E, \sigma, \mu)$ by setting:

$$\sigma(v) = \mu_A(v), \quad \forall v \in V,$$
$$\mu(e) = \mu_B(e), \quad \forall e \in E.$$

Assuming $v_A(v) = 1 - \mu_A(v)$ and $v_B(e) = 1 - \mu_B(e)$, the non-membership functions are determined by the membership functions.

The k-th power $(G'')^k$ transforms to $(G''')^k$ with edge membership degrees:

$$\mu^k(u, v) = \mu_B^k(u, v),$$

for all $(u, v) \in E^k$.

Thus, the Intuitionistic Fuzzy Graph Power reduces to the Fuzzy Graph Power under these conditions. □

Theorem 5.9. *The Fuzzy Graph Power reduces to the classical Graph Power when all membership degrees are binary (0 or 1).*

Proof. Let $G''' = (V, E, \sigma, \mu)$ be a Fuzzy Graph where $\sigma(v) \in \{0, 1\}$ and $\mu(e) \in \{0, 1\}$ for all $v \in V$ and $e \in E$.

The k-th power $(G''')^k$ becomes a classical graph $G^k = (V, E^k)$ where:

$$(u, v) \in E^k \iff \text{there exists a path from } u \text{ to } v \text{ of length } \leq k \text{ in } G.$$

This aligns with the definition of the classical Graph Power. Hence, the Fuzzy Graph Power reduces to the classical Graph Power when membership degrees are binary. □

Funding

No external funding was received for this research.

Acknowledgments

We extend our heartfelt thanks to everyone who provided invaluable support in completing this paper; this work would not have been possible without the contributions of all those involved. We also express our deep respect and gratitude to the authors of prior studies cited in our references. Lastly, we are sincerely thankful to our readers for their time and interest.

Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Selim G Akl and Lin Chen. Efficient parallel algorithms on proper circular arc graphs. *IEICE TRANSACTIONS on Information and Systems*, 79(8):1015–1020, 1996.

[2] Muhammad Akram and Noura Omair Alshehri. Intuitionistic fuzzy cycles and intuitionistic fuzzy trees. *The Scientific World Journal*, 2014(1):305836, 2014.

[3] Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.

[4] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[5] Jørgen Bang-Jensen and Pavol Hell. On chordal proper circular arc graphs. *Discrete Mathematics*, 128(1-3):395–398, 1994.

[6] Manu Basavaraju, L Sunil Chandran, Deepak Rajendraprasad, and Arunselvan Ramaswamy. Rainbow connection number of graph power and graph products. *Graphs and Combinatorics*, 30:1363–1382, 2014.

[7] Asim Basu, Sandip Das, Shamik Ghosh, and Malay Sen. Circular-arc bigraphs and its subclasses. *Journal of Graph Theory*, 73(4):361–376, 2013.

[8] Naveen Belkale and L Sunil Chandran. Hadwiger's conjecture for proper circular arc graphs. *European Journal of Combinatorics*, 30(4):946–956, 2009.

[9] Flavia Bonomo. Self-clique helly circular-arc graphs. *Discrete Mathematics*, 306(6):595–597, 2006.

[10] Flavia Bonomo, Guillermo Durán, Luciano N Grippo, and Martín Darío Safe. Partial characterizations of circular-arc graphs. *Journal of Graph Theory*, 61(4):289–306, 2009.

[11] Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.

[12] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and Prem Kumar Singh. Properties of interval-valued neutrosophic graphs. *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, pages 173–202, 2019.

[13] Chiuyuan Chen, Chin-Chen Chang, and Gerard J Chang. Proper interval graphs and the guard problem. *Discrete Mathematics*, 170(1-3):223–230, 1997.

[14] WL Craine. Characterizations of fuzzy interval graphs. *Fuzzy Sets and Systems*, 68(2):181–193, 1994.

[15] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[16] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[17] Didier Dubois and Henri Prade. *Fundamentals of fuzzy sets*, volume 7. Springer Science & Business Media, 2012.

[18] Guillermo Duran, Agustin Gravano, Ross M McConnell, Jeremy Spinrad, and Alan Tucker. Polynomial time recognition of unit circular-arc graphs. *Journal of Algorithms*, 58(1):67–78, 2006.

[19] Ali Ershadi. *List homomorphisms and bipartite co-circular arc graphs*. Simon Fraser University (Canada), 2012.

[20] Carsten Flotow. On powers of circular arc graphs and proper circular arc graphs. *Discrete Applied Mathematics*, 69(3):199–207, 1996.

[21] Takaaki Fujita. Interval graphs and proper interval graphs in fuzzy and neutrosophic graphs. 2024.

[22] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[23] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[24] Takaaki Fujita. Unit disk graphs in fuzzy and neutrosophic graphs. June 2024. Available under CC BY 4.0 license.

[25] Takaaki Fujita. Advancing uncertain combinatorics through graphization, hyperization, and uncertainization: Fuzzy, neutrosophic, soft, rough, and beyond. 2025.

[26] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[27] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[28] Delbert Fulkerson and Oliver Gross. Incidence matrices and interval graphs. *Pacific journal of mathematics*, 15(3):835–855, 1965.

[29] GA Ganati, VNS Rao Repalle, MA Ashebo, and M Amini. Turiyam graphs and its applications. *Information Sciences Letters*, 12(6):2423–2434, 2023.

[30] A. Nagoor Gani and S. Shajitha Begum. Perfect intuitionistic fuzzy graphs. *Bulletin of Pure & Applied Sciences- Mathematics and Statistics*, pages 145–152, 2011.

[31] Fanica Gavril. Algorithms on circular-arc graphs. *Networks*, 4(4):357–369, 1974.

[32] Fănică Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B*, 16(1):47–56, 1974.

[33] Martin Charles Golumbic. Interval graphs. In *Annals of Discrete Mathematics*, volume 57, pages 171–202. Elsevier, 2004.

[34] Agustın Gravano and Guillermo Durán. The intersection between some subclasses of circular-arc and circle graphs. *Congressus Numerantium*, pages 183–192, 2002.

[35] Udaiprakash I Gupta, Der-Tsai Lee, and JY-T Leung. Efficient algorithms for interval graphs and circular-arc graphs. *Networks*, 12(4):459–467, 1982.

[36] Grzegorz Gutowski, Konstanty Junosza-Szaniawski, Felix Klesen, Paweł Rzążewski, Alexander Wolff, and Johannes Zink. Coloring and recognizing mixed interval graphs. 2023.

[37] Grzegorz Gutowski, Florian Mittelstädt, Ignaz Rutter, Joachim Spoerhase, Alexander Wolff, and Johannes Zink. Coloring mixed and directional interval graphs. In *International Symposium on Graph Drawing and Network Visualization*, pages 418–431. Springer, 2022.

[38] Grzegorz Gutowski[1], Florian Mittelstädt, Ignaz Rutter, Joachim Spoerhase[1], Alexander Wolff, and Johannes Zink. Check for updates coloring mixed and directional interval graphs. In *Graph Drawing and*

Network Visualization: 30th International Symposium, GD 2022, Tokyo, Japan, September 13–16, 2022, Revised Selected Papers, volume 13764, page 418. Springer Nature, 2023.

[39] Tatiana Romina Hartinger. Chordal, interval, and circular-arc product graphs. *Applicable Analysis and Discrete Mathematics*, 10(2):532–551, 2016.

[40] Pinar Heggernes, Daniel Meister, and Yngve Villanger. Induced subgraph isomorphism on interval and proper interval graphs. In *International Symposium on Algorithms and Computation*, pages 399–409. Springer, 2010.

[41] Danny Hermelin, Matthias Mnich, and Erik Jan Van Leeuwen. Parameterized complexity of induced graph matching on claw-free graphs. *Algorithmica*, 70(3):513–560, 2014.

[42] Robert Hickingbotham. Induced subgraphs and path decompositions. *arXiv preprint arXiv:2206.15054*, 2022.

[43] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[44] Benson L Joeris, Min Chih Lin, Ross M McConnell, Jeremy P Spinrad, and Jayme L Szwarcfiter. Linear-time recognition of helly circular-arc models and graphs. *Algorithmica*, 59(2):215–239, 2011.

[45] Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Neutrosophic graphs: a new dimension to graph theory*. Infinite Study, 2015.

[46] Haim Kaplan and Yahav Nussbaum. A simpler linear-time recognition of circular-arc graphs. In *Scandinavian Workshop on Algorithm Theory*, pages 41–52. Springer, 2006.

[47] Haim Kaplan and Yahav Nussbaum. Certifying algorithms for recognizing proper circular-arc graphs and unit circular-arc graphs. *Discrete Applied Mathematics*, 157(15):3216–3230, 2009.

[48] Haim Kaplan and Yahav Nussbaum. A simpler linear-time recognition of circular-arc graphs. *Algorithmica*, 61:694–737, 2011.

[49] Toshinobu Kashiwabara, Sumio Masuda, Kazuo Nakajima, and Toshio Fujisawa. Polynomial time algorithms on circular-arc overlap graphs. *Networks*, 21(2):195–203, 1991.

[50] Anjaly Kishore and MS Sunitha. Chromatic number of fuzzy graphs. *Annals of fuzzy mathematics and informatics*, 7(4):543–551, 2014.

[51] Benjamin Lévêque, David Y Lin, Frédéric Maffray, and Nicolas Trotignon. Detecting induced subgraphs. *Discrete Applied Mathematics*, 157(17):3540–3551, 2009.

[52] Hongxing Li and Vincent C Yen. *Fuzzy sets and fuzzy decision-making*. CRC press, 1995.

[53] YD Liang, SK Dhall, and S Lakshmivarahan. On the problem of finding all maximum weight independent sets in interval and circular-arc graphs. In *1991 Symposium on Applied Computing*, pages 465–466. IEEE Computer Society, 1991.

[54] Min Chih Lin, Ross M McConnell, Francisco J Soulignac, and Jayme L Szwarcfiter. On cliques of helly circular-arc graphs. *Electronic Notes in Discrete Mathematics*, 30:117–122, 2008.

[55] Min Chih Lin, Francisco J Soulignac, and Jayme L Szwarcfiter. Proper helly circular-arc graphs. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 248–257. Springer, 2007.

[56] Min Chih Lin and Jayme L Szwarcfiter. Unit circular-arc graph representations and feasible circulations. *SIAM Journal on Discrete Mathematics*, 22(1):409–423, 2008.

[57] Min Chih Lin and Jayme L Szwarcfiter. Characterizations and recognition of circular-arc graphs and subclasses: A survey. *Discrete Mathematics*, 309(18):5618–5635, 2009.

[58] Wei Lv and Xuanlong Ma. The intersection power graph associated with a finite group. *ScienceAsia*, 47(5), 2021.

[59] Terrance Mankus. *New optimum algorithms for unweighted and weighted interval, circular arc and circle graphs and their models*. University of Illinois at Chicago, 1999.

[60] MLN McAllister. Fuzzy intersection graphs. *Computers & Mathematics with Applications*, 15(10):871–886, 1988.

[61] Terry A McKee and Fred R McMorris. *Topics in intersection graph theory*. SIAM, 1999.

[62] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[63] R Parvathi and MG Karunambigai. Intuitionistic fuzzy graphs. In *Computational Intelligence, Theory and Applications: International Conference 9th Fuzzy Days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings*, pages 139–150. Springer, 2006.

[64] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[65] Sreenanda Raut and Madhumangal Pal. Fuzzy intersection graph: a geometrical approach. *Journal of Ambient Intelligence and Humanized Computing*, pages 1–25, 2022.

[66] Sreenanda Raut, Madhumangal Pal, and Ganesh Ghorai. Fuzzy permutation graph and its complements. *J. Intell. Fuzzy Syst.*, 35:2199–2213, 2018.

[67] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[68] Martín D Safe. Essential obstacles to helly circular-arc graphs. *arXiv preprint arXiv:1612.01513*, 2016.

[69] Martín D Safe. Circularly compatible ones, d-circularity, and proper circular-arc bigraphs. *SIAM Journal on Discrete Mathematics*, 35(2):707–751, 2021.

[70] Rıdvan Şahin. An approach to neutrosophic graph theory with applications. *Soft Computing*, 23(2):569–581, 2019.

[71] CR Saranya, KSP Sowndarya, and Y Lakshmi Naidu. A linear-time algorithm for paired neighbhourhood set on circular-arc graphs. *International Journal of Mathematics Trends and Technology-IJMTT*, 51, 2017.

[72] MK Sen, G Chowdhury, and DS Malik. Fuzzy intersection graphs of fuzzy semigroups. *New Mathematics and Natural Computation*, 2(01):1–10, 2006.

[73] Karen B Singer. *Random intersection graphs*. The Johns Hopkins University, 1996.

[74] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[75] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[76] Florentin Smarandache, Said Broumi, Mohamed Talea, Assia Bakali, and Kishore Kumar. Shortest path problem on single valued neutrosophic graphs. In *International Symposium on Networks, Computers and Communications (ISNCC-2017), Marrakech, Morocco, May 16-18, 2017, www. isncc-conf. org*. ISNCC, 2017.

[77] Florentin Smarandache, WB Kandasamy, et al. Strong neutrosophic graphs and subgraph topological subspaces. 2016.

[78] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[79] Florentin Smarandache, Siti Nurul Fitriah Mohamad, and Roslan Hasni. Novel concepts on domination in neutrosophic incidence graphs with some applications. 2023.

[80] Metawee Songsaeng and Aiyared Iampan. *Neutrosophic set theory applied to UP-algebras*. Infinite Study, 2019.

[81] Francisco J Soulignac. Fully dynamic recognition of proper circular-arc graphs. *Algorithmica*, 71(4):904–968, 2015.

[82] A Sudhakaraiah and V Raghava Lakshmi. Characterization of circular-arc overlap graphs with domination number two. *International Journal of Computer Applications*, 41(3), 2012.

[83] A Sudhakaraiah and V Raghava Lakshmi. Dominator chromatic number of circular-arc overlap graphs. *International Journal of Computer Applications*, 41(3):1–6, 2012.

[84] Alan Tucker. Characterizing circular-arc graphs. 1970.

[85] Alan Tucker. An efficient test for circular-arc graphs. *SIAM Journal on Computing*, 9(1):1–24, 1980.

[86] ALAN CURTISS TUCKER. *Two characterizations of proper circular-arc graphs*. Stanford University, 1969.

[87] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[88] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[89] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

Received: Aug 29, 2024. Accepted: Nov 21, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Modified WASPAS Technique for Triangular Fuzzy Neutrosophic Number Multiple-Attribute Decision-Making to Safe Operation Risk Assessment of Batteries in Energy Storage Power Stations

Lunsen Zou*, Zhiqiang Wang, Jin Wang, Linwei Wang, Liangrui Zhou

CGS POWER GENERATION(GUANGDONG) ENERGY STORAGE TECHNOLOGY CO., LTD, Guangzhou, Guangdong, 510630, China (Zhiqiang Wang, E-mail: cnkjwangzhiqiang@126.com; Jin Wang, E-mail: cnkjwangjin@126.com; Luke, E-mail: 13427505416@163.com; Liangrui Zhou, E-mail: zlrofficial@163.com)

*Corresponding author, E-mail: cnkjzoulunsen@126.com

Lunsen Zou: https://orcid.org/0009-0002-9930-4882

Abstract: With the proposal of China's "dual carbon" goals and the rapid development of the energy storage industry, in recent years, there have been more and more reports on safety accidents of energy storage power stations in the application of the energy storage industry. It can be seen that the pressure and safety hazards of the safe operation of energy storage power stations have significantly increased, and safety issues have become the main bottleneck restricting the good development of the energy storage industry. The safe operation risk assessment of batteries in energy storage power stations could be considered (MADM). Recently, the WASPAS technique has been used to rank the alternatives. The triangular fuzzy neutrosophic sets (TFNSs) express fuzzy information during the safe operation risk assessment of batteries in energy storage power stations. In this paper, the triangular fuzzy neutrosophic number WASPAS (TFNN-WASPAS) approach based on the triangular fuzzy neutrosophic number (TFNNWASPAS) is constructed to put forward MADM with TFNSs. The average method is employed to compute the weight values based on TFNN (TFNNWASPAS) under TFNSs. Finally, a numerical example of safe operation risk assessment of batteries in energy storage power stations and some comparisons are constructed to verify TFNN- WASPAS.

Keywords: Multiple-attribute decision-making (MADM); TFNSs; WASPAS technique; safe operation risk assessment

1. Introduction

Industry-related statistical data shows that as of the end of 2022, the cumulative installed capacity of power storage projects in China has reached 59.4 GW, with a year-on-year increase of 37% in new installed capacity in 2022. Among them, pumped storage accounts

for the largest proportion, with a cumulative installed capacity of 46.1 GW [1, 2]. New energy storage continues to maintain high-speed growth compared to previous years, with a cumulative installed capacity exceeding 10GW and reaching 12.7GW[3, 4]. It is expected that by 2025, China's new energy storage installed capacity will exceed 30GW. In recent years, many provinces in China have required that a certain proportion of energy storage facilities must be installed in newly built new energy projects, with a configuration power ranging from 5% to 30% of the approved capacity of new energy projects, and a storage time of 1 to 4 hours[5, 6]. Compared with other electrochemical energy storage technologies such as flow batteries, lead-acid batteries, and sodium-ion batteries, lithium-ion batteries have the main advantages of high individual energy density, high charging, and discharging efficiency, low self-discharge rate, and fast dynamic response. They are widely used in the energy storage industry[7, 8]. With the continuous deepening of China's energy transformation, battery energy storage power stations have become an important component of ensuring the safe and stable operation of the power system. Battery energy storage is one of the most effective technical means to compensate for the real-time supply-demand deviation of the power system. Along with the rapid development of the energy storage industry, energy storage safety accidents are also common both domestically and internationally[9, 10]. The safety operating pressure and hidden dangers of energy storage power stations have significantly increased. Electrochemical energy storage is a chemical integrated equipment with high energy density, which poses significant safety risks during operation. Safety issues have become the main bottleneck restricting the healthy development of the energy storage industry[11]. In addition, as an emerging industry, energy storage currently has an incomplete and delayed regulatory standard system for energy storage safety. There are risk factors in battery quality and safety management, project planning, construction and operation, maintenance management, and retirement recycling for energy storage power stations. During daily operations, energy storage power stations cannot generally identify and warn of safety risks, and most of them fail to solve problems such as fire protection and emergency response[12, 13]. There are many safety hazards, and the corresponding safety supervision mechanisms are not perfect, leading to frequent safety accidents in energy storage power stations[14, 15]. In the past decade, there have been nearly a hundred energy storage safety accidents worldwide, especially in the past five years. With the explosive development of the global energy storage market, the scale application of energy storage projects continues to increase, and the scale of individual energy storage projects continues to expand, posing increasing risks to the safe operation of energy storage[16]. The safety accidents that have occurred in domestic and foreign energy storage power stations have had a profound impact and a heavy cost, and have also sounded the alarm for the healthy and good development of energy storage power stations in China in the future[17]. For the energy storage industry, safety issues must be taken seriously and solved before achieving large-scale development of energy storage[18]. In the current trend and environment, solving energy storage safety

issues is the top priority for the healthy development of the industry[19]. At present, research on the safety management of battery electrochemical energy storage mainly focuses on battery body materials and technology, as well as risk warnings during the design and construction stages of battery energy storage power stations[20, 21]. At the same time, due to the short operation time of most energy storage power stations, there is not much research specifically focused on safety risk management during the operation period of battery energy storage power stations. In practice, there is a lack of targeted and systematic safety risk management system construction and simple and effective risk management methods. The safety of battery energy storage power stations is not only reflected in the progress of battery body materials and technology but also in the management of safety risks during their operation period[22].

In the field of modern decision-making, MADM is a very important component[23]. MADM has been applied in many fields, such as social enterprises, military affairs, engineering technology, etc., and has been applied in all aspects of life[24, 25]. MADM is the evaluation of a limited number of alternative solutions under a certain number of evaluation attributes (indicators), using certain technical means to aggregate the evaluation information of alternative solutions under multiple evaluation attributes, to select the optimal or non-inferior solution from a limited number of alternative solutions[26-28]. With the development of multi-attribute decision-making and fuzzy uncertainty in real life, it is difficult for people to make accurate judgments in the decision-making process when facing complex decision-making problems, and the evaluation of solutions cannot be measured with accurate numbers[29, 30]. For this phenomenon, Zadeh [31] proposed a fuzzy set in 1965, which uses membership functions to express fuzzy uncertain information in the decision-making process. Once fuzzy set theory was proposed, it attracted many scholars in the industry to explore and study it. Later, with further research, it was found that although fuzzy set theory is a very useful tool for handling uncertain fuzzy information, it only considers membership and cannot handle more fuzzy and uncertain situations. Smarandache [32] implemented the neutrosophic sets (NSs). Biswas, Pramanik, and Giri [33] implemented the triangular fuzzy neutrosophic numbers (TFNNs). Aal, Abd Ellatif and Hassan [34] implemented two ranking techniques under TFNNs. Chakraborty et al. [35] implemented the triangular fuzzy neutrosophic sets (TFNSs). The safe operation risk assessment of batteries in energy storage power stations could be looked at as MADM. Recently, the WASPAS technique [36] and entropy technique [37] was put forward MADM. The TFNSs [33] are put forward expressing fuzzy information during the safe operation risk assessment of batteries in energy storage power stations. In this paper, the TFNN-WASPAS based on TFNN (TFNNWASPAS) is constructed to cope with the MADM under TFNSs. Finally, a numerical example of safe operation risk assessment of batteries in energy storage power stations is constructed and some decision comparisons are utilized to verify TFNN-WASPAS technique. This paper mainly supplies technique guidance and technical support for final realization of safe

operation risk assessment of batteries in energy storage power stations. This technique has far-reaching significance for the MADM of safe operation risk assessment of batteries in energy storage power stations in the construction sector. The main research goal and motivation of this work is outlined: (1) the TFNN-WASPAS technique based on TFNN-WASPAS is constructed; (2) the average technique is employed to manage weight based on TFNN-WASPAS under TFNSs. (3) the TFNN-WASPAS technique is founded to manage the MADM based on TFNN-WASPAS under TFNSs; (4) a numerical example for safe operation risk assessment of batteries in energy storage power stations and some comparative analysis are utilized verify TFNN- WASPAS.

The research framework of such a study is outlined: Sect. 2 lists the TFNSs. Sect. 3 constructs MADM technique based on TFNN-WASPAS model. Sect. 4 verifies the TFNN-WASPAS model through a case study for safe operation risk assessment of batteries in energy storage power stations. Conclusions are constructed in Sect. 5.

2. Preliminaries

Biswas, Pramanik, and Giri [33] constructed the TFNSs.

Definition 1[33]. The TFNSs $s\eta$ is:

$$s\eta = \left\{\left(\theta, s\phi_\eta(\theta), s\varphi_\eta(\theta), s\gamma_\eta(\theta)\right)\middle|\, \theta \in \Theta\right\} \tag{1}$$

where $s\phi_\eta(\theta), s\varphi_\eta(\theta), s\gamma_\eta(\theta)$ represent the truth-membership, the indeterminacy-membership and falsity-membership which is depicted by TFNs.

$$s\phi_\eta(\theta) = \left(s\phi_\eta^L(\theta), s\phi_\eta^M(\theta), s\phi_\eta^U(\theta)\right), 0 \le s\phi_\eta^L(\theta) \le s\phi_\eta^M(\theta) \le s\phi_\eta^U(\theta) \le 1 \tag{2}$$

$$s\varphi_\eta(\theta) = \left(s\varphi_\eta^L(\theta), s\varphi_\eta^M(\theta), s\varphi_\eta^U(\theta)\right), 0 \le s\varphi_\eta^L(\theta) \le s\varphi_\eta^M(\theta) \le s\varphi_\eta^U(\theta) \le 1 \tag{3}$$

$$s\gamma_\eta(\theta) = \left(s\gamma_\eta^L(\theta), s\gamma_\eta^M(\theta), s\gamma_\eta^U(\theta)\right), 0 \le s\gamma_\eta^L(\theta) \le s\gamma_\eta^M(\theta) \le s\gamma_\eta^U(\theta) \le 1 \tag{4}$$

For convenience, $s\eta = \left\{\begin{array}{l}\left(s\phi^L, s\phi^M, s\phi^U\right), \\ \left(s\varphi^L, s\varphi^M, s\varphi^U\right), \left(s\gamma^L, s\gamma^M, s\gamma^U\right)\end{array}\right\}$ is a TFNN that meets the condition $0 \le s\phi^U + s\varphi^U + s\gamma^U \le 3$.

To cope with TFNNs, Biswas, Pramanik, and Giri [33] constructed some novel operations on the TFNNs $s\eta$, $s\eta_1$ and $s\eta_2$.

Definition 2[33]. Let $s\eta_1 = \left\{\begin{array}{l}\left(s\phi_1^L, s\phi_1^M, s\phi_1^U\right), \\ \left(s\varphi_1^L, s\varphi_1^M, s\varphi_1^U\right), \left(s\gamma_1^L, s\gamma_1^M, s\gamma_1^U\right)\end{array}\right\}$,

$s\eta_2 = \left\{\begin{array}{l}\left(s\phi_2^L, s\phi_2^M, s\phi_2^U\right), \\ \left(s\varphi_2^L, s\varphi_2^M, s\varphi_2^U\right), \left(s\gamma_2^L, s\gamma_2^M, s\gamma_2^U\right)\end{array}\right\}$ and $s\eta = \left\{\begin{array}{l}\left(s\phi^L, s\phi^M, s\phi^U\right), \\ \left(s\varphi^L, s\varphi^M, s\varphi^U\right), \left(s\gamma^L, s\gamma^M, s\gamma^U\right)\end{array}\right\}$, the constructed operation laws of TFNNs are designed:

$$(1)\ s\eta_1 \oplus s\eta_2 = \begin{Bmatrix} \left(s\phi_1^L + s\phi_2^L - s\phi_1^L s\phi_2^L, s\phi_1^M + s\phi_2^M - s\phi_1^M s\phi_2^M, s\phi_1^U + s\phi_2^U - s\phi_1^U s\phi_2^U\right), \\ \left(s\varphi_1^L s\varphi_2^L, s\varphi_1^M s\varphi_2^M, s\varphi_1^U s\varphi_2^U\right), \\ \left(s\gamma_1^L s\gamma_2^L, s\gamma_1^M s\gamma_2^M, s\gamma_1^U s\gamma_2^U\right) \end{Bmatrix};$$

$$(2) s\eta_1 \otimes s\eta_2 = \begin{Bmatrix} \left(s\phi_1^L s\phi_2^L, s\phi_1^M s\phi_2^M, s\phi_1^U s\phi_2^U\right), \\ \left(s\varphi_1^L + s\varphi_2^L - s\varphi_1^L s\varphi_2^L, s\varphi_1^M + s\varphi_2^M - s\varphi_1^M s\varphi_2^M, s\varphi_1^U + s\varphi_2^U - s\varphi_1^U s\varphi_2^U\right), \\ \left(s\gamma_1^L + s\gamma_2^L - s\gamma_1^L s\gamma_2^L, s\gamma_1^M + s\gamma_2^M - s\gamma_1^M s\gamma_2^M, s\gamma_1^U + s\gamma_2^U - s\gamma_1^U s\gamma_2^U\right) \end{Bmatrix};$$

$$(3)\ \lambda \times s\eta = \begin{Bmatrix} \left(1-\left(1-s\phi^L\right)^\lambda, 1-\left(1-s\phi^M\right)^\lambda, 1-\left(1-z\phi^U\right)^\lambda\right), \\ \left(\left(s\varphi^L\right)^\lambda, \left(s\varphi^M\right)^\lambda, \left(s\varphi^U\right)^\lambda\right), \\ \left(\left(s\gamma^L\right)^\lambda, \left(s\gamma^M\right)^\lambda, \left(s\gamma^U\right)^\lambda\right) \end{Bmatrix}, \lambda > 0;$$

$$(4)\ s\eta^\lambda = \begin{Bmatrix} \left(\left(s\phi^L\right)^\lambda, \left(s\phi^M\right)^\lambda, \left(s\phi^U\right)^\lambda\right), \\ \left(1-\left(1-s\varphi^L\right)^\lambda, 1-\left(1-s\varphi^M\right)^\lambda, 1-\left(1-s\varphi^U\right)^\lambda\right), \\ \left(1-\left(1-s\gamma^L\right)^\lambda, 1-\left(1-s\gamma^M\right)^\lambda, 1-\left(1-s\gamma^U\right)^\lambda\right) \end{Bmatrix}, \lambda > 0.$$

Then, the constructed operation laws have some properties.

$$(1) s\eta_1 \oplus s\eta_2 = s\eta_2 \oplus s\eta_1, s\eta_1 \otimes s\eta_2 = s\eta_2 \otimes s\eta_1, \left(\left(s\eta_1\right)^{\lambda_1}\right)^{\lambda_2} = \left(s\eta_1\right)^{\lambda_1\lambda_2}; \quad (5)$$

$$(2)\ \lambda\left(s\eta_1 \oplus s\eta_2\right) = \lambda s\eta_1 \oplus \lambda s\eta_2, \left(s\eta_1 \otimes s\eta_2\right)^\lambda = \left(s\eta_1\right)^\lambda \otimes \left(s\eta_2\right)^\lambda; \quad (6)$$

$$(3)\ \lambda_1 s\eta_1 \oplus \lambda_2 s\eta_1 = \left(\lambda_1 + \lambda_2\right) s\eta_1, \left(s\eta_1\right)^{\lambda_1} \otimes \left(s\eta_1\right)^{\lambda_2} = \left(s\eta_1\right)^{\left(\lambda_1+\lambda_2\right)}. \quad (7)$$

Irvanizam et al.[38] put forward the TFNN Hamming distance (TFNNHD) and TFNN Euclid distance (TFNNED).

Definition 3[38]. Let $s\eta_1 = \begin{Bmatrix} \left(s\phi_1^L, s\phi_1^M, s\phi_1^U\right), \\ \left(s\varphi_1^L, s\varphi_1^M, s\varphi_1^U\right), \left(s\gamma_1^L, s\gamma_1^M, s\gamma_1^U\right) \end{Bmatrix}$,

$s\eta_2 = \begin{Bmatrix} \left(s\phi_2^L, s\phi_2^M, s\phi_2^U\right), \\ \left(s\varphi_2^L, s\varphi_2^M, s\varphi_2^U\right), \left(s\gamma_2^L, s\gamma_2^M, s\gamma_2^U\right) \end{Bmatrix}$, the TFNN Hamming distance is constructed:

$$TFNNHD\left(s\eta_1, s\eta_2\right) = \frac{1}{9}\begin{pmatrix} \left|s\phi_1^L - s\phi_2^L\right| + \left|s\phi_1^M - s\phi_2^M\right| + \left|s\phi_1^U - s\phi_2^U\right| \\ + \left|s\varphi_1^L - s\varphi_2^L\right| + \left|s\varphi_1^M - s\varphi_2^M\right| + \left|s\varphi_1^U - s\varphi_2^U\right| \\ + \left|s\gamma_1^L - s\gamma_2^L\right| + \left|s\gamma_1^M - s\gamma_2^M\right| + \left|s\gamma_1^U - s\gamma_2^U\right| \end{pmatrix} \quad (8)$$

Definition 4[38]. Let $s\eta_1=\left\{\begin{matrix}\left(s\phi_1^L,s\phi_1^M,s\phi_1^U\right),\\ \left(s\varphi_1^L,s\varphi_1^M,s\varphi_1^U\right),\left(s\gamma_1^L,s\gamma_1^M,s\gamma_1^U\right)\end{matrix}\right\}$,

$s\eta_2=\left\{\begin{matrix}\left(s\phi_2^L,s\phi_2^M,s\phi_2^U\right),\\ \left(s\varphi_2^L,s\varphi_2^M,s\varphi_2^U\right),\left(s\gamma_2^L,s\gamma_2^M,s\gamma_2^U\right)\end{matrix}\right\}$, the Euclid distance is constructed:

$$ED\left(s\eta_1,s\eta_2\right)=\sqrt{\frac{1}{9}\left(\begin{matrix}\left|s\phi_1^L-s\phi_2^L\right|^2+\left|s\phi_1^M-s\phi_2^M\right|^2+\left|s\phi_1^U-s\phi_2^U\right|^2\\ +\left|s\varphi_1^L-s\varphi_2^L\right|^2+\left|s\varphi_1^M-s\varphi_2^M\right|^2+\left|s\varphi_1^U-s\varphi_2^U\right|^2\\ +\left|s\gamma_1^L-s\gamma_2^L\right|^2+\left|s\gamma_1^M-s\gamma_2^M\right|^2+\left|s\gamma_1^U-s\gamma_2^U\right|^2\end{matrix}\right)} \tag{9}$$

Biswas, Pramanik and Giri [33] combined the concepts of the TFNNs about the score decision values (SDV) and defined THE accuracy decision values (ADV), the SV, and AV of TFNNs to compare two TFNNs.

Definition 5[33]. Let $s\eta_1=\left\{\begin{matrix}\left(s\phi_1^L,s\phi_1^M,s\phi_1^U\right),\\ \left(s\varphi_1^L,s\varphi_1^M,s\varphi_1^U\right),\left(s\gamma_1^L,s\gamma_1^M,s\gamma_1^U\right)\end{matrix}\right\}$,

$s\eta_2=\left\{\begin{matrix}\left(s\phi_2^L,s\phi_2^M,s\phi_2^U\right),\\ \left(s\varphi_2^L,s\varphi_2^M,s\varphi_2^U\right),\left(s\gamma_2^L,s\gamma_2^M,s\gamma_2^U\right)\end{matrix}\right\}$, the SV and AV are defined:

$$SDV\left(s\eta_1\right)=\frac{1}{12}\left[\begin{matrix}8+\left(s\phi_1^L+2s\phi_1^M+s\phi_1^U\right)\\ -\left(s\varphi_1^L+2s\varphi_1^M+s\varphi_1^U\right)\\ -\left(s\gamma_1^L+2s\gamma_1^M+s\gamma_1^U\right)\end{matrix}\right],SDV\left(s\eta_1\right)\in\left[0,1\right] \tag{10}$$

$$SDV\left(s\eta_2\right)=\frac{1}{12}\left[\begin{matrix}8+\left(s\phi_2^L+2s\phi_2^M+s\phi_2^U\right)\\ -\left(s\varphi_2^L+2s\varphi_2^M+s\varphi_2^U\right)\\ -\left(s\gamma_2^L+2s\gamma_2^M+s\gamma_2^U\right)\end{matrix}\right],SDV\left(s\eta_2\right)\in\left[0,1\right] \tag{11}$$

$$ADV\left(s\eta_1\right)=\frac{1}{4}\left(\begin{matrix}\left(s\phi_1^L+2s\phi_1^M+s\phi_1^U\right)\\ -\left(s\gamma_1^L+2s\gamma_1^M+s\gamma_1^U\right)\end{matrix}\right),ADV\left(s\eta_1\right)\in\left[-1,1\right] \tag{12}$$

$$ADV\left(s\eta_2\right)=\frac{1}{4}\left(\begin{matrix}\left(s\phi_2^L+2s\phi_2^M+s\phi_2^U\right)\\ -\left(s\gamma_2^L+2s\gamma_2^M+s\gamma_2^U\right)\end{matrix}\right),ADV\left(s\eta_2\right)\in\left[-1,1\right] \tag{13}$$

For two TFNNs $s\eta_1=\left\{\begin{matrix}\left(s\phi_1^L,s\phi_1^M,s\phi_1^U\right),\\ \left(s\varphi_1^L,s\varphi_1^M,s\varphi_1^U\right),\left(s\gamma_1^L,s\gamma_1^M,s\gamma_1^U\right)\end{matrix}\right\}$,

$s\eta_2=\left\{\begin{matrix}\left(s\phi_2^L,s\phi_2^M,s\phi_2^U\right),\\ \left(s\varphi_2^L,s\varphi_2^M,s\varphi_2^U\right),\left(s\gamma_2^L,s\gamma_2^M,s\gamma_2^U\right)\end{matrix}\right\}$, then

(1) *if* $SDV\left(s\eta_1\right) \prec SDV\left(s\eta_2\right)$, *then* $s\eta_1 \prec s\eta_2$;
(2) *if* $SDV\left(s\eta_1\right) = SDV\left(s\eta_2\right), ADV\left(s\eta_1\right) \prec ADV\left(s\eta_2\right)$, *then* $s\eta_1 \prec s\eta_2$;
(3) *if* $SDV\left(s\eta_1\right) = SDV\left(s\eta_2\right), ADV\left(s\eta_1\right) = ADV\left(s\eta_2\right)$, *then* $s\eta_1 = s\eta_2$.

3. TFNN-WASPAS technique for MADM with TFNSs

The TFNN-WASPAS technique is put forward MADM. Let $\left\{SX_1, SX_2, \ldots, SX_m\right\}$ be alternatives, $\left\{SG_1, SG_2, \ldots, SG_n\right\}$ be attributes. The TFNN-WASPAS technique is put forward MADM (See Figure 1).

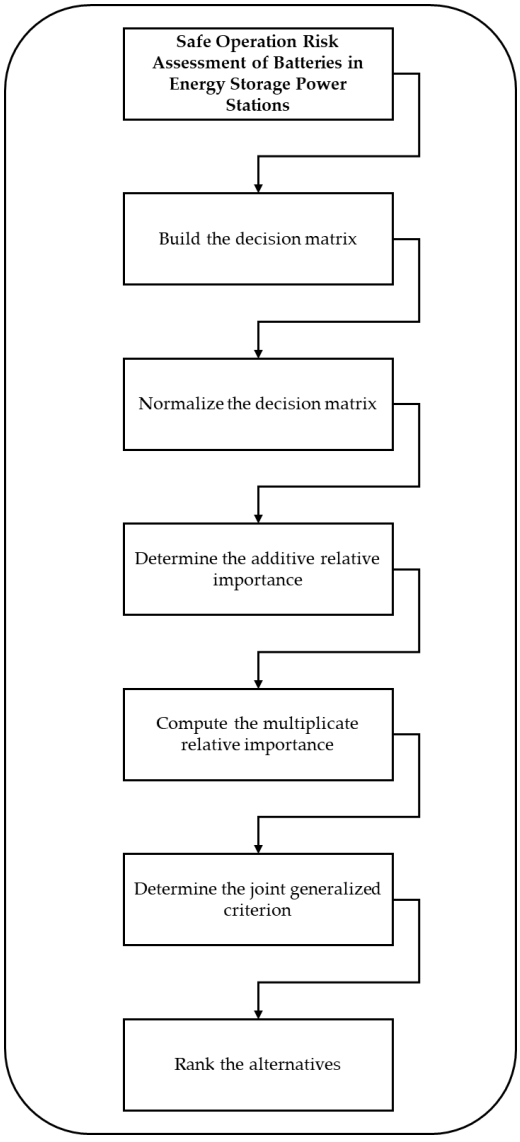

Safe Operation Risk Assessment of Batteries in Energy Storage Power Stations
Build the decision matrix
Normalize the decision matrix
Determine the additive relative importance
Compute the multiplicate relative importance
Determine the joint generalized criterion
Rank the alternatives

Figure 1. TFNN-WASPAS technique for MADM

Step 1. Build the decision matrix.

$$Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{m1} & \cdots & y_{mn} \end{bmatrix} \tag{14}$$

Where $i = 1, \ldots, m; j = 1, \ldots, n$

Step 2. Normalize the decision matrix

$$P_{ij} = \frac{SDV(y_{ij})}{\max_i SDV(y_{ij})} \tag{15}$$

$$P_{ij} = \frac{\min_i SDV(y_{ij})}{SDV(y_{ij})} \tag{16}$$

Step 3. Determine the additive relative importance

$$U_i^{(1)} = \sum_{j=1}^{n} w_j P_{ij} \tag{17}$$

Step 4. Compute the multiplicate relative importance

$$U_i^{(2)} = \prod_{j=1}^{n} \left(P_{ij}\right)^{w_j} \tag{18}$$

Step 5. Determine the joint generalized criterion.

$$U_i = 0.5 \left(U_i^{(1)} + U_i^{(2)}\right) \tag{19}$$

Step 6. Rank the alternatives.

4. Numerical example and comparative analysis

With the proposal of the dual-stage goal of carbon reduction in China, "peaking carbon" and "carbon neutrality" have become current research hotspots, and new energy-related industries are experiencing faster development. Building a new power system with new energy as the main body can be foreseen as a key task for development in the next thirty years. However, as a large number of new energy generation equipment such as wind power and photovoltaics occupy an increasing proportion of the overall power grid, the randomness, volatility, and intermittency of new energy generation also pose significant risks to the smooth operation of the power system. Energy storage, as a key technology that can improve the consumption level of new energy generation, enhance the flexibility of power systems, and support the smooth and safe operation of new power systems, will undoubtedly be the next trillion-level emerging market. The energy storage industry is facing significant opportunities and challenges. Building a new type of energy storage in the large power grid can effectively improve the regulation capacity of the power system and enhance the system's ability to absorb and store renewable energy. It is an important guarantee for establishing a new type of power system mainly based on new energy and has important practical significance for building a clean, low-carbon, safe, and efficient energy system. With the advancement of energy storage technology and the urgent increase in demand, new energy storage power stations have the characteristics of flexible site selection and layout, short construction period, and fast regulation response speed, which have led to the rapid development of new energy storage power station construction and increasing large-scale applications. The safe operation risk assessment of batteries in energy storage power stations could be put forward by the MADM. Then, a numerical example is done for safe operation risk assessment of batteries in energy storage power stations through the TFNN-WASPAS technique. Five energy storage power stations are assessed in line with 27 attributes as shown in Table 1.

Table 1. The weights of criteria.

	Criteria	Weights
C_1	Temperature Sensitivity	0.039035
C_2	Maintenance Complexity	0.038284
C_3	Human Error Probability	0.033374
C_4	Cybersecurity Risks	0.035291
C_5	Load Demand Fluctuation	0.038474
C_6	State of Health	0.039258
C_7	Natural Disasters	0.035884
C_8	Power Outages or Fluctuations	0.035291
C_9	Vibration and Shock Resistance	0.039035
C_{10}	Humidity and Corrosion Resistance	0.038619
C_{11}	Emergency Response Preparedness	0.033262
C_{12}	Chemical Stability	0.037084
C_{13}	Fault Tolerance	0.039976
C_{14}	Energy Management System Reliability	0.038507
C_{15}	Compliance with Standards	0.033485
C_{16}	System Redundancy	0.036412
C_{17}	Thermal Stability	0.040536
C_{18}	Toxicity of Materials	0.037723
C_{19}	Regulatory Penalties	0.033485
C_{20}	Monitoring and Diagnostics	0.036412
C_{21}	Electrical Safety	0.040536
C_{22}	End-of-Life Disposal Risks	0.038284
C_{23}	Training and Competency	0.033485
C_{24}	Grid Compatibility	0.036412
C_{25}	Fire and Explosion Risk	0.039976
C_{26}	Recycling and Reusability	0.037084
C_{27}	Cost of Safety Measures	0.034796

Step 1: We begin by constructing the decision matrix using Equation (14), which is displayed in Table 2. Then, we calculate the criteria weights, as shown in Table 1.

Step 2: Next, we normalize the decision matrix using Equation (15), and the results are shown in Table 3.

Step 3: After that, we calculate the additive relative importance based on Equation (19).

Step 4: Then, we determine the multiplicative relative importance using Equation (20).

Step 5: We proceed by calculating the joint generalized criterion with Equation (21).

Step 6: Finally, we rank the alternatives, as illustrated in Figure 2.

Table 2. The decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_2	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_3	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_4	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_5	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_6	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_7	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_8	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_9	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{10}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{11}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{12}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{13}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{14}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{15}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{16}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{17}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{18}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{19}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{20}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{21}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{22}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{23}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)

C_{24}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{25}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{26}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{27}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
	A_1	A_2	A_3	A_4	A_5
C_1	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_2	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_3	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_4	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_5	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_6	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_7	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_8	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_9	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{10}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{11}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{12}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{13}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{14}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{15}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{16}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{17}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{18}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{19}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{20}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{21}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{22}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{23}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{24}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{25}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{26}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{27}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_2	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_3	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_4	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_5	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_6	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_7	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_8	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_9	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{10}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{11}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{12}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{13}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{14}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{15}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{16}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{17}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{18}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{19}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{20}	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{21}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{22}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{23}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)
C_{24}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)
C_{25}	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
C_{26}	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(1,3,5;0.5,1.5,2.5;1.2,2.7,4.5)
C_{27}	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(0.5,1.5,2.5;0.3,1.3,2.2;0.7,1.7,2.2)	(0.3,1.2,2.8;0.5,1.5,2.5;0.8,1.7,2.7)

Table 3. The normalized decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	0.918595	1	0.658472	0.776431	0.878483
C_2	0.597938	0.686089	0.942722	0.686089	1
C_3	0.781704	0.739381	0.848384	1	0.848384
C_4	1	0.749556	0.976498	0.907335	0.722004
C_5	0.703651	1	0.880889	1	0.956333
C_6	0.670574	0.857836	0.939242	0.788534	1
C_7	1	0.79227	0.933857	0.779095	0.73
C_8	1	0.974485	0.985044	0.985044	0.78384
C_9	0.597938	0.942722	0.620756	1	0.828165
C_{10}	0.632165	0.686089	0.942722	0.686089	1
C_{11}	0.753489	1	0.929172	0.739381	0.781704
C_{12}	1	0.620756	0.686089	0.808701	0.67468
C_{13}	0.935311	0.990903	0.839543	1	0.84864
C_{14}	0.609347	0.885443	0.808701	0.632165	1
C_{15}	0.828708	0.78384	1	0.959529	0.914354
C_{16}	1	0.805444	0.845238	0.956333	0.690476
C_{17}	0.686089	0.942722	0.686089	1	0.828165
C_{18}	0.634268	0.797078	0.869939	0.788534	1
C_{19}	0.767596	0.848384	1	0.834277	0.781704
C_{20}	1	0.847251	0.856431	0.856431	0.681498
C_{21}	0.751423	0.942722	0.620756	1	0.828165
C_{22}	0.597938	0.686089	0.942722	0.686089	1
C_{23}	0.763332	0.883832	1	0.722004	0.763332
C_{24}	1	0.805444	0.845238	0.956333	0.690476
C_{25}	0.956333	1	0.805444	1	0.956333
C_{26}	0.597938	0.762832	0.731959	0.697498	1

C_{27}	0.73	1	0.956333	0.690476	0.73

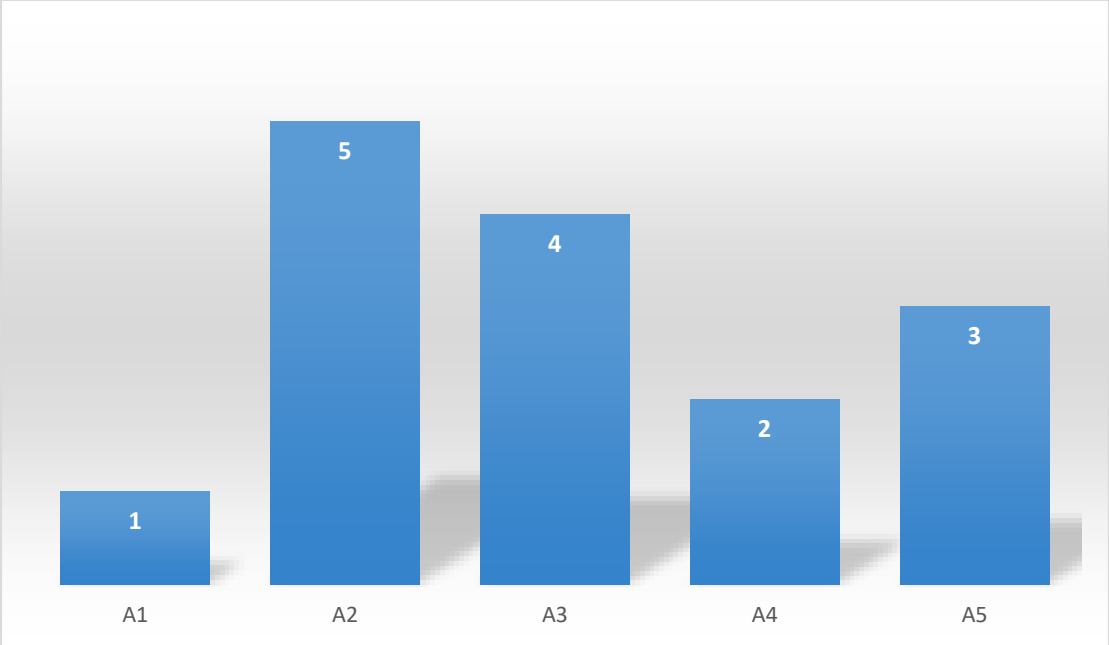


Figure 2. The rank of alternatives.

4.1 Sensitivity analysis

We adjusted the criteria weights across 28 different cases, as shown in Figure 3. In the first case, we assigned equal weights to all criteria. In the other cases, we set the weight of the first criterion to 0.05, while the remaining criteria had equal weights. Next, we applied the TFNN-WASPAS method for each set of criteria weights, as illustrated in Figure 4. Our results show that Alternative 3 is the best option, while Alternative 1 is the worst.

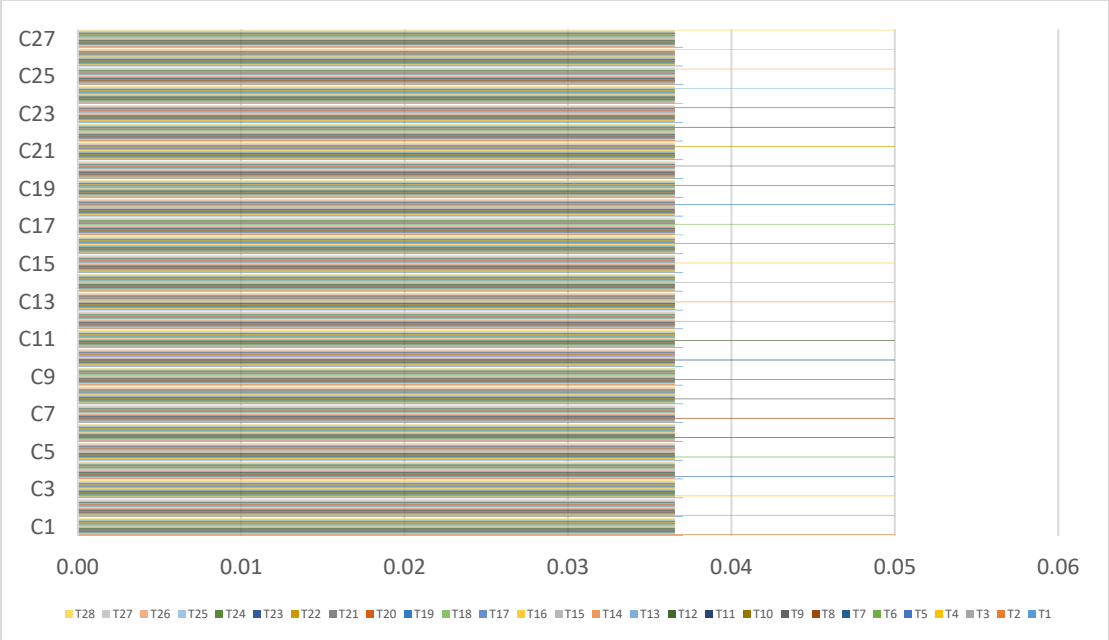


Figure 3. The criteria weights.

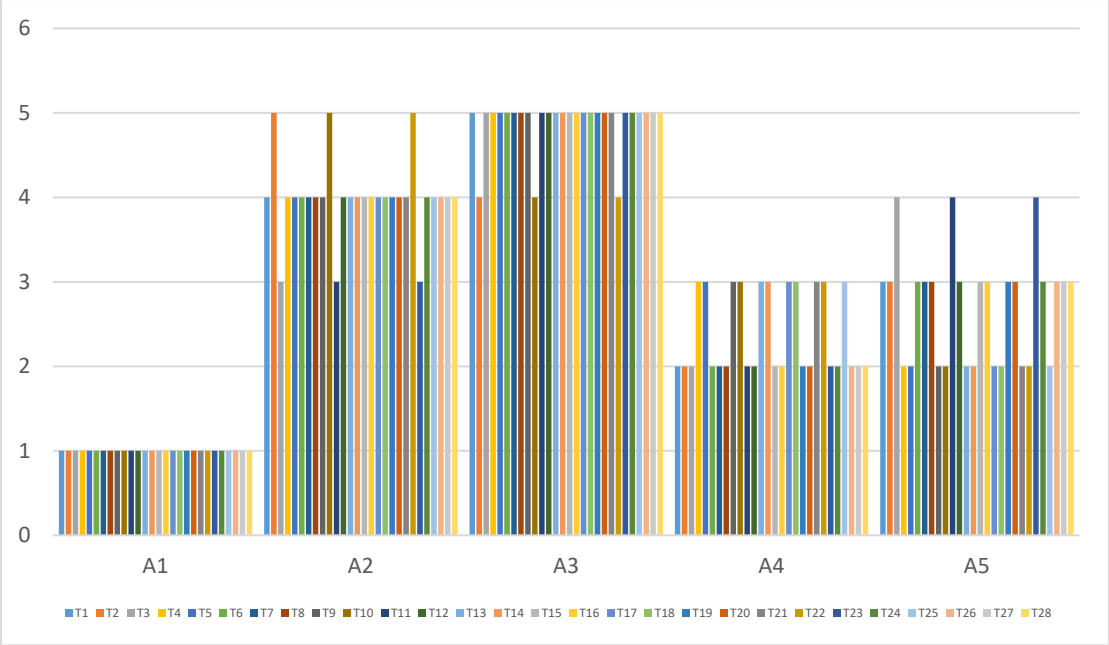


Figure 4. The rank of alternatives under different weights.

4.2 Comparative Study

This study compares the proposed methodology by various MADM methods such as TFNN-TOPSIS, TFNN-VIKOR, TFNN-COPRAS, and TFNN-EDAS methods. These methods are compared under the weights of criteria by the proposed method. Figure 5 shows comparative study. We show the proposed method is robust compared with other MADM methods.

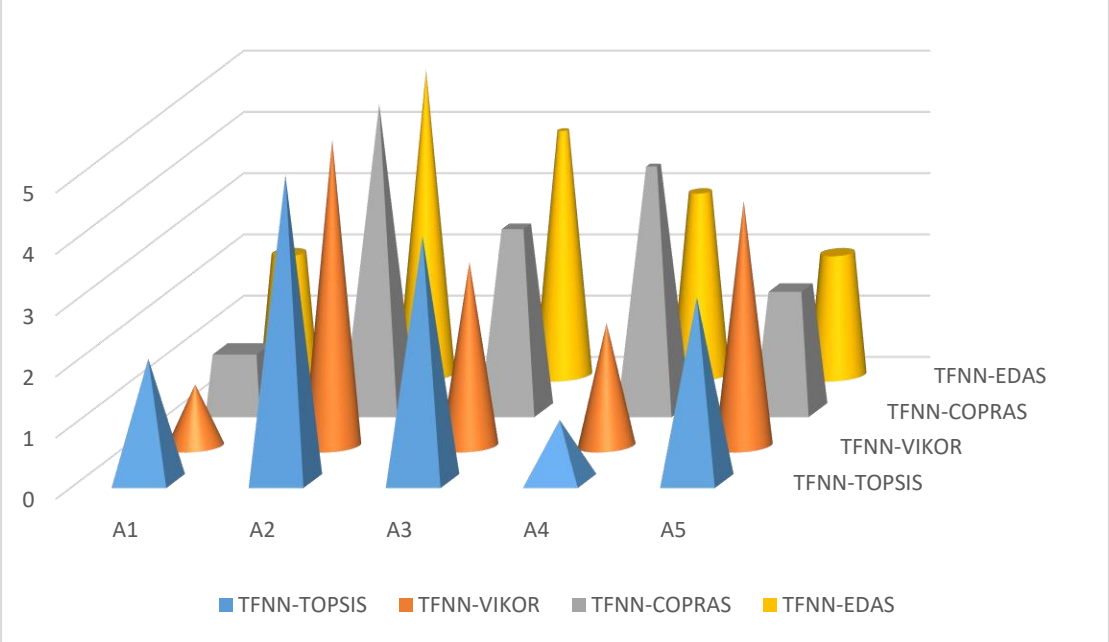


Figure 5 The comparative study.

5. Conclusion

With the rapid development of related technologies, the scale of batteries in energy storage power stations continues to increase, but they face huge challenges in terms of safety. The more than 30 incidents of energy storage power plant accidents in South Korea and the fire and explosion accidents at energy storage power plants in Fengtai District, Beijing have had a profound impact and sounded the alarm for the subsequent development of energy storage power plants in China. The safety of batteries is not only reflected in breakthroughs in materials and technology but also in the management of operational risks. Risk assessment, as an important step in the safe operation risk management of energy storage power station batteries, is currently less studied, and the results of the assessment directly affect subsequent risk control. Proper and accurate evaluation can effectively prevent and reduce the occurrence of accidents. The safe operation risk assessment of batteries in energy storage power stations is considered a MADM problem. Recently, the WASPAS technique and the average method have been applied to address MADM challenges. To better express uncertainty during the risk assessment process, TFNSs have been utilized. In this paper, a TFNN-WASPAS approach based on TFNN-WASPAS is proposed to handle MADM with TFNSs. The average method is used to calculate weight values through TFNNWASPAS under TFNSs. A numerical example is provided to evaluate the safe operation risks of batteries in energy storage power stations, and comparative analyses are conducted to validate the effectiveness of the TFNN-WASPAS method. The key contributions of this work are summarized as follows: (1) the construction of the TFNN- WASPAS technique based on TFNN-WASPAS; (2) the use of the average method to manage weights within the TFNN-WASPAS framework under TFNSs; (3) the application of the TFNN-WASPAS technique to address MADM problems using TFNN-WASPAS under TFNSs; and (4) the provision of a numerical example and comparative analysis for the safe operation risk assessment of batteries in energy storage power stations to verify the proposed method.

References

[1] L.B. Yang, H. Ribberink, Investigation of the potential to improve DC fast charging station economics by integrating photovoltaic power generation and/or local battery energy storage system, Energy, 167 (2019) 246-259.

[2] H. He, F.X. Peng, Z.N. Gao, X. Liu, S.B. Hu, W. Zhou, H. Sun, A multi-objective risk scheduling model of an electrical power system-containing wind power station with wind and energy storage integration, Energies, 12 (2019) 21.

[3] Y. Yang, C. Lian, C. Ma, Y.S. Zhang, Research on energy storage optimization for large-scale pv power stations under given long-distance delivery mode, Energies, 13 (2020) 20.

[4] A. Hussain, V.H. Bui, H.M. Kim, Optimal sizing of battery energy storage system in a fast ev charging

station considering power outages, Ieee Transactions on Transportation Electrification, 6 (2020) 453-463.

[5] C. Balasundar, C.K. Sundarabalan, N.S. Srinath, J. Sharma, J.M. Guerrero, Interval type2 fuzzy logic-based power sharing strategy for hybrid energy storage system in solar powered charging station, Ieee Transactions on Vehicular Technology, 70 (2021) 12450-12461.

[6] F. Zhang, A.H. Fu, L. Ding, Q.W. Wu, Mpc based control strategy for battery energy storage station in a grid with high photovoltaic power penetration, International Journal of Electrical Power & Energy Systems, 115 (2020) 9.

[7] M.R. Khalid, I.A. Khan, S. Hameed, M.S.J. Asghar, J.S. Ro, A comprehensive review on structural topologies, power levels, energy storage systems, and standards for electric vehicle charging stations and their impacts on grid, Ieee Access, 9 (2021) 128069-128094.

[8] M. Javidsharifi, H. Pourroshanfekr, T. Kerekes, D. Sera, S. Spataru, J.M. Guerrero, Optimum sizing of photovoltaic and energy storage systems for powering green base stations in cellular networks, Energies, 14 (2021) 21.

[9] H. Saboori, S. Jadid, M. Savaghebi, Optimal management of mobile battery energy storage as a self-driving, self-powered and movable charging station to promote electric vehicle adoption, Energies, 14 (2021) 19.

[10] M.A.H. Rafi, J. Bauman, A comprehensive review of dc fast-charging stations with energy storage: Architectures, power converters, and analysis, Ieee Transactions on Transportation Electrification, 7 (2021) 345-368.

[11] B.J. Sun, A multi-objective optimization model for fast electric vehicle charging stations with wind, pv power and energy storage, Journal of Cleaner Production, 288 (2021) 17.

[12] Y.X. Ma, Z.C. Hu, Y.H. Song, Hour-ahead optimization strategy for shared energy storage of renewable energy power stations to provide frequency regulation service, Ieee Transactions on Sustainable Energy, 13 (2022) 2331-2342.

[13] N. Kumar, T. Kumar, S. Nema, T. Thakur, A multiobjective planning framework for ev charging stations assisted by solar photovoltaic and battery energy storage system in coupled power and transportation network, International Journal of Energy Research, 46 (2022) 4462-4493.

[14] W.W. Zhao, T.T. Zhang, H. Kildahl, Y.L. Ding, Mobile energy recovery and storage: Multiple energy-powered evs and refuelling stations, Energy, 257 (2022) 10.

[15] Y.L. Wu, L. Ge, X.D. Yuan, X.Y. Fu, M.S. Wang, Adaptive power control based on double-layer q-learning algorithm for multi-parallel power conversion systems in energy storage station, Journal of Modern Power Systems and Clean Energy, 10 (2022) 1714-1724.

[16] G.W. Ding, L.J. Li, Y. Li, Q.S. Zhang, C. Luo, Q. Chen, X.H. Zheng, Strategy of 5g base station energy storage participating in the power system frequency regulation, Arabian Journal for Science and Engineering, 48 (2023) 14537-14548.

[17] J.H. Yue, X.Y. Xia, Y. Zhang, T. Xia, A state-of-health estimation and prediction algorithm for lithium-ion battery of energy storage power station based on information entropy of characteristic data, Journal of Electrical Engineering & Technology, 18 (2023) 1757-1768.

[18] B.X. Zhu, Y.Z. Wang, H. Guo, N. Yang, L. Lu, Improved model of base station power system for the optimal capacity planning of photovoltaic and energy storage system, Electronics, 12 (2023) 15.

[19] J.G. Han, S.W. Lin, B.Y. Pu, Hierarchical energy management of dc microgrid with photovoltaic power generation and energy storage for 5g base station, Sustainability, 16 (2024) 19.

[20] P. Stanko, M. Tkac, M. Kajanova, M. Roch, Impacts of electric vehicle charging station with photovoltaic system and battery energy storage system on power quality in microgrid, Energies, 17 (2024) 22.

[21] N. Pei, X.L. Song, Z. Zhang, F. Peña-Mora, Optimizing the operation and allocating the cost of shared energy storage for multiple renewable energy stations in power generation side, Energy Conversion and Management, 302 (2024) 17.

[22] X.Y. Xia, J.H. Yue, Y. Guo, C.G. Lv, X.Y. Zeng, Y.K. Xia, G.Q. Chen, Technologies for energy storage power stations safety operation: Battery state evaluation survey and a critical analysis, Ieee Access, 12 (2024) 31334-31356.

[23] D. Pamucar, G. Duran-Romero, M. Yazdani, A.M. Lopez, A decision analysis model for smart mobility system development under circular economy approach, Socio-Economic Planning Sciences, 86 (2023) 24.

[24] H. Garg, Z. Ali, T. Mahmood, M.R. Ali, Topsis-method based on generalized dice similarity measures with hamy mean operators and its application to decision-making process, Alexandria Engineering Journal, 65 (2023) 383-397.

[25] D. Rani, H. Garg, Multiple attributes group decision-making based on trigonometric operators, particle swarm optimization and complex intuitionistic fuzzy values, Artificial Intelligence Review, 56 (2023) 1787-1831.

[26] J. Ye, B.Z. Sun, Q. Bao, C. Che, Q.C. Huang, X.L. Chu, A new multi-objective decision-making method with diversified weights and pythagorean fuzzy rough sets, Computers & Industrial Engineering, 182 (2023) 15.

[27] J. Ye, B.Z. Sun, X.L. Chu, J.M. Zhan, J.X. Cai, Valued outranking relation-based heterogeneous multi-decision multigranulation probabilistic rough set and its use in medical decision-making, Expert Systems with Applications, 228 (2023) 18.

[28] C. Kahraman, S.C. Onar, B. Oztaysi, A novel spherical fuzzy critic method and its application to prioritization of supplier selection criteria, Journal of Intelligent & Fuzzy Systems, 42 (2022) 29-36.

[29] H. Sun, Z. Yang, Q. Cai, G.W. Wei, Z.W. Mo, An extended exp-todim method for multiple attribute decision making based on the z-wasserstein distance, Expert Systems with Applications, 214 (2023) 14.

[30] H.Y. Zhang, H.J. Wang, G.W. Wei, Spherical fuzzy todim method for magdm integrating cumulative prospect theory and critic method and its application to commercial insurance selection, Artificial Intelligence Review, (2023) https://doi.org/10.1007/s10462-10023-10409-10463.

[31] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[32] F. Smarandache, A unifying field in logics: Neutrosophic logic, Multiple-Valued Logic, 8 (1999).

[33] P. Biswas, S. Pramanik, B.C. Giri, Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making, Neutrosophic Sets and Systems, 12 (2016) 20-40.

[34] S.I.A. Aal, M.M. Abd Ellatif, M.M. Hassan, Proposed model for evaluating information systems quality based on single valued triangular neutrosophic numbers, International Journal of Mathematical Sciences & Computing, 4 (2018) 1-14.

[35] A. Chakraborty, S.P. Mondal, A. Ahmadian, N. Senu, S. Alam, S. Salahshour, Different forms of triangular neutrosophic numbers, de-neutrosophication techniques, and their applications, Symmetry-Basel, 10 (2018) 27.

[36] E.K. Zavadskas, J. Antucheviciene, J. Saparauskas, Z. Turskis, Mcdm methods waspas and multimoora: Verification of robustness of methods when assessing alternative solutions, Economic Computation and Economic Cybernetics Studies and Research, 47 (2013) 5-20.

[37] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[38] I. Irvanizam, N.N. Zi, R. Zuhra, A. Amrusi, H. Sofyan, An extended mabac method based on triangular fuzzy neutrosophic numbers for multiple-criteria group decision making problems, Axioms, 9 (2020) 18.

Received: Aug 28, 2024. Accepted: Nov 22, 2024

Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

An Approach to Numerical Solutions for Refined Neutrosophic Differential Problems of High-Orders

Ahmed Salem Heilat

Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan

ahmed-heilat@yahoo.com

ABSTRACT

The main goal of this paper is to studying a new numerical method for finding the refined neutrosophic numerical solutions to some refined neutrosophic differential problems of high orders. We use high-degrees refined neutrosophic polynomials to get the approximated solutions, where we apply our method on two different refined neutrosophic boundary value problems with numerical tables to compare the novel findings with classical ones.

Keywords: refined neutrosophic set, refined neutrosophic Differential Equations, numerical solution, refined neutrosophic approximation.

Introduction

The primary goal of numerical analysis is to provide simple algorithms and methods for finding numerical solutions to differential equations, specifically those equations that are of higher order, because it is difficult to find their solutions using traditional methods [1-2, 8]. We find in previous research works a broad discussion of some high-order differential equations [3,9], where researchers have modeled many numerical methods with the aim of finding the best approximate numerical solutions for these differential equations that are specialized in describing many natural phenomena in many scientific fields such as physics, computer science, And even chemistry. Neutrosophic logic is a revolutionary logic introduced by Smarandache [6] as a new generalization of fuzzy logic that takes into account the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena. It has been used to study many traditional mathematical concepts, such as algebraic

structures, analysis, and even computer science [14, 17-18]. In [10-11, 13], many numerical algorithms for solving differential equations and numerically complex problems have been discussed by Neutrosophic sets, and researchers have obtained many good approximate results. Recently Hatamleh and AL-Husban et al., Discussed the new structure of such a neutrosophic set and its application [19-39]. This has motivated us to study a new numerical method for finding the refined neutrosophic numerical solutions to some refined neutrosophic differential problems of high orders. We use high-degree refined neutrosophic polynomials to get the approximated solutions, where we apply our method to two different refined neutrosophic boundary value problems with numerical tables to compare the novel findings with classical ones.

Main Discussion

Refined Neutrosophic Approximations by Polynomials:

The main idea is to use the refined neutrosophic interval $[a + cI_1 + mI_2, b + dI_1 + nI_2]$, as follows:

$k = 0,1, \dots, N \quad , \quad x_k = a + cI_1 + mI_2 + kI_1 + hI_2$

where $h = {(b - a + (d - c)I_1 + (n - m)I_2)}/{n}$ step length.

Let's take $P(X + YI_1 + ZI_2)$ polynomial as an approximation for solving the problem $u(x + yI_1 + zI_2)$ and for every $x + yI_1 + zI_2 \in [\mathrm{x_k} + y_kI_1 + z_kI_2\ , \mathrm{x_{k+1}} + y_{k+1}I_1 + z_{k+1}I_2]$ this approximation is given as:

$$P(x + yI_1 + zI_2) = \sum_{i=0}^{17} \frac{(x + yI_1 + zI_2 - (\mathrm{x_k} + y_kI_1 + z_kI_2))^i}{i!} P_k^{(i)} + \frac{(x + yI_1 + zI_2 - (\mathrm{x_k} + y_kI_1 + z_kI_2))^{18}}{18!} C_{k,1} + \frac{(x + yI_1 + zI_2 - (\mathrm{x_k} + y_kI_1 + z_kI_2))^{19}}{19!} C_{k,2} + \frac{\left(x + yI_1 + zI_2 - \left(\mathrm{x_k} + y_kI_1 + z_kI_2\right)\right)^{20}}{20!} C_{k,3} + \frac{\left(x + yI_1 + zI_2 - \left(\mathrm{x_k} + y_kI_1 + z_kI_2\right)\right)^{21}}{21!} C_{k,4} + \frac{\left(x + yI_1 + zI_2 - \left(\mathrm{x_k} + y_kI_1 + z_kI_2\right)\right)^{22}}{22!} C_{k,5}\ ; k = 0,1, \dots, N - 1 \qquad (1)$$

Where $P_k^{(i)} = P^{(i)}(X_k + Y_kI_1 + + Z_kI_2)\ , (i = 0,1, \dots ,17)$

The approximation $P(X + YI_1 + ZI_2)$ satisfies the following conditions:

- $P_k^{(m)}(\mathrm{x_k} + y_k I_1 + z_k I_2) = u^{(m)}(\mathrm{x_k} + y_k I_1 + z_k I_2)\,, k = 0,1, \dots, N-1; m = 0,1, \dots, 17$
- $P_k^{(m)}(\mathrm{x_{k+1}} + y_{k+1} I_1 + z_{k+1} I_2) = P_{k+1}^{(m)}(\mathrm{x_{k+1}} + y_{k+1} I_1 + z_{k+1} I_2)\,, k = 0,1, \dots . . N-2, m = 0,1, \dots, 17$

We define five aggregation points in each partial domain as follows:

$$X_{k+z_j} = X_k + Y_k I_1 + Z_k I_2 + h I_1 + z_j I_2\,, (j = 1,2, \dots, 5), \tag{2}$$

The given points are related by the relation (2) defined by the form:

$$0 < z_1 + z_1' I_1 + z_1'' I_2 < z_2 + z_2' I_1 + z_2'' I_2 < z_3 + z_3' I_1 + z_3'' I_2 < z_4 + z_4' I_1 + z_4'' I_2 < z_5 + z_5' I_1 + z_5'' I_2 = 1 + I_1 + I_2 \tag{3}$$

Numerical solution of the refined neutrosophic boundary value problem:

Assuming that $u(x + yI_1 + zI_2)$ is a single solution to the problem of boundary values, then this solution is related to the solutions denoted by $\{\mathrm{W}_i(x + yI_1 + zI_2)\}_{\mathrm{i=0}}^{7}$, so that for seven real constants $\mathrm{c_1, c_2, \dots, c_7}$,we have:

$$\mathrm{u}(x + yI_1 + zI_2) = \mathrm{W_0}(x + yI_1 + zI_2) + \sum_{k=1}^{17} c_k \mathrm{W}_k(x + yI_1 + zI_2) \tag{4}$$

$$\mathrm{W_0^{(18)}}(x + yI_1 + zI_2) + \sum_{i=1}^{18} q_i(x + yI_1 + zI_2)\mathrm{W_0^{(18-i)}}(x + yI_1 + zI_2) = f(x + yI_1 + zI_2),\ a + cI_1 + mI_2 \le x + yI_1 + zI_2 \le\ b + dI_1 + nI_2, \tag{5}$$

$$\mathrm{W_0^{(2j)}}(a + cI_1 + mI_2) = \alpha_i\,, \mathrm{W_0^{(2j+1)}}(a + cI_1 + mI_2) = 0, j = 0,1, \dots ., 6 \tag{6}$$

Then,

$$\mathrm{W_1^{(18)}}(x + yI_1 + zI_2) + \sum_{i=1}^{14} q_i(x + yI_1 + zI_2)\mathrm{W_1^{(18-i)}}(x + yI_1 + zI_2) = 0\,, a + cI \le x + yI_1 + zI_2 \le b + dI_1 + nI_2\,, \tag{7}$$

$$\mathrm{W_1^{(2j)}}(a + cI_1 + mI_2) = 0\,, \mathrm{W_1'}(a + cI_1 + mI_2) = 1\,, V_1^{(2j+1)}(a) = 0\,, j = 1,2, \dots, 6 \tag{8}$$

$$\mathrm{W_2^{(18)}}(x + yI_1 + zI_2) + \sum_{i=1}^{18} q_i(x + yI_1 + zI_2)\mathrm{W_2^{(18-i)}}(x + yI_1 + zI_2) = 0\,, a + cI_1 + mI_2 \le x + yI_1 + zI_2 \le b + dI_1 + nI_2\,, \tag{9}$$

$$\mathrm{W}_2^{(j)}(a+cI_1+mI_2)=0\,,\mathrm{W}_2^{(3)}(a+cI_1+mI_2)=1\,,j=0,1,\dots.,13,$$
$$j\neq 3, \tag{10}$$

$$\mathrm{W}_3^{(18)}(x+yI_1+zI_2)+\sum_{i=1}^{18}q_i(x+yI_1+zI_2)\mathrm{W}_3^{(18-i)}(x+yI_1+zI_2)=0\,,a+cI_1+mI_2\leq x+yI_1+$$
$$zI_2\leq b+dI_1+nI_2, \tag{11}$$

So that

$$\mathrm{W}_3^{(j)}(a+cI_1+mI_2)=0\,,\mathrm{W}_3^{(5)}(a+cI_1+mI_2)=1\,,j=0,1,\dots,17, \tag{12}$$

$$\mathrm{W}_4^{(18)}(x+yI_1+zI_2)+\sum_{i=1}^{18}q_i(x+yI_1+zI_2)\mathrm{W}_4^{(18-i)}(x+yI_1+zI_2)=0\,,\ a+cI_1+mI_2\leq x+yI_1+$$
$$zI_2\leq b+dI_1+nI_2\,, \tag{13}$$

$$\mathrm{W}_4^{(j)}(a+cI_1+mI_2)=0,\mathrm{W}_4^{(7)}(a+cI_1+mI_2)=1\,,j=0,1,\dots,17\,, \tag{14}$$

$$W_5^{(14)}(x+yI_1+zI_2)+\sum_{i=1}^{14}q_i(x+yI_1+zI_2)\mathrm{W}_5^{(14-i)}(x+yI_1+zI_2)=0\,,a+cI_1+mI_2\leq x+yI_1+$$
$$zI_2\leq b+dI_1+nI_2, \tag{15}$$

And So on.

Now we will prove that the function $u(x+yI_1+zI_2)=V_0(x+yI_1+zI_2)+\sum_{k=1}^{18}c_kV_k(x+yI_1+zI_2)$ is the only solution to the problem of the boundary value for the real constants

$c_i\ for\ all\ possible\ values\ of\ i$ we have:

$$u^{(18)}(x+yI_1+zI_2)$$
$$=W_0^{(18)}(x+yI_1+zI_2)$$
$$+\sum_{k=1}^{7}c_kW_k^{(18)}=-\sum_{i=1}^{18}q_i(x+yI_1+zI_2)W_0^{(18)}(x+yI_1+zI_2)+f(x+yI_1+zI_2I)$$
$$+\sum_{k=1}^{7}c_k\left[-\sum_{i=1}^{18}q_i(x+yI_1+zI_2)W_k^{(18-i)}(x+yI_1+zI_2)\right]$$

$$= -\sum_{i=1}^{18} q_i(x + yI_1 + zI_2)\left[W_0^{(18-i)}(x + yI) + \sum_{k=1}^{18} c_k W_k^{(18-i)}(x + yI_1 + zI_2)\right] + f(x + yI_1 + zI_2)$$

$$= -\sum_{i=1}^{18} q_i(x + yI_1 + zI_2)\left[u^{(18-i)}(x + yI_1 + zI_2) + f(x + yI_1 + zI_2)\right.$$

Where: $u^i(x + yI_1 + zI_2) = W_0^{(i)}(x + yI_1 + zI_2) + \sum_{k=1}^{18} c_k W_k^{(i)}(x + yI)$, $i = 0,1, \ldots .,17$

$$u^{(2j)}(a + cI_1 + mI_2)$$

$$= W_0^{(2j)}(a + cI_1 + mI_2)$$

$$+ \sum_{k=1}^{18} c_k W_k^{(2j)}(a + cI_1 + mI_2) = \alpha_j + \sum_{k=1}^{18} c_k(0) = \alpha_j \,, \quad (j = 0, \ldots .,12)$$

$$u^{(2j)}(a + cI_1 + mI_2) = W_0^{(2j)}(a + cI_1 + mI_2) + \sum_{k=1}^{18} c_k W_k^{(2j)}(a + cI_1 + mI_2) = \alpha_j \,, (j = 0, \ldots ,12)$$

And to achieve the rest of the conditions (2) at the end of the domain, we put the following system of equations:

$$u^{(2j)}(b + dI_1 + nI_2)$$

$$= W_0^{(2j)}(b + dI_1 + nI_2) + \sum_{k=1}^{18} c_k W_k^{(2j)}(b + dI_1 + nI_2) \equiv \beta_j \,, \quad (j = 0,1, \ldots ,16) \,, \qquad (16)$$

By solving the system of linear equations (16),we get:

$$C = W_b^{-1} B.$$

The derivatives as a sum of solutions of elementary value problems as follows:

$P^{(i)}\left(\mathrm{x_k} + y_k I_1 + z_k I_2\right) = P_0^{(i)}\left(\mathrm{x_k} + y_k I_1 + z_k I_2\right) + \sum_{j=1}^{14} c_j P_j^{(i)}\left(\mathrm{x_k} + y_k I_1 + z_k I_2\right)$, $i = 0,1, \ldots .,13; k =$ $0,1, \ldots . , N$. (17)

Existence of the numerical solution:

Let's take the eighteenth-order differential equation in the following case:

$$\begin{cases} \mathrm{u}^{(18)}(x+yI_1+zI_2)=\mathrm{F}\left[x+yI_1+zI_2\,,\mathrm{u}(x+yI_1+zI_2)\,,\acute{\mathrm{u}}(x+yI_1+zI_2)\ldots,\mathrm{u}^{(17)}(x+yI_1+zI_2)\right] \\ \mathrm{u}^{(\mathrm{j})}(\mathrm{a})=\mathrm{u}_{\mathrm{j}}\ ,\ \mathrm{j}=0,1,\ldots,13 \end{cases} \quad (18)$$

Assuming that $\mathrm{F}:[a+cI_1+mI_2\mathrm{I},b+dI_1+nI_2]\times\mathrm{C}[a+cI_1+mI_2,b+dI_1+nI_2]\times\ldots\times\mathrm{C}^{17}[a+cI_1+mI_2,b+dI_1+nI_2]\to\mathrm{R}(I_1,I_2)$ is a sufficiently smooth function, the function F is said to satisfy the Lipschitz condition if the following Lipschitz inequality holds:

$$\left|F(x+yI_1+zI_2\,,u_0,u_1,\ldots,u_{17})-F\left(x+yI_1+zI_2\,,\ddot{y}_0\,,\ddot{y}_1,\ldots,\ddot{y}_{17}\right)\right|\le L\sum_{i=0}^{17}\left|u_i-\ddot{y}_i\right|,$$

$$\forall(x+yI_1+zI_2,u_0,u_1,\ldots,u_{17})\,,(x+yI_1+zI_2,\ddot{y}_0,\ddot{y}_1,\ldots,\ddot{y}_{17})\in[a+cI_1+mI_2,b+dI]\times R(I)^{17}$$

Where L is called the Lipschitz constant of the function F.

We apply the approximate polynomial (5) and its derivatives with summation points (7)-(6) to the problem of differential equations (18), we obtain the following set of algebraic equations:

$$C_{k,1}+C_{k,1}I_1+C_{k,1}I_2+\left(hz_j\right)(C_{k,2}+C_{k,2}I_1+C_{k,2}I_2)+\frac{\left(hz_j\right)^2}{2!}(C_{k,3}+C_{k,3}I_1+C_{k,3}I_2)+\frac{\left(hz_j\right)^3}{3!}(C_{k,3}+C_{k,3}I_1+C_{k,3}I_2)+\frac{\left(hz_j\right)^4}{4!}(C_{k,5}+C_{k,5}I_1+C_{k,5}I_2)=$$

$$\mathrm{F}\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\,,\mathrm{P}\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\right),\acute{\mathrm{P}}\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\right),\ldots,\mathrm{P}^{(17)}\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\right)\right)\,,\mathrm{j}=1,2,\ldots,5,\ \mathrm{k}=0,1,\ldots,\mathrm{N}-1, \quad (19)$$

$$P^{(i)}(a+cI_1+mI_2)=P_i\,,i=0,1,\ldots,17. \quad (20)$$

We rewrite the sentence of equations (19) in the Matrix formula as follows:

$$A\bar{C}_k=\hat{F}_k\,. \quad (21)$$

Where: $\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2=F\left[\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\,,P\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\right),\acute{P}\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\right),\ldots,P^{(17)}\left(\mathrm{x}_{\mathrm{k+z_j}}+\mathrm{y}_{\mathrm{k+z_j}}I_1+\mathrm{y}_{\mathrm{k+z_j}}I_2\right)\right]$,

J=1,2,…,5

$$\bar{C}_k = \begin{bmatrix} C_{k,1} + C_{k,1}I_1 + C_{k,1}I_2 \\ h(C_{k,2} + C_{k,2}I_1 + C_{k,2}I_2) \\ h^2(C_{k,3} + C_{k,3}I_1 + C_{k,3}I_2) \\ h^3(C_{k,4} + C_{k,4}I_1 + C_{k,4}I_2) \\ h^4(C_{k,5} + C_{k,5}I_1 + C_{k,5}I_2) \end{bmatrix} , \hat{F}_k = \begin{bmatrix} F_{k+z_1}(1 + I_1 + I_2) \\ F_{k+z_2}(1 + I_1 + I_2) \\ F_{k+z_3}(1 + I_1 + I_2) \\ F_{k+z_4}(1 + I_1 + I_2) \\ F_{k+1}(1 + I_1 + I_2) \end{bmatrix} , A$$

$$= \begin{bmatrix} (1 + I_1 + I_2) & z_1(1 + I_1 + I_2) & \frac{z_1^2}{2} + (1 + I_1 + I_2) & \frac{z_1^3}{6} + (1 + I_1 + I_2) & \frac{z_1^4}{24} + (1 + I_1 + I_2) \\ (1 + I_1 + I_2) & z_2(1 + I_1 + I_2) & \frac{z_2^2}{2} + (1 + I_1 + I_2) & \frac{z_2^3}{6} + (1 + I_1 + I_2) & \frac{z_2^4}{24} + (1 + I_1 + I_2 \\ (1 + I_1 + I_2) & z_4(1 + I_1 + I_2) & \frac{z_4^2}{2} + (1 + I_1 + I_2) & \frac{z_4^3}{6} + (1 + I_1 + I_2) & \frac{z_4^4}{24} + (1 + I_1 + I_2) \\ (1 + I_1 + I_2) & (1 + I_1 + I_2) & \frac{1}{2} + (1 + I_1 + I_2) & \frac{1}{6} + (1 + I_1 + I_2) & \frac{1}{24} + (1 + I_1 + I_2) \end{bmatrix}$$

Numerical tests:

We test the technique proposed in this research by applying it to find numerical solutions to some problems

Problem 1: let's take the question of the Linear Differential Equation of the following 14th order [9]:

$$\mathrm{u}^{(18)}(x + yI_1 + zI_2) = \exp(-(x + yI_1 + zI_2))\, u(x + yI_1 + zI_2)\,, \quad 0 \le x + yI_1 + zI_2 \le 1,$$

According to the limiting conditions:

$$u^{(2i)}(0) = 1\,, u^{(2i)}(1) = e\,, i = 0,1,\dots,6$$

In Table (1) we compare the numerical solutions of our proposed method with the cubic slide methods [9] using step h=0.1, and in Table (2) we compare the absolute errors $\left|u\left((x + yI_1 + zI_2)_i\right) - P\left((x + yI_1 + zI_2)_i\right)\right|$ in the numerical solution of our proposed method with the two slide methods in [9].

Table (1): comparisons of numerical solutions of our proposed method with two methods in [9] with a step h=0.1

x_i	exact solution	Cubic Poly.Sol	Our method
0.1	1.10517+1.56I_1+1.88965I_2	------	1.21109807+I_1+I_2

0.2	*1.2214027581*+1.22376I_1+1.34776465I_2	1.2214020778+1.09876I_1+I_2	1.11377894+I_1+I_2
0.3	*1.3498588075*+1.52376I_1+1.8810015I_2	------	1.209114987+I_1+I_2
0.4	*1.491824697*+1.1003256I_1+1.2138965I_2	1.4918236132+1.098387I_1+I_2	*1.333255673*+I_1+I_2
0.5	*1.6487212702*+1.5556834I_1+1.22136I_2	------	1.1100134761128+I_1+I_2
0.6	1.822118800390509+1.9887I_1+1.5464I_2	1.8221176295+1.120187I_1+1.86657I_2	1.78541009432+I_1+I_2
0.7	2.0137527074704766+2.12020301I_1+2.34990I_2	------	2.0112743866+I_1+I_2
0.8	2.225540928492468+2.405986I_1+2.90986I_2	2.2255400739+2.235579140I_1+2.60098165I_2	2.225400982684+I_1+I_2
0.9	2.45960311115695+2.346687I_1+2.103056065I_2	------	2.23114583+I_1+I_2

Table (2): comparisons of absolute errors in the numerical solution of our method with two methods in [9].

x_i	Cubic Poly. Sol.	Cubic Non – Poly.Sol.	Our method
0.1	------	------	4.23189 E-18+I_1+I_2
0.2	3.71E-04+I_1+I_2	6.80E-07+I_1+I_2	2.66785 E-18+I_1+I_2
0.3	------	------	2.22316 E-18+I_1+I_2
0.4	5.92E-04+I_1+I_2	1.08E-06+I_1+I_2	2.5665 E-18+I_1+I_2
0.5	------	------	2.12145 E-18+I_1+I_2
0.6	6.35E-04+I_1+I_2	1.17E-06+I_1+I_2	4.22341 E-20+I_1+I_2
0.7	------	------	6.6678593 E-20+I_1+I_2

0.8	4.59E-04+I_1+I_2	8.54E-07+I_1+I_2	8.1001231 E-20+I_1+I_2
0.9	------	------	8.320193 E-20+I_1+I_2

Problem 2: let's take the question of the nonlinear differential equation of the following 18th order:

$$u^{(18)}(x+yI_1+zI_2)+u(x+yI_1+zI_2)u^{(4)}(x+yI_1+zI_2)-\grave{u}(x+yI_1+zI_2)u^{\grave{}\grave{}\grave{}}(x+yI_1+zI_2)=$$

$$3/2+x^2+14\cos(x+yI_1+zI_2)-3/2\cos(2(x+yI_1+zI_2))-(x+yI_1+zI_2)\sin(x+yI_1+zI_2), 0\le x+yI_1+zI_2\le$$

According to the initial conditions:

$$u(0)=0\,,\grave{u}(0)=0\,,u^{\grave{}}(0)=2\,,u^{\grave{}\grave{}\grave{}}(0)=0\,,u^{(4)}(0)=-4\,,u^{(5)}(0)=0\,,u^{(6)}(0)=6\,,$$

$$u^{(7)}(0)=0\,,u^{(8)}(0)=-8\,,u^{(9)}(0)=0\,,u^{(10)}(0)=10,u^{(11)}(0)=0,u^{(12)}(0)=-12,u^{(13)}(0)=0$$

We put in Table (3) some values of the numerical solution of our proposed method in the domain [0, 5] using the step h=0.1, and also we put in Table (4) the absolute errors in the numerical solution and the derivatives of the solution of our proposed method.

Table (3): the numerical solution of our proposed method with the exact solution in step h=0.1

x_i	exact solution	Numerical solution by our proposed method
0.5	0.23886541015+0.21143I_1+I_2	0.23443259+0.1300159I_1+0.11323259I_2
1.0	0.8433255+0.8411001I_1+0.112098I_2	0.81366948078966+I_1+I_2
1.5	1.4962376+1.556401I_1+1.092213I_2	1.492319060816+I_1+I_2
2.0	1.8185948536513634+1.0977655I_1+1.09808715I_2	1.8185948536513+I_1+I_2
2.5	1.496180360259891+1.03981735I_1+I_2	1.496180360259892+I_1+I_2
3.0	0.4233600241796016+1.0115615I_1+I_2	0.42853651363417962+I_1+I_2
3.5	-1.2277412969136694+*1.6698*I_1+I_2	-1.2277066709134+I_1+I_2
4.0	-3.027209981231713+1.0214215I_1+I_2	-3.0272099812302096+I_1+I_2
4.5	-4.398885294929365+1.06670321+I_2	-4.3363484593+I_1+I_2
5.0	-4.794621373315692+1.44579301I_1+I_2	-4.70667077071+I_1+I_2

Table (4): absolute errors in the numerical solution and derivatives of the solution of our proposed method.

x_i	$\lvert u(x_i) - P(x_i)\rvert$	$\lvert \acute{u}(x_i) - \acute{P}(x_i)\rvert$	$\lvert u^{''}(x_i) - P^{''}(x_i)\rvert$
0.5	8.32667 E-17+I_1+I_2	1.11022 E-18+I_1+I_2	6. 36334 E-119+I_1+I_2
1.0	1.11022 E-16+I_1+I_2	4.44089 E-18+I_1+I_2	6.38378 E-18+I_1+I_2
1.5	2.4329301 E-16+I_1+I_2	3.41120985 E-18+I_1+I_2	1.00855252 E-18+I_1+I_2
2.0	4.44089 E-16+I_1+I_2	5.27356 E-18+I_1+I_2	3.55271 E-17+I_1+I_2
2.5	1.111209302 E-15+I_1+I_2	9.10383 E-17+I_1+I_2	5.06262 E-15+I_1+I_2
3.0	2.29301 E-14+I_1+I_2	1.10578 E-17+I_1+I_2	5.03855221 E-15+I_1+I_2
3.5	2. 930495 E-13+I_1+I_2	8.9112096 E-17+I_1+I_2	3.46878 E-15+I_1+I_2
4.0	1.593024 E-12+I_1+I_2	5.4694 E-15+I_1+I_2	1.852363 E-13+I_1+I_2
4.5	8.3493055 E-12+I_1+I_2	2.61120946 E-14+I_1+I_2	8.1185522 E-13+I_1+I_2
5.0	3.86216 E-11+I_1+I_2	1.11120998 E-13+I_1+I_2	3.038363 E-13+I_1+I_2

Conclusion

In this paper is we studied a new numerical method for finding the refined neutrosophic numerical solutions to some refined neutrosophic differential problems of high orders. We used high-degrees refined neutrosophic polynomials to get the approximated solutions, where we applied our method on two different refined neutrosophic boundary value problems with numerical tables to compare the novel findings with classical ones.

References

1. A B B A S W.M. Fathy , M. Mostafa, Hesham A. M. A. Galerkin Methood for Nonlinear Higher –Order Boundary Value Problems Based on chebyshev Polynomials, Journal of physics : Conference Series , 2128 (2021) 012035.

2. Al-Hayani W. (2015) Adomian Decomposition Method with Green's Function for Solving Twelfth–Order Boundary Value Problems. Applied Mathematical Sciences, Vol. 9, No 8, 353-368.

3. AMINR., Shah , I . Khan , M. Asif , K.M.Abualnaja, E.E. Mahmoud , and A. Abdel–Aty. A powerful numerical technique for treating twelfth-order boundary value problems (2020). Open Physics, 18, 1048-1062.

4. ARIFEEN Sh . U., S . Haq , A. Ghafoor , A . Ullah , P. Kumam and P . Chaipanya (2021) . Numerical Solutions Of Higher Order Boundary Value Problems Via Wavelet Approach , Advances in Difference Equations , 2021:347 ,1-15.

5. FALADE , K . I , A .(2019) Numerical Approach for Solving High –Order Boundary Value Problems , I .J. Mathematical Sciences and Computing , Vol . 3, pp.1-16.

6. MB Zeina, M Abobala, On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry, Neutrosophic Sets and Systems, 2023.

7. IFTIKHAR M .,H.U.REHMAN, and M. YOUNIS , Solution of Thirteenth Order Boundary Value Problems by Differential Transformation Method, Asian Journal of Mathematics and Applications, Vol. (2014) , 11 pages.

8. ISLAM N. (2020) Bezier Polynomials and its Applications with the Tenth and Twelfth Order Boundary Value Problems . Journal of Information and Computing Science. 15(1), pp.052-066.

9. Khalid A.A and M. N.Naeem , (2018) Cubic Spline Solution of linear fourteenth order boundary value problems . Ceylon Journal of Science 47(3), 253-261.

10. Abubaker, Ahmad A, Hatamleh, Raed, Matarneh, Khaled, Al-Husban, Abdallah. (2024). On the Numerica Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, Inernational Journal of Neutrosophic Science,25(2),197-205. (Doi: https://doi.org/10.54216/IJNS.250217).

11. Hatamleh, R. (2025). On The Numerical Solutions of the Neutrosophic One-Dimensional Sine-Gordon System. International Journal of Neutrosophic Science, 25(3), 25-36.DOI: https://doi.org/10.54216/IJNS.250303

12. Kanth. A. S. V. R.; Reddy.Y.N.(2004).Higher order finite difference method for a class of singular boundary value problems. Appl. Math. Comput., 155, 249-258.

13. Abobala, M., Sankari, H., Zeina, M. (2024). On novel security systems based on the 2-cyclic refined integers and the foundations of 2-cyclic refined number theory. Journal of Fuzzy Extension and Applications, *5*(1), 69-85. doi: 10.22105/jfea.2024.423818.1321.

14. Ahmed Hatip, Necati Olgun. (2023). On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic Modules. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 46-52 (Doi: https://doi.org/10.54216/JNFS.070205).

15. Karak, M., Mahata, A., Rong, M., Mukherjee, S., Mondal, S. P., Broumi, S., & Roy, B. (2023). A Solution Technique of Transportation Problem in Neutrosophic Environment. Neutrosophic Systems With Applications, *3*, 17-34. https://doi.org/10.61356/j.nswa.2023.5.

16. Singh, S., & Sharma, S. (2024). Divergence Measures and Aggregation Operators for Single-Valued Neutrosophic Sets with Applications in Decision-Making Problems. Neutrosophic Systems With Applications, 20, 27-44. https://doi.org/10.61356/j.nswa.2024.20281.

17. Sankari, H., and Abobala, M. (2020). Neutrosophic Linear Diophantine Equations With two variables. Neutrosophic Sets and Systems, Vol. 38, pp. 22-30.

18. Hasan Sankari, Mohammad Abobala. (2023). On a Generalization of RSA Crypto-system By Using 2-Cyclic Refined Integers. Journal of Cybersecurity and Information Management, 12(1), 67-73 (Doi: https://doi.org/10.54216/JCIM.120106).

19. Dima Alrwashdeh , Talat Alkhouli , Ahmed Soiman Rashed Alhawiti , Ali Allouf , Hussein Edduweh , Abdallah Al-Husban, On Two Novel Generalized Versions of Diffie-Hellman Key Exchange Algorithm Based on Neutrosophic and Split-Complex Integers and their Complexity Analysis, International Journal of Neutrosophic Science, Vol. 25 Issue. 2 PP. 01-10, (2025), Doi:https://doi.org/10.54216/IJNS.250201.

20. A. S. Heilat, H. Zureigat, R. Hatamleh, B. Batiha, New Spline Method for Solving Linear Two-Point Boundary Value Problems European Journal of Pure and Applied Mathematics 14 (4), 1283-1294, 2021.

21. Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Mowafaq Omar Al-Qadri, Abdallah Al-Husban,(2024), On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2\leq n\leq 3$, Neutrosophic Sets and Systems, Vol. 68,pp.8-25.

22. Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems,vol.67, pp. 169-178.(DOI:10.5281/zenodo.11151930).

23. Heilat, A. S., Karoun, R. C., Al-Husban, A., Abbes, A., Al Horani, M., Grassi, G., & Ouannas, A. (2023). The new fractional discrete neural network model under electromagnetic radiation: Chaos, control and synchronization. Alexandria Engineering Journal, 76, 391-409.

24. Al-Husban, A., Karoun, R. C., Heilat, A. S., Al Horani, M., Khennaoui, A. A., Grassi, G., & Ouannas, A. (2023). Chaos in a two dimensional fractional discrete Hopfield neural network and its control. Alexandria Engineering Journal, 75, 627-638.

25. Heilat, A.S., Batiha,B ,T.Qawasmeh ,Hatamleh R.(2023). Hybrid Cubic B-spline Method for Solving A Class of Singular Boundary Value Problems. European Journal of Pure and Applied Mathematics, 16(2), 751-762. https://doi.org/10.29020/nybg.ejpam.v16i2.4725.

26. Heilat, A.S.,Qazza,A.,Hatamleh,R.Saadeh,R., Alomari,M.W.(2023). An application of Hayashi's inequality in numerical integration.Open Mathematics, 21(1), 20230162. https://doi.org/10.1515/math2023-0162.

27. Heilat, A. S. (2023). Cubic Trigonometric B-spline Method for Solving a Linear System of Second Order Boundary Value Problems. European Journal of Pure and Applied Mathematics, 16(4), 2384-2396. https://doi.org/10.29020/nybg.ejpam.v16i4.4947

28. Heilat, A. S., Hailat, R. S. (2020). Extended cubic B-spline method for solving a system of non-linear second-order boundary value problems. J. Math. Comput. Sci, 21, 231-242. http://dx.doi.org/10.22436/jmcs.021.03.06.

29. Edduweh, Hussein. , Salem, Ahmed. , Razouk, Lama. , A., Sara. , Atallah, Ahmed. , Al-Husban, Abdallah. On The Weak Fuzzy Complex Differential Equations and Some Types of the 1st Order 1st degree WFC-ODEs. International Journal of Neutrosophic Science, 25(3), 2025, pp. 450-468. DOI: https://doi.org/10.54216/IJNS.250338.

30. Hatamleh, R., Zolotarev, V. A. (2016). Triangular Models of Commutative Systems of Linear Operators Close to Unitary Operators. Ukrainian Mathematical Journal, 68(5), 791–811. https://doi.org/10.1007/s11253-016-1258-6.

31. Hatamleh, R. (2003). On the Form of Correlation Function for a Class of Nonstationary Field with a Zero Spectrum. Rocky Mountain Journal of Mathematics, 33(1) 159-173. https://doi.org/10.1216/rmjm/1181069991.

32. Hatamleh, R., Zolotarev,V.A. (2015). On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component. Journal of Mathematical Physics, Analysis, Geometry,11(2), 174–186. https://doi.org/10.15407/mag11.02.174.

33. Hatamleh, R., Zolotarev, V. A. (2014). On Two-Dimensional Model Representations of One Class of Commuting Operators. Ukrainian Mathematical Journal, 66(1), 122–144. https://doi.org/10.1007/s11253-014-0916-9.

34. R.Hatamleh,V.A. Zolotarev.(2017).On the abstract inverse scattering problem for traces class pertubations. Journal of Mathematical Physics, Analysis, Geometry 13(1),1-32. https://doi.org/10.15407/mag13.01.003.

35. Batiha, B., Ghanim, F., Alayed, O., Hatamleh, R., Heilat, A. S., Zureigat, H., & Bazighifan, O. (2022). Solving Multispecies Lotka–Volterra Equations by the Daftardar-Gejji and Jafari Method. International Journal of Mathematics and Mathematical Sciences, 2022, 1–7. (Doi:https://doi.org/10.1155/2022/1839796).

36. Hamadneh, T., Hioual, A., Alsayyed, O., AL-Khassawneh, Y. A., Al-Husban, A., & Ouannas, A. (2023). Local stability, global stability, and simulations in a fractional discrete glycolysis reaction–diffusion model. Fractal and Fractional, 7(8), 587.

37. Hamadneh, T., Alomari, M. W., Al-Shbeil, I., Alaqad, H., Hatamleh, R., Heilat, A. S., & Al-Husban, A. (2023). Refinements of the Euclidean operator radius and Davis–Wielandt radius-type inequalities. Symmetry, 15(5), 1061.

38. T.Qawasmeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh.(2023).Further accurate numerical radius inequalities. Axiom-MDPI, 12 (8), 801.

39. Ayman Hazaymeh, Ahmad Qazza, Raed Hatamleh, Mohammad W Alomari, Rania Saadeh.(2024). On Further Refinements of Numerical Radius Inequalities, Axiom-MDPI, 12(9),807.

40. Hamadneh, T., Alomari, M. W., Al-Shbeil, I., Alaqad, H., Hatamleh, R., Heilat, A. S., & Al-Husban, A. (2023). Refinements of the Euclidean operator radius and Davis–Wielandt radius-type inequalities. Symmetry, 15(5), 1061.

Received: Aug 28, 2024. Accepted: Nov 22, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Lagrange-Integrated GRA Model for Urban Landscape Sculpture Site Selection with Neutrosophic Decision-Making

Qiwei Liu[1], Jiayin Li[2], Chang Liu[1]*

[1]School of Art and Design, Changchun Guanghua University, Changchun City, Jilin Province, China, 130031(E-mail: liuqiwei198104@163.com)

[2]Faculty of Social Sciences and Liberal Arts, UCSI University, 56000, Malaysia (E-mail: lijiayin2287@163.com)

*Corresponding author, E-mail: 15144177977@163.com (https://orcid.org/0009-0003-6086-0643)

Abstract: Urban landscape sculpture plays a crucial role in preserving cultural heritage, enhancing city vitality, and reflecting the lives of urban citizens. With the continued development of landscape sculpture in modern urban environments, the process of site selection has become increasingly important. A well-chosen location can establish a positive interaction between sculptures and urban activities. Traditionally, site selection methods have been largely theoretical, lacking quantitative foundations, which made practical application challenging for designers. To address this, site selection for urban landscape sculptures can be approached as a multiple-attribute decision-making (MADM) problem. Grey Relational Analysis (GRA) is an effective method for handling MADM. Additionally, the probabilistic simplified Neutrosophic set (PSNS) is a useful tool for expressing uncertainty during site selection evaluations. This study proposes an optimization technique using the Lagrange function to determine attribute weights, along with the PSNN-GRA technique for MADM. A numerical example of site selection for urban landscape sculptures is used to validate the PSNN-GRA method through comparative analysis.

Keywords: Multiple-attributes decision-making (MADM); probabilistic simplified neutrosophic sets (PSNSs); GRA technique; site selection evaluation

1. Background

Landscape sculpture can play a certain role in shaping the image of a city. Therefore, at the beginning of design, landscape sculpture should take the city's image as an important reference, and the expression of landscape sculpture should be adapted to urban culture, economy, and environmental factors [1]. Firstly, focus on the personalized display of urban images. In the design stage of landscape sculpture, it is necessary to understand the profound cultural background of the city. Only by mastering the historical and cultural characteristics of the city can landscape sculpture design stand out and showcase different images that are different from other cities. Secondly, showcases urban culture and vitality [2]. The impact of landscape sculpture on the promotion and promotion of urban image is immeasurable. To design landscape sculptures that are more in line with the city's image, it is necessary to add urban cultural elements to the design to present the spiritual connotation and inexhaustible vitality of the city. Thirdly, make reasonable use of the natural environment. Landscape sculpture design needs to consider environmental factors such as urban climate and geographical location, and its shape and color presentation should be adapted to the urban environment [3]. At the same time, landscape sculpture design should consider the surrounding natural environment and not cause damage to the original natural environment. The

application of urban landscape sculpture started relatively late, and there is limited experience in site selection planning for landscape sculpture [4]. Compared with the speed of urban development, the site selection planning for urban landscape sculpture is relatively lagging. In practice, it mainly draws on some foreign experiences in site selection planning for urban landscape sculpture. Due to differences in urban culture, history, development positioning, and functions, it is not advisable to fully draw on the site selection planning experience of others [5]. If planning is not based on the characteristics and needs of the city itself, urban landscape sculpture will find it difficult to play its functions and cannot meet the cultural and spiritual needs of urban landscape architecture. Urban landscape sculpture site selection planning is a process of controlling the relationship between landscape sculpture and urban spatial environment. The quality of site selection planning is related to whether the relationship between landscape sculpture and urban spatial environment is reasonable. The urban spatial environment mainly includes the material environment and the cultural environment. As the carrier space of urban landscape sculpture, the sculpture should be appropriately selected and integrated into the urban spatial environment. As a part of the urban spatial environment, it should coordinate and unify the urban landscape sculpture with the spatial environment [6]. Therefore, before planning the location of urban landscape sculpture, it is necessary to clarify the relationship between the two. The urban spatial environment is the carrying space of landscape sculpture, therefore, urban landscape sculpture acts on the spatial environment [7]. A qualified landscape sculpture site selection plan can create rich urban spatial environmental significance, and fully reflect the spiritual quality and cultural connotations of the urban area. Landscape sculpture is arranged in the form of points, lines, and surfaces in the urban spatial environment, playing a certain role in decorating the urban spatial environment. Therefore, in the overall layout planning process, more attention is paid to urban cultural elements and, the reasonable layout of landscape sculpture, shaping the humanistic culture, historical culture, and spiritual culture of the city, thereby improving the quality of the urban spatial environment[8]. Secondly, through the interpretation of urban landscape sculpture, one can understand urban customs, historical processes, etc., thereby understanding the connotation of urban spatial environment. Therefore, urban landscape sculpture not only originates from the urban spatial environment but also acts on the urban spatial environment [9]. There is a certain constraint relationship between urban spatial environment and landscape sculpture. Firstly, it restricts the site selection and design of urban landscape sculpture. The site selection of urban landscape sculptures should be in line with the functions of urban space. Different functions and forms of urban space result in different locations for landscape sculpture site selection. For example, urban scenic spots are divided into cultural and natural scenic spots, which are further divided into historical and cultural scenic spots, religious and cultural scenic spots, and classical garden scenic spots. The landscape sculptures set up in historical and cultural scenic spots should mainly be sculptures related to the corresponding historical and cultural aspects of the scenic spot [10]. Through the interpretation and appreciation of the sculptures, people can understand the relevant historical and cultural aspects, and thus play the cultural function of the scenic spot. Secondly, the urban spatial environment constrains the color, scale, and theme of landscape sculpture. These elements of landscape sculpture are limited by the characteristics of the urban spatial environment. The color of landscape sculpture should be unified with the urban spatial environment, the scale should be coordinated with the urban spatial environment, and the theme should be consistent with the cultural characteristics of the urban spatial environment.

MADM is a decision-making process that prioritizes alternative solutions in line with existing decision information in a certain way[11, 12]. Its theory and models are widely used in different fields such as venture capital decision-making, project evaluation, and industrial sector development evaluation [13, 14]. In recent decades, MADM has been widely applied to engineering, the economy, technology, and the military. The site selection evaluation for urban landscape sculpture is a MADM. Recently, the GRA model [15-18] was employed and recommended the MADM. The

PSNSs [19] are employed as a technique for characterizing uncertain information during the site selection evaluation for urban landscape sculpture. In this study, the PSNN-GRA model is recommended to solve the MADM with PSNSs. Finally, a numerical study for site selection evaluation for urban landscape sculpture is recommended to validate the PSNN-GRA model.

The study is driven by four key motivations:

(1) Extension of the GRA model to PSNSs: Traditional GRA models are limited in handling uncertainty. By extending GRA to PSNSs, the study aims to improve the ability to manage uncertain and imprecise information during the decision-making process. This enhancement provides a more accurate and comprehensive evaluation of complex scenarios.

(2) Optimization of attribute weights: A new optimization technique based on the Lagrange function is introduced to calculate attribute weights objectively. This method ensures that the weights assigned to different criteria in the decision-making process are more balanced and reflective of the actual importance of each attribute.

(3) Development of the PSNN-GRA model: The PSNN-GRA model is proposed as a solution to MADM problems using PSNSs. This model offers a robust and flexible framework for decision-making, particularly in situations characterized by high levels of uncertainty, making it a valuable tool for complex evaluations.

(4)Validation through a numerical study: To verify the effectiveness of the PSNN-GRA model, a numerical example related to site selection for urban landscape sculptures is conducted. Comparative analysis is used to demonstrate the performance of the approach, highlighting its practical applications in real-world decision-making scenarios.

The overall structure is as follows: The PSNSs are discussed in Sect. 2. The PSNN-GRA model is introduced as the recommended MADM method in Sect. 3. A case study for urban landscape sculpture site selection is presented in Sect. 4, demonstrating the PSN-GRA technique along with a sensitivity analysis. Sect. 5 concludes the paper.

2. Preliminary Remarks

Wang et al. [20] recommended the SVNSs

Definition 1 [20]. The SVNSs are recommended:

$$V=\left\{\left(\theta,T\left(\theta\right),I\left(\theta\right),F\left(\theta\right)\right)\middle|\theta\in\Theta\right\} \tag{1}$$

where $T\left(\theta\right),I\left(\theta\right),F\left(\theta\right)$ is membership, indeterminacy-membership and falsity-membership, $T\left(\theta\right),I\left(\theta\right),F\left(\theta\right)\in\left[0,1\right],\ 0\le T\left(\theta\right)+I\left(\theta\right)+F\left(\theta\right)\le 3.$

Altun, Sahin and Guler [19] recommended the PSNSs.

Definition 2[19]. The PSNSs is recommended:

$$PV=\left\{\left(\theta,T\left(\theta\right)\left(PT\left(\theta\right)\right),I\left(\theta\right)\left(PI\left(\theta\right)\right),F\left(\theta\right)\left(PF\left(\theta\right)\right)\right)\middle|\theta\in\Theta\right\} \tag{2}$$

where $T\left(\theta\right),I\left(\theta\right),F\left(\theta\right)$ is truth-membership, indeterminacy-membership and falsity-membership, $T\left(\theta\right),I\left(\theta\right),F\left(\theta\right)\in\left[0,1\right]$, $0\le T\left(\theta\right)+I\left(\theta\right)+F\left(\theta\right)\le 3$, $0\le PT\left(\theta\right),PI\left(\theta\right),PF\left(\theta\right)\le 1$, the $PT\left(\theta\right),PI\left(\theta\right),PF\left(\theta\right)$ is possibility values of $T\left(\theta\right),I\left(\theta\right),F\left(\theta\right)$. The probabilistic simplified neutrosophic number (PSNN) is listed as $PV=\left(T\left(PT\right),I\left(PI\right),VF\left(PF\right)\right)$.

Definition 3[19]. Let $PV_1=\left(T_1\left(PT_1\right),I_1\left(PI_1\right),F_1\left(PF_1\right)\right)$, $PV_2=\left(T_2\left(PT_2\right),I_2\left(PI_2\right),F_2\left(PF_2\right)\right)$, the basic operations are recommended:

$$(1)\ PV_1\oplus PV_2=\begin{pmatrix} T_1+T_2-T_1\cdot T_2\left(2!\dfrac{PT_1\cdot PT_2}{PT_1+PT_2}\right), \\ I_1\cdot I_2\left(2!\dfrac{PI_1\cdot PI_2}{PI_1+PI_2}\right),F_1\cdot F_2\left(2!\dfrac{PF_1\cdot PF_2}{PF_1+PF_2}\right)\end{pmatrix};$$

$$(2)\ PV_1\otimes PV_2=\begin{pmatrix} T_1\cdot T_2\left(2!\dfrac{PT_1\cdot PT_2}{PT_1+PT_2}\right), \\ I_1+I_2-I_1\cdot I_2\left(2!\dfrac{PI_1\cdot PI_2}{PI_1+PI_2}\right), \\ F_1+F_2-F_1\cdot F_2\left(2!\dfrac{PF_1\cdot PF_2}{PF_1+PF_2}\right)\end{pmatrix};$$

$$(3)\ \lambda PV=\left(1-\left(1-T\right)^{\lambda}\left(PT\right),\left(I\right)^{\lambda}\left(PI\right),\left(F\right)^{\lambda}\left(PF\right)\right),\lambda>0;$$

$$(4)\ \left(PV\right)^{\lambda}=\left(\left(T\right)^{\lambda}\left(PT\right),1-\left(1-I\right)^{\lambda}\left(PI\right),1-\left(1-F\right)^{\lambda}\left(PF\right)\right),\lambda>0.$$

Definition 4[19]. Let $PV_1=\left(T_1\left(PT_1\right),I_1\left(PI_1\right),F_1\left(PF_1\right)\right)$ and $PV_2=\left(T_2\left(PT_2\right),I_2\left(PI_2\right),F_2\left(PF_2\right)\right)$, the probabilistic simplified Neutrosophic number Logarithmic distance (PSNNLD) between $PV_1=\left(T_1\left(PT_1\right),I_1\left(PI_1\right),F_1\left(PF_1\right)\right)$ and $PV_2=\left(T_2\left(PT_2\right),I_2\left(PI_2\right),F_2\left(PF_2\right)\right)$ is recommended:

$$PSNNLD\left(PV_1,PV_2\right)$$

$$=\frac{1}{3}\begin{pmatrix} \left(T_1\times PT_1\right)\log\dfrac{\left(T_1\times PT_1\right)}{\dfrac{\left(T_1\times PT_1\right)+\left(T_2\times PT_2\right)}{2}}+\left(T_2\times PT_2\right)\log\dfrac{\left(T_2\times PT_2\right)}{\dfrac{\left(T_1\times PT_1\right)+\left(T_2\times PT_2\right)}{2}} \\ +\left(I_1\times PI_1\right)\log\dfrac{\left(I_1\times PI_1\right)}{\dfrac{\left(I_1\times VI_1\right)+\left(I_2\times PI_2\right)}{2}}+\left(I_2\times PI_2\right)\log\dfrac{\left(I_2\times PI_2\right)}{\dfrac{\left(I_1\times PI_1\right)+\left(I_2\times PI_2\right)}{2}} \\ +\left(F_1\times PF_1\right)\log\dfrac{\left(F_1\times PF_1\right)}{\dfrac{\left(F_1\times PF_1\right)+\left(F_2\times PF_2\right)}{2}}+\left(F_2\times PF_2\right)\log\dfrac{2\left(F_2\times PF_2\right)}{\dfrac{\left(F_1\times PF_1\right)+\left(F_2\times PF_2\right)}{2}}\end{pmatrix} \tag{3}$$

3. PSN-GRA method for MADM problem with Lagrange function

The PSNN-GRA is recommended for MADM under PSNSs in light of the extended GRA method[21, 22] with a completely unknown weight in line with the Lagrange function. Let be alternatives, and $VG=\left\{VG_1,VG_2,\cdots,VG_n\right\}$ be attributes with weight vw, where $vw_j\in\left[0,1\right]$,

$\sum_{j=1}^{n} vw_j = 1$. Suppose that the decision information is depicted with PSNNs $VR = \left(VR_{ij}\right)_{m\times n} = \left(T_{ij}\left(PT_{ij}\right), I_{ij}\left(PI_{ij}\right), F_{ij}\left(PF_{ij}\right)\right)_{m\times n}$. Then, the PSNN-GRA technique is recommended MADM with Lagrange function (See Figure 1).

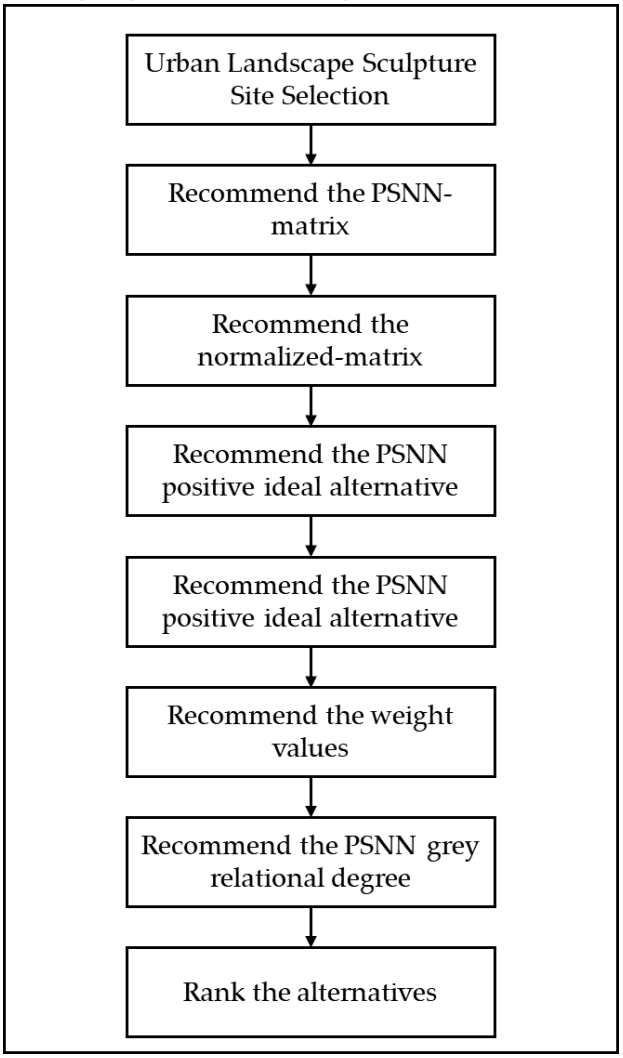

Urban Landscape Sculpture Site Selection
Recommend the PSNN-matrix
Recommend the normalized-matrix
Recommend the PSNN positive ideal alternative
Recommend the PSNN positive ideal alternative
Recommend the weight values
Recommend the PSNN grey relational degree
Rank the alternatives

Figure 1. PSNN-GRA framework for MADM with Lagrange function

Step 1. Recommend the PSNN-matrix $VR = \left(VR_{ij}\right)_{m\times n} = \left(T_{ij}\left(PT_{ij}\right), I_{ij}\left(PI_{ij}\right), F_{ij}\left(PF_{ij}\right)\right)_{m\times n}$.

Step 1. Recommend the normalized matrix $NVR = \left[NVR_{ij}\right]_{m\times n}$ based on $VR = \left(VR_{ij}\right)_{m\times n}$.

$$NVR_{ij} = \left(T_{ij}^{N}\left(PT_{ij}^{N}\right), I_{ij}^{N}\left(PI_{ij}^{N}\right), F_{ij}^{N}\left(PF_{ij}^{N}\right)\right) = \begin{cases} \left(T_{ij}\left(PT_{ij}\right), I_{ij}\left(PI_{ij}\right), F_{ij}\left(PF_{ij}\right)\right), & VG_j \text{ is the benefit attributes} \\ \left(F_{ij}\left(PF_{ij}\right), 1-I_{ij}\left(1-PI_{ij}\right), T_{ij}\left(PT_{ij}\right)\right), & VG_j \text{ is the cost attributes} \end{cases} \tag{4}$$

Step 2. Recommend the PSNN positive ideal alternative (PSNNPIA) [23]:

$$PSNNPIA = \left\{PSNNPIA_j\right\},\ j = 1, 2, \ldots, n. \tag{5}$$

$$PSNNPIA_j = \left(T_j^N\left(PT_j^N\right), I_j^N\left(PI_j^N\right), F_j^N\left(PF_j^N\right)\right), j = 1,2,\ldots,n. \tag{6}$$

$$\begin{aligned} &SV\left(T_j^N\left(PT_j^N\right), I_j^N\left(PI_j^N\right), F_j^N\left(PF_j^N\right)\right) \\ &= \max_i SV\left(T_{ij}^N\left(PT_{ij}^N\right), I_{ij}^N\left(PI_{ij}^N\right), F_{ij}^N\left(PF_{ij}^N\right)\right) \end{aligned} \tag{7}$$

where

$$\begin{aligned} &SV\left(T_j^N\left(PT_j^N\right), I_j^N\left(PI_j^N\right), F_j^N\left(PF_j^N\right)\right) \\ &= \frac{T_j^N \cdot PT_j^N + I_j^N \cdot PI_j^N + F_j^N \cdot PF_j^N}{3} \end{aligned} \tag{8}$$

$$\begin{aligned} &SV\left(T_{ij}^N\left(PT_{ij}^N\right), I_{ij}^N\left(PI_{ij}^N\right), F_{ij}^N\left(PF_{ij}^N\right)\right) \\ &= \frac{T_{ij}^N \cdot PT_{ij}^N + I_{ij}^N \cdot PI_{ij}^N + F_{ij}^N \cdot PF_{ij}^N}{3} \end{aligned} \tag{9}$$

Step 3. Recommend the PSNN grey relational coefficients (PSNNGRC) from PSNNPIA:

$$GRC\left(\xi_{ij}^+\right) = \frac{\left(\min\limits_{1\le i\le m} PSNNLD\left(NVR_{ij}, PSNNPIA_j\right) + \rho \max\limits_{1\le i\le m} PSNNLD\left(NVR_{ij}, PSNNPIA_j\right)\right)}{\left(PSNNLD\left(NVR_{ij}, PSNNPIA_j\right) + \rho \max\limits_{1\le i\le m} PSNNLD\left(NVR_{ij}, PSNNPIA_j\right)\right)} \tag{10}$$

In generally, $\rho \in [0,1]$, usually $\rho = 0.5$.

Step 4. Recommend the weight values.

If the attribute weights are completely unknown, firstly, the weight should be obtained. Thus, the PSNN deviation sum (PSNNDS) in light with PSNNGRC is recommended:

$$PSNNDS_i\left(vw\right) = \sum_{j=1}^{n}\left[\left(1 - GRC\left(\xi_{ij}^+\right)\right)vw_j\right]^2 \tag{11}$$

The multiple-objective optimization technique (MOOT-1) is recommended for the weight values:

$$\left(\text{MOOT-1}\right)\begin{cases} \min PSNNDS_i\left(vw\right) = \sum\limits_{j=1}^{n}\left[\left(1 - GRC\left(\xi_{ij}^+\right)\right)vw_j\right]^2, \\ \sum\limits_{j=1}^{n} vw_j = 1 \end{cases} \tag{12}$$

Because every alternative is the same, there is no preference for these alternatives. Then, the MOOT-1 is recommended the same weights in the single-objective optimization technique (SOOT-1):

$$\left(\text{SOOT-1}\right)\begin{cases} \min PSNNDS\left(vw\right) = \sum\limits_{i=1}^{m} PSNNDS_i\left(vw\right) \\ = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\left[\left(1 - GRC\left(\xi_{ij}^+\right)\right)vw_j\right]^2 \\ s.t. \quad \sum\limits_{j=1}^{n} vw_j = 1 \end{cases} \tag{13}$$

To cope up with the SOOT-1 technique, the Lagrange function is recommended:

$$LF\left(vw,\lambda\right)=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(1-GRC\left(\xi_{ij}^{+}\right)\right)vw_{j}\right]^{2}+2\lambda\left(\sum_{j=1}^{n}vw_{j}-1\right)\right) \tag{14}$$

where λ is the Lagrange multiplier.

Differentiating Eq. (12) forward $vw_{j}\left(j=1,2,\cdots,n\right)$ and λ, and supposing that these partial derivatives are zero, the weight values are obtained below:

$$vw_{j}=\frac{\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(1-GRC\left(\xi_{ij}^{+}\right)\right)^{2}\right)^{-1}\right]^{-1}}{\sum_{i=1}^{m}\left(1-GRC\left(\xi_{ij}^{+}\right)\right)^{2}} \tag{15}$$

Step 5. Recommend the grey relational degree (GRD) from PSNNPIA:

$$PSNNGRD\left(\xi_{i}^{+}\right)=\sum_{j=1}^{n}vw_{j}GRC\left(\xi_{ij}^{+}\right) \tag{16}$$

Step 6. In light with $GRD\left(\xi_{i}^{+}\right)$. The larger $GRD\left(\xi_{i}^{+}\right)$, the better the alternative is.

4. Case Study and Sensitivity Analysis of Urban Sculpture Site Selection Using Neutrosophic Decision-Making

In this section, we conduct a case study and sensitivity analysis to explore the application of Neutrosophic decision-making for urban sculpture site selection. We examine how different decision-making methods and criteria weights influence the ranking of potential sites, providing insights into the robustness of the proposed approach.

4.1. Case study

With the continuous development of China's economy and the acceleration of urbanization, people's production and living standards have been improving year by year, and their pursuit of quality of life has also become increasingly high. As for urban landscape sculpture, it has already become a part of people's daily lives. On the one hand, urban landscape sculptures meet people's emotional needs for the city they rely on for their livelihood; On the other hand, urban landscape sculptures also carry the cultural history of a city. However, at present, there are some problems in the construction of urban landscape sculptures. Some sculptures cannot truly reflect the historical and cultural characteristics of the city, and there is a sense of inconsistency with the local cultural customs. In addition, a single design cannot meet the diverse development of the current era, and it lacks artistic and aesthetic sense. These problems need to be solved. Landscape sculpture is a unique way of presenting urban culture. When appreciating landscape sculpture, people associate it with urban culture and can understand and appreciate it through landscape sculpture. Landscape sculpture design usually presents urban culture in a strong artistic form. Therefore, landscape sculpture design helps to promote the enhancement of urban cultural image. On the one hand, every city has a long history, and a unique urban culture has been formed in the continuous development of cities. Urban culture is an important reference for landscape sculpture design, conveying specific cultural connotations. Therefore, landscape sculpture design has a certain cultural color and has become an important form of showcasing urban culture. On the other hand, landscape sculpture serves as a business card to showcase the cultural image of a city, which also includes the historical connotation

of the city. Through landscape sculpture, people can understand the historical customs and changes of the city. Landscape sculpture, like other art or cultural industries, is a product of urban development to a certain stage. The level and quality of landscape sculpture are symbols of a city's comprehensive strength in economy, culture, and other aspects. Only when the urban economy is mature and stable can public cultural construction be gradually promoted to create a good urban cultural atmosphere. The role of landscape sculpture in promoting urban economic image is mainly reflected in its ability to enhance urban attractiveness. Firstly, landscape sculpture can showcase a well-developed urban image, thereby attracting more investment and talent, promoting local economic development, and providing many driving forces for urban economic development. Secondly, landscape sculpture can effectively promote the development of the urban tourism industry. Landscape sculpture itself is a tourism resource with direct economic benefits. Beautifully designed and meaningful landscape sculptures can bring people a profound artistic and cultural impression. Therefore, the artistic and cultural value of landscape sculpture can promote the development of the tourism industry and inject new vitality into urban economic development. At the same time, while promoting the development of the tourism industry, it can also generate income for tourism-related industries such as catering and transportation, drive economic construction in multiple fields of the city, and accelerate the pace of urban modernization. The site selection evaluation for urban landscape sculpture is MADM. In this work, the PSNN-GRA is recommended for site selection evaluation for urban landscape sculpture. Six urban landscape sculpture sites are assessed with 18 attributes as shown in Table 1.

Table 1. Criteria weights.

Criteria		**Weights**
C_1	Harmony with Surroundings	0.058909
C_2	Public Transportation Connectivity	0.057197
C_3	Operational Costs	0.051321
C_4	Community Engagement	0.055844
C_5	Safety Considerations	0.054315
C_6	Historical Significance	0.055856
C_7	Visibility and Prominence	0.060893
C_8	Environmental Compatibility	0.055626
C_9	Cost of Installation	0.051988
C_{10}	Event Hosting Capability	0.055785
C_{11}	Green Space Preservation	0.054823
C_{12}	Structural Stability	0.054775
C_{13}	Pedestrian Accessibility	0.057391
C_{14}	Weather Resistance	0.055549
C_{15}	Maintenance Requirements	0.053175
C_{16}	Potential Economic Benefits	0.05079
C_{17}	Scenic Value	0.058761
C_{18}	Parking Availability	0.057002

We begin by constructing the decision matrix that links the criteria and alternatives, as shown in Table 2. Next, we normalize the decision matrix, which is presented in Table 3. After that, we identify the positive ideal alternatives. Then, we calculate the criteria weights, as shown in Table 1. Following this, we compute grey relational scores. Finally, we rank the alternatives, as illustrated in Figure 2.

Table 2. The decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_2	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_3	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_4	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_5	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
C_6	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}
C_7	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_8	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_9	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}

C_{10}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_{11}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
C_{12}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}
C_{13}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_{14}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_{15}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_{16}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_{17}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_{18}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_2	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_3	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_4	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_5	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_6	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}
C_7	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_8	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_9	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_{10}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_{11}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
C_{12}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}
C_{13}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_{14}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_{15}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_{16}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_{17}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_{18}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
	A_1	A_2	A_3	A_4	A_5	A_6
C_1	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_2	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_3	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_4	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_5	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
C_6	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_7	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_8	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_9	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_{10}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_{11}	{0.26(0.5),0.39(0.2),0.54(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}
C_{12}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_{13}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.95(0.3),0.86(0.3),0.19(0.4)}
C_{14}	{0.95(0.3),0.86(0.3),0.19(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_{15}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}
C_{16}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.26(0.5),0.39(0.2),0.54(0.3)}
C_{17}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}
C_{18}	{0.43(0.3),0.81(0.4),0.32(0.3)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.34(0.4),0.53(0.1),0.84(0.5)}	{0.13(0.2),0.74(0.1),0.48(0.7)}	{0.86(0.4),0.16(0.2),0.65(0.4)}	{0.43(0.3),0.81(0.4),0.32(0.3)}

Table 3. The normalized decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	-0.17896	-1	-0.34699	-0.4153	0	-0.06967
C_2	-0.09574	-1	-0.875	-0.92819	0	-0.9734
C_3	-0.30769	-0.88718	0	-1	-0.88718	-0.8
C_4	-0.03306	0	-0.92231	-1	-0.41157	-0.31901
C_5	-1	0	-0.4782	-0.32771	-0.4782	-0.14627
C_6	-1	-0.14925	-0.50959	-0.75906	0	-0.14712
C_7	-0.04775	-0.94382	-1	0	-0.4382	-0.01966
C_8	-0.06557	-0.8051	-0.59927	-0.66667	0	-1
C_9	-0.02023	-0.67341	0	-0.69075	-1	-0.8526
C_{10}	-0.10256	-0.27863	-0.40855	-1	-0.8	0
C_{11}	-0.82017	-0.47395	-1	0	-0.30756	-0.31933
C_{12}	-1	-0.14925	-0.89979	-0.75906	0	-0.14712
C_{13}	-1	-0.16393	-0.35155	-0.4408	-0.6357	0
C_{14}	0	-1	-0.30986	-0.45599	-0.64437	-0.64437
C_{15}	0	-0.58333	-0.60109	-0.81967	-0.33333	-1
C_{16}	0	-0.68223	-0.51881	-0.36706	-0.9144	-1
C_{17}	-0.46753	-0.12987	-1	-0.60519	-0.42338	0
C_{18}	-0.19262	-1	-0.58333	-0.31967	0	-0.35656

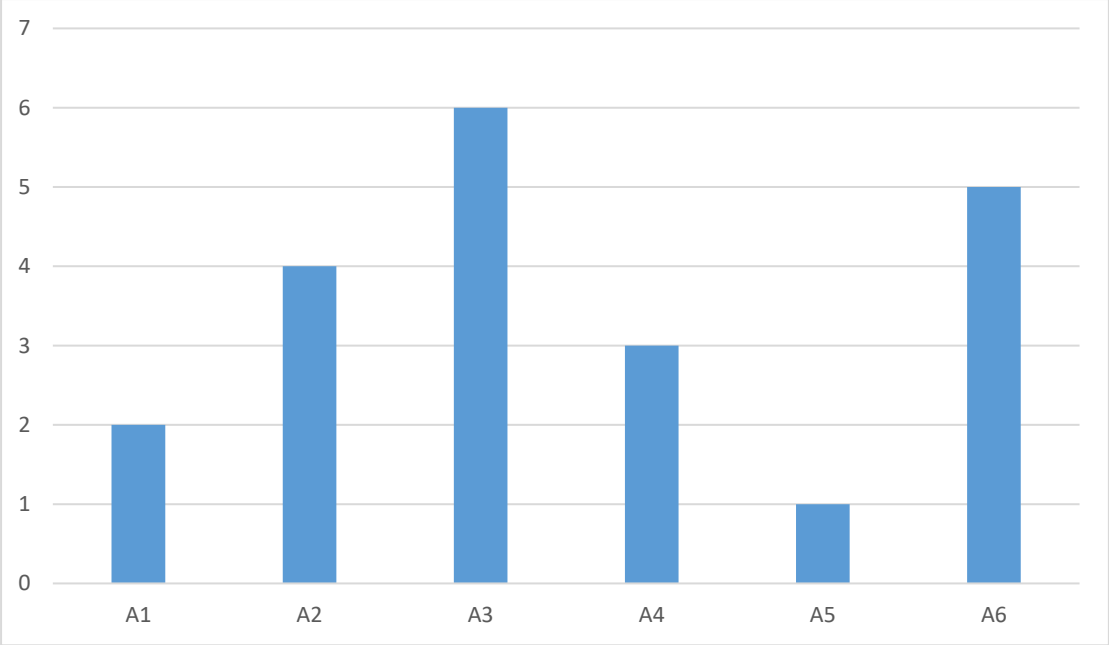


Figure 2. The rank of alternatives.

Table 4. The different criteria weights.

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}	Y_{16}	Y_{17}	Y_{18}	Y_{19}
C_1	0.0555556	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_2	0.0555556	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_3	0.0555556	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_4	0.0555556	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_5	0.0555556	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_6	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_7	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_8	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_9	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_{10}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_{11}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_{12}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118
C_{13}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118	0.054118
C_{14}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118	0.054118
C_{15}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118	0.054118
C_{16}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118	0.054118
C_{17}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08	0.054118
C_{18}	0.0555556	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.054118	0.08

4.2. Sensitivity analysis

In real-life decision-making, different decision-makers often have different levels of risk tolerance, which can impact how the identification coefficient is chosen. This, in turn, may change the ranking of the alternatives. To address this, we perform a sensitivity analysis in this section. We adjust the criteria weights across 19 different cases, as shown in Table 4, and then apply the proposed method with these different weights. Finally, we rank the alternatives, as illustrated in Figure 3.

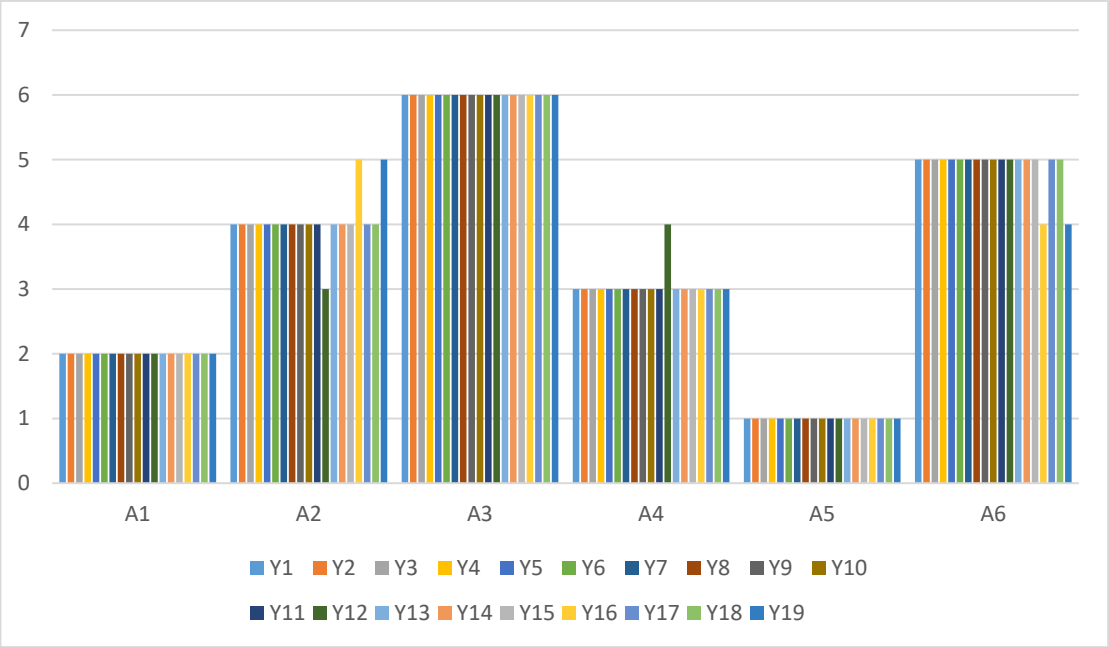


Figure 3. The rank of alternatives under different weights.

5. Conclusions and future research directions

In recent years, economic development has driven continuous progress in politics, culture, technology, and education, and the pace of construction of urbanization is also getting faster and faster. Urban landscape sculpture is an important component of urban construction, and the aesthetics contained in urban landscape sculpture symbolize the essence of urban culture on the one hand; On the other hand, it is also a representative of the city's landmark. At present, the integration of planning urban landscape sculptures and urban environment has become a hot topic of concern in urban construction. The site selection evaluation for urban landscape sculpture is viewed as MADM. In this study, the PSNN-GRA model is recommended to solve the MADM under PSNSs. Finally, a numerical study for site selection evaluation for urban landscape sculpture is recommended to validate the PSNN-GRA technique. The main research motivations of this study are recommended: (1) the GRA model are extended to PSNSs; (2) an optimization technique is recommended to obtain the attribute weights based on Lagrange function; (3) the PSNN-GRA model is recommended to solve the MADM with PSNSs; (4) Finally, a numerical study for site selection evaluation for urban landscape sculpture is recommended to validate the PSNN-GRA technique with comparative analysis.

Although this study provides an effective solution for urban landscape sculpture site selection through the PSNN-GRA method, there are still some research limitations and shortcomings. These limitations mainly lie in the complexity of the model, insufficient adaptability to dynamic environments, and the lack of consideration for social and public participation. To enhance the comprehensiveness of the research and its practical applicability, future studies can explore and improve in the following three directions. Thus, the research limitations are obtained: (1) Balance between model complexity and practical applicability: Although the PSNN-GRA method effectively deals with uncertainty in multi-attribute decision-making, its construction and computational process are relatively complex. This complexity, especially in large-scale decision problems, may pose operational challenges for designers and decision-makers. The intricate computational steps and high technical threshold may limit its widespread application in practical projects. (2) Lack of adaptability to dynamic environments: The proposed method in this study is

primarily based on a static decision-making environment. However, in reality, urban environments and social activities are often in a state of dynamic change. The PSNN-GRA model does not account for the influence of time, making it less adaptable to long-term or continuously evolving decision-making scenarios, where flexibility is needed to respond to dynamic environmental changes. (3) Insufficient consideration of social and public participation: Although the study mentions that the site selection of sculptures should promote interaction with urban activities, the specific model lacks sufficient consideration of public participation and social feedback. Public opinions and preferences play a crucial role in actual site selection decisions, and neglecting these factors may lead to outcomes that are inconsistent with real needs, affecting the social acceptance and effectiveness of the sculpture.

Although the PSNN-GRA method proposed in this study offers significant advantages in urban landscape sculpture site selection, there is still room for improvement. Future research can focus on optimizing the model's complexity, adaptability to dynamic environments, and public participation to further enhance its practical applicability and decision-making effectiveness. These improvements aim to provide better decision support in more complex and dynamic real-world scenarios. Thus, future research directions are outlined:

(1) Simplifying model complexity and improving practical applicability

Given the complexity of the current model, future research can focus on simplifying the computational process of the PSNN-GRA method and developing more intuitive and user-friendly tools or algorithms to improve its practicality in real-world applications. By optimizing and simplifying the model, it can be applied in a wider range of scenarios, lowering the barrier to use and making it more accessible for non-expert users.

(2) Introducing dynamic decision-making models

Considering the dynamic nature of urban environments, future research can explore incorporating time factors into the decision-making model and develop methods that adapt to dynamic changes. For example, dynamic grey relational analysis or temporal neutrosophic sets could be integrated into the PSNN-GRA framework to handle the evolving urban environment and social needs, enhancing the model's applicability in long-term decision-making.

(3) Enhancing public participation and social feedback mechanisms

To better reflect the needs and preferences of the public, future research can integrate public participation mechanisms and incorporate public feedback into the decision-making process. By leveraging social feedback data, such as surveys or social media data, more inclusive decision-making models can be developed. This would improve the social acceptance of site selection decisions and ensure that urban landscape sculptures better serve the public, enhancing their overall effectiveness.

Acknowledgment

The work was supported by the 2023 Social Science Research Planning Project of Education Department of Jilin Province: Research on Contemporary Design Transformation of Traditional Rural Residential Buildings in Jilin Province (JJKH20231473SK).

Reference

[1] S.Y. Huai, F. Chen, S. Liu, F. Canters, T. van de Voorde, Using social media photos and computer vision to assess cultural ecosystem services and landscape features in urban parks, Ecosystem Services, 57 (2022) 14.

[2] C.Y. Li, Landscape design of the urban bus station based on the urban characteristics and eco-logical environment of smart city, china, Fresenius Environmental Bulletin, 29 (2020) 3300-3307.

[3] Y. Wang, Y. Chen, Analysis of the function of urban sculpture in landscape environment, in: 3rd

International Conference on Civil Engineering, Architecture and Building Materials (CEABM 2013), Trans Tech Publications Ltd, Jinan, PEOPLES R CHINA, 2013, pp. 2036-2039.

[4] J.E. Zheng, Producing chinese urban landscapes of public art: The urban sculpture scene in shanghai, China Quarterly, 239 (2019) 775-803.

[5] N.D. Senoge, C.T. Downs, The use of cellular telecommunication towers as nesting sites by pied crows in an urban mosaic landscape, Urban Ecosystems, 26 (2023) 881-892.

[6] Q. Mao, J.X. Sun, Y.Y. Deng, Z.L. Wu, H.T. Bai, Assessing effects of multi-scale landscape pattern and habitats attributes on taxonomic and functional diversity of urban river birds, Diversity-Basel, 15 (2023) 18.

[7] K.K. Josiah, C.T. Downs, Facets of the nesting ecology of a ground-nesting bird, the spotted thick-knee, in an urban mosaic landscape, Global Ecology and Conservation, 44 (2023) 15.

[8] M. Ciach, P. Tetkowski, I. Fedyn, Local-scale habitat configuration makes a niche for wildlife encroaching into an urban landscape: Grubbing sites of wild boar in a city matrix, Urban Ecosystems, 26 (2023) 629-639.

[9] C. Chahboun, S. Hanane, B. Chahboun, A. Qninba, Modulating the effect of landscape composition on urban pylon use by white storks: The role of clustering strength in nesting supports, Urban Ecosystems, (2023) 11.

[10] D. Mikolajewski, V.D. Iii, N.F. Sonti, C.C. Pinchot, C.E. Flower, L.A. Roman, T.L.E. Trammell, Restoring the iconic ulmus americana to urban landscapes: Early tree growth responds to aboveground conditions, Urban Forestry & Urban Greening, 74 (2022) 14.

[11] D. Pamucar, A.E. Torkayesh, S. Biswas, Supplier selection in healthcare supply chain management during the covid-19 pandemic: A novel fuzzy rough decision-making approach, Annals of Operations Research, 328 (2023) 977-1019.

[12] H. Garg, K. Ullah, K. Ali, M. Akram, M.N. Abid, Multi-attribute decision-making based on sine trigonometric aggregation operators for t-spherical fuzzy information, Soft Computing, (2023) 15.

[13] J. Ye, B.Z. Sun, X.L. Chu, J.M. Zhan, J.X. Cai, Valued outranking relation-based heterogeneous multi-decision multigranulation probabilistic rough set and its use in medical decision-making, Expert Systems with Applications, 228 (2023) 18.

[14] M. Akram, C. Kahraman, K. Zahid, Extension of topsis model to the decision-making under complex spherical fuzzy information, Soft Computing, 25 (2021) 10771-10795.

[15] J. Deng, Introduction to grey system theory, The Journal of Grey System, 1 (1989) 1-24.

[16] G. Wei, New method of grey relational analysis to multiple attribute decision making with intervals, Systems Engineering and Electronics, 28 (2006) 1358-1359+1383.

[17] H.Y. Zhang, G.W. Wei, X.D. Chen, Sf-gra method based on cumulative prospect theory for multiple attribute group decision making and its application to emergency supplies supplier selection, Engineering Applications of Artificial Intelligence, 110 (2022) 13.

[18] S.J. Fu, H.J. Fu, Modeling and topsis-gra algorithm for autonomous driving decision-making under 5g-v2x infrastructure, Cmc-Computers Materials & Continua, 75 (2023) 1051-1071.

[19] F. Altun, R. Sahin, C. Guler, Multi-criteria decision making approach based on promethee with probabilistic simplified neutrosophic sets, Soft Computing, 24 (2020) 4899-4915.

[20] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

[21] G.W. Wei, Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making, Expert Systems with Applications, 38 (2011) 11671-11677.

[22] F. Lei, J. Lu, G. Wei, J. Wu, C. Wei, Y. Guo, Gra method for waste incineration plants location problem with probabilistic linguistic multiple attribute group decision making, Journal of Intelligent & Fuzzy Systems, 39 (2020) 2909-2920.

[23] C. Liu, Gra method for probabilistic simplified neutrosophic madm and application to talent training quality evaluation of segmented education, Journal of Intelligent & Fuzzy Systems, 44 (2023) 8637-8647.

Received: Aug 25 2024. Accepted: Nov 24, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

Enhanced MADM Strategy with Heptapartitioned Neutrosophic Distance Metrics

Mohamed eassa[1,2], M. Myvizhi[3], Mohamed Elkholy[4], Ahmed Abdelhafeez[1,2], Hussam Elbehiery[1]

[1] Computer Science Department, Faculty of Information System and Computer Science, October 6 University, Giza, 12585, Egypt; mohamed.eassa.cs@o6u.edu.eg; aahafeez.scis@o6u.edu.eg; H.elbehiry.csis@o6u.edu.eg

[2] Applied Science Research Center. Applied Science Private University, Amman, Jordan

[3]Assistant Professor, Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore, Tamilnadu, India; myvizhi.m@kpriet.ac.in

[4]Faculty of Computer Science& Engineering, Alamein International University; melkholy@aiu.edu.eg

Abstract: In this work, a novel methodological approach to multi-attribute decision-making problems is developed and the notion of Heptapartitioned Neutrosophic Set Distance Measures (HNSDM) is introduced. By averaging the Penta-Partitioned Neutrosophic Distance Measures under particular conditional criteria, the suggested technique ranks the options. Additionally, we explore the fundamental characteristics of HNSDM and provide several illustrative theorems. The Heptapartitioned Neutrosophic Set Distance Measures are formulated based on the Hepta Neutrosophic Set. Lastly, a practical numerical example is provided to illustrate the efficacy of the suggested methodology.

Keywords: Heptapartitioned set, Heptapartitioned Neutrosophic set, Neutrosophic sine distance measure, Heptapartitioned Neutrosophic set and MADM strategy.

1. Introduction

Neutrosophic sets have emerged as a powerful tool for handling uncertainty, indeterminacy, and inconsistency in various fields of research [5]. In recent years, classical set theory has been insufficient for modeling complex uncertain systems [5]. Florentin Smarandache introduced the concept of Neutrosophic set theory, or Neutral knowledge, in 1998 [1, 4], provides a robust framework for handling indeterminacy and inconsistency [6]. Recent studies have explored neutrosophic extensions of distance measures, including sine distance measure (SDM) and Hepta Hypothesis Distance Measures (HHDM) [2]. Mostly we choose the leaders by voting i.e.., by election. Every election gives voters the choice to support candidate A, support opponent B, or abstain from voting altogether. The fraction of voters who choose not to cast ballots may determine the election's outcome by selecting options A through B; this is known as "neutral or indeterminist" [3].

The distance measure, which is used to examine the relationship between two distinct neutrosophic sets, is computed [3]. This research investigates the intersection of these concepts, proposing neutrosophic sine distance measures [NSDM] and Neutrosophic Hepta Hypothesis Distance Measure, Sine Distance Measures [SDM] quantify the similarity or dissimilarity between sets or vectors using the sine function [9], [10], [11], [12], [13], [14].

One of the most crucial clustering methods for figuring out how closely two things are related is the distance measure [11]. Medical diagnosis decision-making is developing into a broad area of study. Distance metrics have played an important role in decision science throughout the past few decades. Over the past few decades, numerous researchers have created a variety of distance metrics that are utilized in medical diagnostics [12].

There has been a lot of interest in hesitant fuzzy linguistic term sets (HELTSs) because of their special capacity and efficacy to convey ambiguity and uncertainty in the decision-making process. We can research and create various distance and similarity metrics for HFLTSs to improve their applicability [14]. A distance measure between intuitionistic fuzzy sets [IFSs] is defined axiomatically. We can calculate IFS's distance and similarity [15J]. Order is simply defined by signed-distance ranking, which allows us to describe ordering using both positive and negative values [21].

2. Preliminaries

Definition 1 [22]

Let X be a non-empty set. A Heptapartitioned neutrosophic set A having the form $A = \{(x, T_A(x), M_A(x), C_A(x), I_A(x), U_A(x), F_A(x), K_A(x): x \in X\}$, where $T_A(x), M_A(x), C_A(x), I_A(x), U_A(x), F_A(x), K_A(x) \in [0,1]$represent an absolute truth-membership (namely $T_A(x)$), a relative truth-membership (namely $M_A(x)$), a contradiction membership (namely $C_A(x)$), a ignorance-membership (namely $I_A(x)$), an unknown membership (namely $U_A(x)$), an absolute falsity-membership (namely $F_A(x)$) and a relative falsity membership (namely $K_A(x)$) respectively of each $x \epsilon X$ to the set A such that $0 \leq T_A(x) + M_A(x) + C_A(x) + I_A(x) + U_A(x) + F_A(x) + K_A(x) \leq 7$ for all $x \in X$.

The collection of all hepta-partitioned neutrosophic sets of X is represented as HNS(X).

Definition 2 [22]

Let $X = \{x_1, x_2, \ldots . x_n\}$ be a distinct, limited set. If the following axioms are met, a mapping d: $NS(X) \times NS(X) \to [0,1]$ is considered a distance measure between two neutrosophic sets.

$(i) d(A,B) \geq 0$ for all $A, B \in NS(X)$.

$(ii) d(A,B) = 0$ is and only if $A = B$ for all $A, B \in NS(X)$.

$(iii) d(A,B) = d(B,A)$ for all $A, B \in NS(X)$.

(iv) If $A \subseteq B \subseteq C$ for all $A, B, C \in NS(X)$, then $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$.

Definition 3 [22]

The normalized hamming distance of two single-valued neutrosophic sets, A and B, is described as

$$d_1(A,B) = \frac{1}{3n}\sum_{j=1}^{n}\left(\left|\mu_A(x_j) - \mu_B(x_j)\right| + \left|\sigma_A(x_j) - \sigma_B(x_j)\right| + \left|\gamma_A(x_j) - \gamma_B(x_j)\right|\right).$$

Definition 4 [22]

Two single-valued neutrosophic sets A and B are separated by the normalized Euclidean distance, which is defined as

$$d_2(A,B) = \left\{ \frac{1}{3n} \sum_{j=1}^{n} (\mu_A(x_j) - \mu_B(x_j))^2 + (\sigma_A(x_j) - \sigma_B(x_j))^2 + (\gamma_A(x_j) - \gamma_B(x_j))^2 \right\}^{\frac{1}{2}}.$$

Definition 5 [23]

Two single-valued neutrosophic sets, A and B, have distance measures that are defined as

$$D_1(A,B) = \frac{1}{3n} \sum_{j=1}^{n} \left(\left| \mu_A(x_j)^2 - \mu_B(x_j)^2 \right| + \left| \sigma_A(x_j)^2 - \sigma_B(x_j)^2 \right| + \left| \gamma_A(x_j)^2 - \gamma_B(x_j)^2 \right| \right)$$

and

$$D_2(A,B) = \frac{1}{3n} \sum_{j=1}^{n} \left| \left(\mu_A(x_j)^2 - \mu_B(x_j)^2 \right) - \left(\sigma_A(x_j)^2 - \sigma_B(x_j)^2 \right) - \left(\gamma_A(x_j)^2 - \gamma_B(x_j)^2 \right) \right|$$

Definition 6 [18]

A mapping $S: NS(X) \times NS(X) \to [0,1]$ is considered a measure of similarity between two Neutrosophic sets if it satisfies the subsequent requirements:

$(i) S(A,B) \geq 0$ for all $A, B \in NS(X)$.

$(ii) S(A,B) = 1$ if and only if $A = B$ for all $A, B \in NS(X)$.

$(iii) S(A,B) = S(B,A)$ for all $A, B \in NS(X)$.

(iv) If $A \subseteq B \subseteq C$ for all $A, B, C \in NS(X)$, then $S(A,C) \leq S(A,B)$ and $S(A,C) \leq S(B,C)$.

3. Heptapartitioned Neutrosophic Distance Measures

Definition 7

Let $X = \{x_1, x_2 \dots \dots . x_n\}$ be a distinct, limited set. A mapping $d: HNS(X) * HNS(X) \to [0,1]$ is said if it meets the following criteria, it can be considered a Heptapartitioned neutrosophic distance measure between two Hepta partitioned neutrosophic sets:

1. $d(A,B) \geq 0$ for all $A, B \in HNS(X)$.

2. $d(A,B) = 0$ if and only if $A = B$ for all $A, B \in HNS(X)$.

3. $d(A,B) = d(B,A)$ for all $A, B \in HNS(X)$.

4. If $A \subseteq B \subseteq C$ for all $A, B, C \in HNS(X)$, then $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$.

Definition 8

Let $X = \{x_1, x_2 \dots \dots . x_n\}$ be a discrete confined set. Let $A = \{x_j, T_A(x_j), M_A(x_j), C_A(x_j), I_A(x_j), U_A(x_j), F_A(x_j), K_A(x_j): x_j \in X\}$ and $B = \{x_j, T_B(x_j), M_B(x_j), C_B(x_j), I_B(x_j), U_B(x_j), F_B(x_j), K_B(x_j): x_j \in X\}$ be two Heptapartitioned neutrosophic sets. Then for $i = 1,2,3,4,5,6,7$, define a mapping $d_i: HNS(X) * HNS(X) \to [0,1]$ as

$$1.\ d_1(A,B) = \frac{1}{7n}\sum_{j=1}^{n}\big(\big|T_A(x_j) - T_B(x_j)\big| + \big|M_A(x_j) - M_B(x_j)\big| + \big|C_A(x_j) - C_B(x_j)\big| + \big|I_A(x_j) - I_B(x_j)\big| + \big|U_A(x_j) - U_B(x_j)\big| + \big|F_A(x_j) - F_B(x_j)\big| + \big|K_A(x_j) - K_B(x_j)\big|\big).$$

$$2.\ d_2(A,B) = \left\{\frac{1}{7n}\sum_{j=1}^{n}\left(\left(T_A(x_j) - T_B(x_j)\right)^2 + \left(M_A(x_j) - M_B(x_j)\right)^2 + \left(C_A(x_j) - C_B(x_j)\right)^2 + \left(I_A(x_j) - I_B(x_j)\right)^2 + \left(U_A(x_j) - U_B(x_j)\right)^2 + \left(F_A(x_j) - F_B(x_j)\right)^2 + \left(K_A(x_j) - K_B(x_j)\right)^2\right)\right\}^{1/2}.$$

$$3.\ d_3(A,B) = \frac{1}{7n}\sum_{j=1}^{n}\big(\big|(T_A(x_j))^2 - (T_B(x_j))^2\big| + \big|(M_A(x_j))^2 - (M_B(x_j))^2\big| + \big|(C_A(x_j))^2 - (C_B(x_j))^2\big| + \big|(I_A(x_j))^2 - (I_B(x_j))^2\big| + \big|(U_A(x_j))^2 - (U_B(x_j))^2\big| + \big|(F_A(x_j))^2 - (F_B(x_j))^2\big| + \big|(K_A(x_j))^2 - (K_B(x_j))^2\big|\big).$$

$$4.\ d_4(A,B) = \frac{1}{7n}\sum_{j=1}^{n}\Big(\big|(T_A(x_j))^2 - (T_B(x_j))^2\big| + \big|(M_A(x_j))^2 - (M_B(x_j))^2\big| + \big|(C_A(x_j))^2 - (C_B(x_j))^2\big| - \Big|\left(I_A(x_j)\right)^2 - \left(I_B(x_j)\right)^2\Big| - \Big|\left(U_A(x_j)\right)^2 - \left(U_B(x_j)\right)^2\Big| - \Big|\left(F_A(x_j)\right)^2 - \left(F_B(x_j)\right)^2\Big| - \big|(K_A(x_j))^2 - (K_B(x_j))^2\big|\Big).$$

5. $d_5(A,B)$

$$= \frac{7}{5n}\sum_{j=1}^{n} \frac{\begin{pmatrix} \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+ \\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\} \end{pmatrix}}{1+\begin{pmatrix} \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+ \\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\} \end{pmatrix}}$$

$$6.\ d_6(A,B) = \frac{7}{5n}\sum_{J=1}^{n} \frac{\begin{pmatrix} sin\left\{\frac{\pi}{6}\left(T_A(x_j)-T_B(x_j)\right)^2\right\}+sin\left\{\frac{\pi}{6}\left(M_A(x_j)-M_B(x_j)\right)^2\right\} \\ +sin\left\{\frac{\pi}{6}\left(C_A(x_j)-C_B(x_j)\right)^2\right\}+sin\left\{\frac{\pi}{6}\left(I_A(x_j)-I_B(x_j)\right)^2\right\} \\ +sin\left\{\frac{\pi}{6}\left(U_A(x_j)-U_B(x_j)\right)^2\right\}+sin\left\{\frac{\pi}{6}\left(F_A(x_j)-F_B(x_j)\right)^2\right\} \\ +sin\left\{\frac{\pi}{6}\left(K_A(x_j)-K_B(x_j)\right)^2\right\} \end{pmatrix}}{1+\begin{pmatrix} sin\left\{\frac{\pi}{6}\left(T_A(x_j)-T_B(x_j)\right)^2\right\}+sin\left\{\frac{\pi}{6}\left(M_A(x_j)-M_B(x_j)\right)^2\right\} \\ +sin\left\{\frac{\pi}{6}\left(C_A(x_j)-C_B(x_j)\right)^2\right\}+sin\left\{\frac{\pi}{6}\left(I_A(x_j)-I_B(x_j)\right)^2\right\} \\ +sin\left\{\frac{\pi}{6}\left(U_A(x_j)-U_B(x_j)\right)^2\right\}+sin\left\{\frac{\pi}{6}\left(F_A(x_j)-F_B(x_j)\right)^2\right\} \\ +sin\left\{\frac{\pi}{6}\left(K_A(x_j)-K_B(x_j)\right)^2\right\} \end{pmatrix}}$$

7. $d_7(A,B)$

$$= \frac{9}{7n}\sum_{j=1}^{n} \frac{\begin{pmatrix} \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+ \\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\} \end{pmatrix}}{1+\begin{pmatrix} \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\} \\ +\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+ \\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\} \end{pmatrix}}$$

Theorem 1: A Hepta-partitioned neutrosophic measure of distance between two Hepta-partitioned Neutrosophic sets B and A is given for each i=1,2,3,4,5,d_i (A,B).

Proof: Only d_4 (A,B) is shown to be a Heptapartitioned neutrosophic distance measure here; the other d_i (A,B) have comparable arguments for i=1,2,5.

1. $d_4(A,B) \geq 0$ is trivially true from the definition of Heptapartitioned neutrosophic sets.

2. $d_4(A,B) = 0$ if and only if $\frac{1}{7n}\sum_{j=1}^{n}\Big(\left|(T_A(x_j))^2 - (T_B(x_j))^2\right| + \left|(M_A(x_j))^2 - (M_B(x_j))^2\right| + \left|(C_A(x_j))^2 - (C_B(x_j))^2\right| - \left|\left(I_A(x_j)\right)^2 - \left(I_B(x_j)\right)^2\right| - \left|\left(U_A(x_j)\right)^2 - \left(U_B(x_j)\right)^2\right| - \left|\left(F_A(x_j)\right)^2 - \left(F_B(x_j)\right)^2\right| - \left|(K_A(x_j))^2 - (K_B(x_j))^2\right|\Big) = 0,$

if and only if, for all $x_j \in X, \Big(\left|\left(T_A(x_j)\right)^2 - \left(T_B(x_j)\right)^2\right| + \left|\left(M_A(x_j)\right)^2 - \left(M_B(x_j)\right)^2\right| + \left|\left(C_A(x_j)\right)^2 - \left(C_B(x_j)\right)^2\right| - \left|\left(I_A(x_j)\right)^2 - \left(I_B(x_j)\right)^2\right| - \left|\left(U_A(x_j)\right)^2 - \left(U_B(x_j)\right)^2\right| - \left|\left(F_A(x_j)\right)^2 - \left(F_B(x_j)\right)^2\right| - \left|\left(K_A(x_j)\right)^2 - \left(K_B(x_j)\right)^2\right|\Big) = 0.$

if, for all $x_j \in X$, $\left(T_A(x_j)\right)^2 - \left(T_B(x_j)\right)^2 = 0$; $\left(M_A(x_j)\right)^2 - \left(M_B(x_j)\right)^2 = 0$;

$\left(C_A(x_j)\right)^2 - \left(C_B(x_j)\right)^2 = 0$; $\left(I_A(x_j)\right)^2 - \left(I_B(x_j)\right)^2 = 0$; $\left(U_A(x_j)\right)^2 - \left(U_B(x_j)\right)^2 = 0$; $\left(F_A(x_j)\right)^2 - \left(F_B(x_j)\right)^2 = 0$; $\left(K_A(x_j)\right)^2 - \left(K_B(x_j)\right)^2 = 0.$

if and only if, $T_A(x_j) = T_B(x_j)$; $M_A(x_j) = M_B(x_j)$; $C_A(x_j) = C_B(x_j)$; $I_A(x_j) = I_B(x_j)$; $U_A(x_j) = U_B(x_j)$; $F_A(x_j) = F_B(x_j)$; $K_A(x_j) = K_B(x_j)$, for all $x_j \in X$, iff $A = B$.

$$3.\ d_4(A,B) = \frac{1}{7n}\sum_{j=1}^{n}\Big(\Big|\Big(\left(T_A(x_j)\right)^2 - \left(T_B(x_j)\right)^2\Big) + \Big(\left(M_A(x_j)\right)^2 - \left(M_B(x_j)\right)^2\Big) + \Big(\left(C_A(x_j)\right)^2 - \left(C_B(x_j)\right)^2\Big) - \Big(\left(I_A(x_j)\right)^2 - \left(I_B(x_j)\right)^2\Big) - \Big(\left(U_A(x_j)\right)^2 - \left(U_B(x_j)\right)^2\Big) - \Big(\left(F_A(x_j)\right)^2 - \left(F_B(x_j)\right)^2\Big) - \Big(\left(K_A(x_j)\right)^2 - \left(K_B(x_j)\right)^2\Big)\Big|\Big)$$

$$= \frac{1}{7n}\sum_{j=1}^{n}\Big(\Big|\Big(\left(T_B(x_j)\right)^2 - \left(T_A(x_j)\right)^2\Big) + \Big(\left(M_B(x_j)\right)^2 - \left(M_A(x_j)\right)^2\Big) + \Big(\left(C_B(x_j)\right)^2 - \left(C_A(x_j)\right)^2\Big) - \Big(\left(I_B(x_j)\right)^2 - \left(I_A(x_j)\right)^2\Big) - \Big(\left(U_B(x_j)\right)^2 - \left(U_A(x_j)\right)^2\Big) - \Big(\left(F_B(x_j)\right)^2 - \left(F_A(x_j)\right)^2\Big) - \Big(\left(K_B(x_j)\right)^2 - \left(K_A(x_j)\right)^2\Big)\Big|\Big)$$

$$= d_4(B, A).$$

4. If $A \subseteq B \subseteq C$, then $T_A(x_j) \leq T_B(x_j) \leq T_C(x_j)$; $M_A(x_j) \leq M_B(x_j) \leq M_C(x_j)$; $C_A(x_j) \leq C_B(x_j) \leq C_C(x_j)$; $I_A(x_j) \geq I_B(x_j) \geq I_C(x_j)$; $U_A(x_j) \geq U_B(x_j) \geq U_C(x_j)$; $F_A(x_j) \geq F_B(x_j) \geq F_C(x_j)$; $K_A(x_j) \geq K_B(x_j) \geq K_C(x_j)$, for all $x_j \in X$.

Then, we have the following inequalities:

$$\left(T_A(x_j)\right)^2 - \left(T_C(x_j)\right)^2 \leq \left(T_A(x_j)\right)^2 - \left(T_B(x_j)\right)^2, \left(T_A(x_j)\right)^2 - \left(T_C(x_j)\right)^2 \leq \left(T_B(x_j)\right)^2 - \left(T_C(x_j)\right)^2$$

$$\left(M_A(x_j)\right)^2 - \left(M_C(x_j)\right)^2 \leq \left(M_A(x_j)\right)^2 - \left(M_B(x_j)\right)^2, \left(M_A(x_j)\right)^2 - \left(M_C(x_j)\right)^2 \leq \left(M_B(x_j)\right)^2 - \left(M_C(x_j)\right)^2$$

$$\left(C_A(x_j)\right)^2 - \left(C_C(x_j)\right)^2 \leq \left(C_A(x_j)\right)^2 - \left(C_B(x_j)\right)^2, \left(C_A(x_j)\right)^2 - \left(C_C(x_j)\right)^2 \leq \left(C_B(x_j)\right)^2 - \left(C_C(x_j)\right)^2$$

$$\left(I_A(x_j)\right)^2 - \left(I_C(x_j)\right)^2 \geq \left(I_A(x_j)\right)^2 - \left(I_B(x_j)\right)^2, \left(I_A(x_j)\right)^2 - \left(I_C(x_j)\right)^2 \geq \left(I_B(x_j)\right)^2 - \left(I_C(x_j)\right)^2$$

$$\left(U_A(x_j)\right)^2 - \left(U_C(x_j)\right)^2 \geq \left(U_A(x_j)\right)^2 - \left(U_B(x_j)\right)^2, \left(U_A(x_j)\right)^2 - \left(U_C(x_j)\right)^2 \geq \left(U_B(x_j)\right)^2 - \left(U_C(x_j)\right)^2$$

$$\left(F_A(x_j)\right)^2 - \left(F_C(x_j)\right)^2 \geq \left(F_A(x_j)\right)^2 - \left(F_B(x_j)\right)^2, \left(F_A(x_j)\right)^2 - \left(F_C(x_j)\right)^2 \geq \left(F_B(x_j)\right)^2 - \left(F_C(x_j)\right)^2$$

$$\left(K_A(x_j)\right)^2 - \left(K_C(x_j)\right)^2 \geq \left(K_A(x_j)\right)^2 - \left(K_B(x_j)\right)^2, \left(K_A(x_j)\right)^2 - \left(K_C(x_j)\right)^2 \geq \left(K_B(x_j)\right)^2 - \left(K_C(x_j)\right)^2$$

From these inequalities we have,

$$\frac{1}{7n}\sum_{j=1}^{n}\left(\left|\left(\left(T_A(x_j)\right)^2-\left(T_C(x_j)\right)^2\right)+\left(\left(M_A(x_j)\right)^2-\left(M_C(x_j)\right)^2\right)\right.$$
$$+\left(\left(C_A(x_j)\right)^2-\left(C_C(x_j)\right)^2\right)-\left(\left(I_A(x_j)\right)^2-\left(I_C(x_j)\right)^2\right)$$
$$-\left(\left(U_A(x_j)\right)^2-\left(U_C(x_j)\right)^2\right)-\left(\left(F_A(x_j)\right)^2-\left(F_C(x_j)\right)^2\right)$$
$$\left.-\left(\left(K_A(x_j)\right)^2-\left(K_C(x_j)\right)^2\right)\right|\Big)$$

$$\geq \frac{1}{7n}\sum_{j=1}^{n}\left(\left|\left(\left(T_B(x_j)\right)^2-\left(T_C(x_j)\right)^2\right)+\left(\left(M_B(x_j)\right)^2-\left(M_C(x_j)\right)^2\right)\right.$$
$$+\left(\left(C_B(x_j)\right)^2-\left(C_C(x_j)\right)^2\right)-\left(\left(I_B(x_j)\right)^2-\left(I_C(x_j)\right)^2\right)$$
$$-\left(\left(U_B(x_j)\right)^2-\left(U_C(x_j)\right)^2\right)-\left(\left(F_B(x_j)\right)^2-\left(F_C(x_j)\right)^2\right)$$
$$\left.-\left(\left(K_B(x_j)\right)^2-\left(K_C(x_j)\right)^2\right)\right|\Big)$$

and

$$\frac{1}{7n}\sum_{j=1}^{n}\left(\left|\left(\left(T_A(x_j)\right)^2-\left(T_C(x_j)\right)^2\right)+\left(\left(M_A(x_j)\right)^2-\left(M_C(x_j)\right)^2\right)\right.$$
$$+\left(\left(C_A(x_j)\right)^2-\left(C_C(x_j)\right)^2\right)-\left(\left(I_A(x_j)\right)^2-\left(I_C(x_j)\right)^2\right)$$
$$-\left(\left(U_A(x_j)\right)^2-\left(U_C(x_j)\right)^2\right)-\left(\left(F_A(x_j)\right)^2-\left(F_C(x_j)\right)^2\right)$$
$$\left.-\left(\left(K_A(x_j)\right)^2-\left(K_C(x_j)\right)^2\right)\right|\Big)$$

$$\geq \frac{1}{7n}\sum_{j=1}^{n}\left(\left|\left(\left(T_A(x_j)\right)^2-\left(T_B(x_j)\right)^2\right)+\left(\left(M_A(x_j)\right)^2-\left(M_B(x_j)\right)^2\right)\right.$$
$$+\left(\left(C_A(x_j)\right)^2-\left(C_B(x_j)\right)^2\right)-\left(\left(I_A(x_j)\right)^2-\left(I_B(x_j)\right)^2\right)$$
$$-\left(\left(U_A(x_j)\right)^2-\left(U_B(x_j)\right)^2\right)-\left(\left(F_A(x_j)\right)^2-\left(F_B(x_j)\right)^2\right)$$
$$\left.-\left(\left(K_A(x_j)\right)^2-\left(K_B(x_j)\right)^2\right)\right|\Big)$$

Therefore, $d_4(A,C)\geq d_4(B,C)$ and $d_4(A,C)\geq d_4(A,B)$.

Hence $d_4(A,B)$ is a Heptapartitioned neutrosophic distance measure.

Theorem 2: For $i=1,2,3,4,5,6,7, d_i(A,C)\leq d_i(A,B)+d_i(B,C)$ is true for $A,B,C\in HNS(X)$.

Proof: Here, we only demonstrate the triangle inequality for i=7. In a similar manner, we may demonstrate it for i=1, 2, and 3.

Let $A, B, C \in HNS(X)$ then for the real numbers, the following inequalities hold true.

$$\left|T_A(x_j) - T_C(x_j)\right| \leq \left|T_A(x_j) - T_B(x_j)\right| + \left|T_B(x_j) - T_C(x_j)\right|, \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j) - T_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j) - T_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j) - T_C(x_j)\right|\right)\right\}, \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j) - M_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j) - M_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j) - M_C(x_j)\right|\right)\right\}, \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j) - C_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j) - C_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j) - C_C(x_j)\right|\right)\right\}, \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j) - I_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j) - I_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j) - I_C(x_j)\right|\right)\right\}, \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j) - U_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j) - U_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j) - U_C(x_j)\right|\right)\right\}, \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j) - F_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j) - F_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j) - F_C(x_j)\right|\right)\right\}, \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j) - K_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j) - K_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j) - K_C(x_j)\right|\right)\right\}$$

Then,

$$\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j) - T_C(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j) - M_C(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j) - C_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j) - I_C(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j) - U_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j) - F_C(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j) - K_C(x_j)\right|\right)\right\}$$

$$\leq \sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j) - T_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j) - T_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j) - M_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j) - M_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j) - C_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j) - C_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j) - I_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j) - I_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j) - U_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j) - U_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j) - F_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j) - F_C(x_j)\right|\right)\right\}$$
$$+ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j) - K_B(x_j)\right|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j) - K_C(x_j)\right|\right)\right\}$$

So,

$$\frac{1}{\begin{array}{r}1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_C(x_j)\right|\right)\right\}+\end{array}} \geq \frac{1}{\begin{array}{r}1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}\end{array}}$$

Then,

$$1-\frac{1}{1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_C(x_j)\right|\right)\right\}+}\leq 1-\frac{1}{1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}}$$

Hence,

$$\frac{\begin{array}{c}\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_C(x_j)\right|\right)\right\}\end{array}}{\begin{array}{l}1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_C(x_j)\right|\right)\right\}\end{array}}$$

$$\leq \frac{\begin{array}{c}\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}\end{array}}{\begin{array}{l}1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}\end{array}} \leq$$

$$\frac{\begin{matrix}\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}\end{matrix}}{\begin{matrix}1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}++\\ \sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}\end{matrix}}+\frac{\begin{matrix}\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}\end{matrix}}{\begin{matrix}1+\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\\ \sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}\end{matrix}}\Rightarrow$$

$$\frac{7}{5n}\sum_{j=1}^{n}\frac{\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_C(x_j)\right|\right)\right\}}{1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_C(x_j)\right|\right)\right\}}\leq\frac{7}{5n}\sum_{j=1}^{n}\frac{\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}}{1+\sin\left\{\frac{\pi}{6}\left(\left|T_A(x_j)-T_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_A(x_j)-M_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_A(x_j)-C_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_A(x_j)-I_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_A(x_j)-U_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_A(x_j)-F_B(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_A(x_j)-K_B(x_j)\right|\right)\right\}}+$$

$$\frac{7}{5n}\sum_{j=1}^{n}\frac{\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}}{1+\sin\left\{\frac{\pi}{6}\left(\left|T_B(x_j)-T_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|M_B(x_j)-M_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|C_B(x_j)-C_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|I_B(x_j)-I_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|U_B(x_j)-U_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|F_B(x_j)-F_C(x_j)\right|\right)\right\}+\sin\left\{\frac{\pi}{6}\left(\left|K_B(x_j)-K_C(x_j)\right|\right)\right\}}$$

Therefore, $d_5(A,C) \leq d_5(A,B) + d_5(B,C)$ is true for $A,B,C \in HNS(X)$.

4. Methodology:

In this section, a sine metric heptapartitioned neutrosophic distance measure is used to present a systematic approach to solving neutrosophic multiple attributes decision making issues. The steps required to determine the pertinent traits and options are described in the approach below.

Step 1: Selection of the problem field

Table 1 Attributes vs Alternatives

	A_1	A_2	A_m
R_1	(r_{11})	(r_{12})	(r_{1m})
R_2	(r_{21})	(r_{22})	(r_{2m})
R_p	(r_{p1})	(r_{p2})	(r_{pm})

Table 2 Alternatives vs Decision Attributes

	R_1	R_2	R_P
B_1	(a_{11})	(a_{12})	(a_{1p})
B_2	(a_{21})	(a_{22})	(a_{2p})
B_n	(a_{n1})	(a_{n2})	(a_{np})

Table 3 Distance Measure Table

B_kA_j	B_1	B_2	B_n
A_1	$d(A_1,B_1)$	$d(A_1,B_2)$	$d(A_1,B_n)$
A_2	$d(A_2,B_1)$	$d(A_2,B_2)$	$d(A_2,B_n)$
A_m	$d(A_m,B_1)$	$d(A_m,B_2)$	(A_m,B_n)

Select the attribute from the table of distance measures. The optimal attribute for alternative A_j is $B_k, k = 1,2,\ldots\ldots n$ for the substitutes $A_j, j = 1,2,\ldots.m$, which is determined by the options' lowest distance measure value A_j and the attributes B_k,Nd.

In order to demonstrate a practical implementation of our Heptapartitioned Neutrosophic Distance Measure within the suggested approach, we present a numerical illustration of hospital selection for specifically diseases in this section.

4.1 Numerical illustration: Utilization of the sine metric to the hexapartitioned measure of neutrosophic distance

Hospitals play a vital role in managing and treating a wide range of diseases, catering to the diverse needs of patients. General hospitals provide comprehensive care for common illnesses and emergencies, offering essential diagnostic and treatment services. Specialized hospitals, on the other hand, focus on specific medical fields such as cardiology, oncology, orthopaedics, and neurology,

making them ideal for treating complex and chronic conditions. The selection of a hospital often depends on factors like its reputation, the expertise of its medical staff, available technology, cost, and proximity. Diseases, whether acute or chronic, require different levels of care acute conditions like appendicitis necessitate immediate intervention, while chronic illnesses like diabetes benefit from long-term management and specialized care. Advances in healthcare, such as telemedicine and multidisciplinary approaches, are further enhancing the treatment landscape, making it easier to address a variety of medical needs effectively.

Step 1 Problem field selection.

Let $A = \{A_1, A_2, A_3\}$Be the set of treatments and $B = \{B_1, B_2, B_3\}$Be the set of a selection of hospitals and $R = \{R_1, R_2, R_3, R_4, R_5\}$ be the set of specified diseases. Table 4 shows the specified set of diseases and Table 5 shows the selection of hospitals and specified diseases.

Step 2 Proximity Analysis for and Attributes Alternatives

The measures of distance between each disease concerning and each hospital

$$d_7(B_k A_j) = \frac{9}{7n}\sum_{j=1}^{n} \frac{\begin{pmatrix} \sin\left\{\frac{\pi}{6}\left(|T_A(x_j) - T_B(x_j)|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(|M_A(x_j) - M_B(x_j)|\right)\right\} \\ + \sin\left\{\frac{\pi}{6}\left(|C_A(x_j) - C_B(x_j)|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(|I_A(x_j) - I_B(x_j)|\right)\right\} \\ + \sin\left\{\frac{\pi}{6}\left(|U_A(x_j) - U_B(x_j)|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(|F_A(x_j) - F_B(x_j)|\right)\right\} + \\ \sin\left\{\frac{\pi}{6}\left(|K_A(x_j) - K_B(x_j)|\right)\right\} \end{pmatrix}}{1 + \begin{pmatrix} \sin\left\{\frac{\pi}{6}\left(|T_A(x_j) - T_B(x_j)|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(|M_A(x_j) - M_B(x_j)|\right)\right\} \\ + \sin\left\{\frac{\pi}{6}\left(|C_A(x_j) - C_B(x_j)|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(|I_A(x_j) - I_B(x_j)|\right)\right\} \\ + \sin\left\{\frac{\pi}{6}\left(|U_A(x_j) - U_B(x_j)|\right)\right\} + \sin\left\{\frac{\pi}{6}\left(|F_A(x_j) - F_B(x_j)|\right)\right\} + \\ \sin\left\{\frac{\pi}{6}\left(|K_A(x_j) - K_B(x_j)|\right)\right\} \end{pmatrix}}$$

Table 4 Treatments Vs Specified Diseases

	R_1	R_2	R_3	R_4	R_5
A_1	(0.5,0.5,0.9,0.8, 0.4,0.1,0)	(0.4,0.6,0.2,0.1, 0.5,0.2,0.3)	(0.8,0.2,0.6,0.4, 0.2,0,0.1)	(0.3,0.2,0.1,0.5, 0.3,0.2,0.1)	(0.3,0.3,0.3,0.2, 0.4,0.1,0.1)
A_2	(0.8,0.1,0.2,0.2, 0.4,0.3,0.4)	(0.3,0.6,0.7,0.4, 0.6,0.4,0.3)	(0.5,0.5,0.5,0.5, 0.5,0.5,0.5)	(0.2,0.2,0.4,0.2, 0.1,0.3,0.1)	(0.7,0.8,1,0.4, 0.5,0.2,0.2)
A_3	(0.2,0,0.1,0.2, 0.4,0.4,0.3)	(0.8,0.4,0.1,0, 0.1,0.1,0.2)	(0.3,0.6,0.7,0.1, 0.2,0.1,0.3)	(0.9,0.1,0.4,0.6, 0.7,0.8,0.7)	(0.3,0.2,0.3,0.7, 0.4,0.2,0.1)

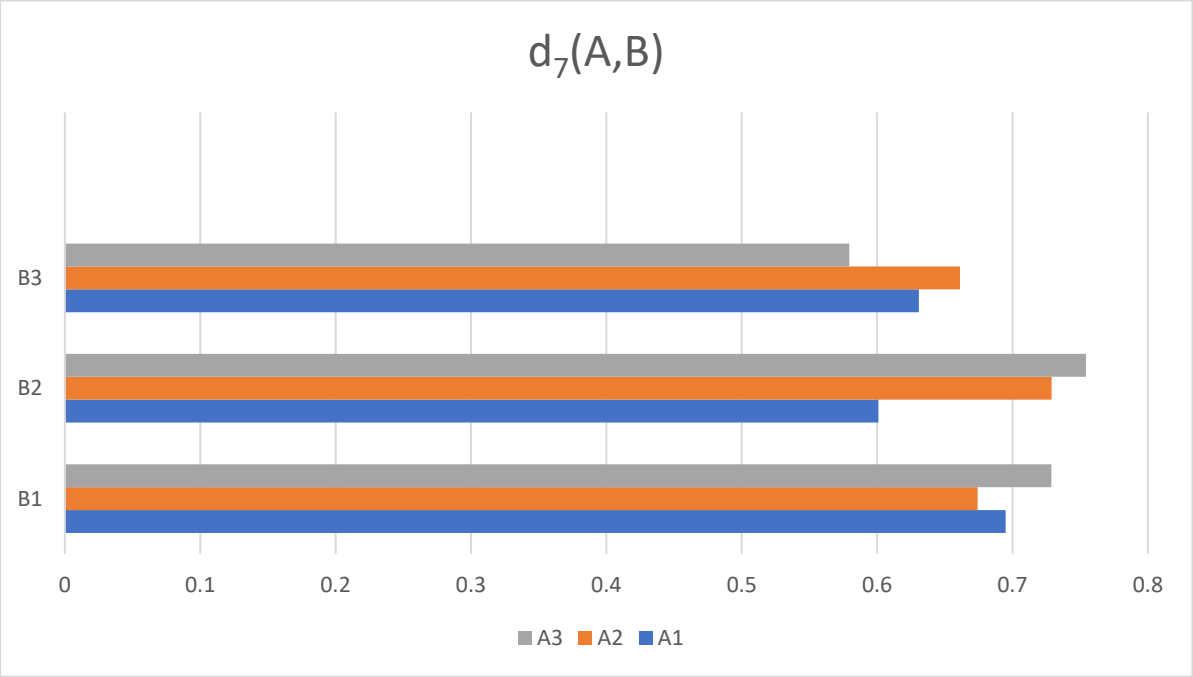
Table 5 Specified Diseases Vs Hospitals

	B_1	B_2	B_3
R_1	(0.8,0.9,0.1,0.7,0.5,0.6,0.2)	(0.4,0.5,1,0.5,0.6,1,0.1)	(0.1,0.1,0.1,0.1,0.1,0.1,0.1)
R_2	(0.7,0.9,0.3,0.8,0.9,0.1,0.3)	(0.8,0.9,0.7,0.2,0.3,0,1)	(0.6,0.3,0.1,0.1,0.5,0.5,0.4)
R_3	(1,1,1,0,0,0,0)	(0.6,0.4,0.2,0.3,0.2,0.1,0)	(0,0,0,0,1,1,1)
R_4	(0,0.7,1,0.1,0.4,0.4,0.3)	(0.9,0.8,0.6,0.4,0.4,0.2,0.2)	(0.8,0.1,0.3,0.3,0.1,0.4,0.2)

R_5	(0.5,0.8,0.9,0.9,0.7,0.6,0.1)	(0.3,0.2,0.2,0.1,0.1,0.3,1)	(0,0,0,0.2,0.2,0.2,0.1)

Table 6 Distance Analysis Matrix

	A_1	A_2	A_3	w_r
B_1	0.69494	0.67421	0.72870	0.699283
B_2	0.60093	0.72888	0.75425	0.694687
B_3	0.63083	0.66120	0.57944	0.623823



4.2 Comparison Study and Simulation

This section presents a comparative analysis and simulation of Tables 6 through 12.

Table 7 Distance Analysis Matrix $d_1(A,B)$

	A_1	A_2	A_3	w_r
B_1	0.337143	**0.317143**	0.37143	0.34191
B_2	0.38286	0.37143	**0.32286**	0.35905
B_3	0.53429	0.32	**0.27714**	0.37714

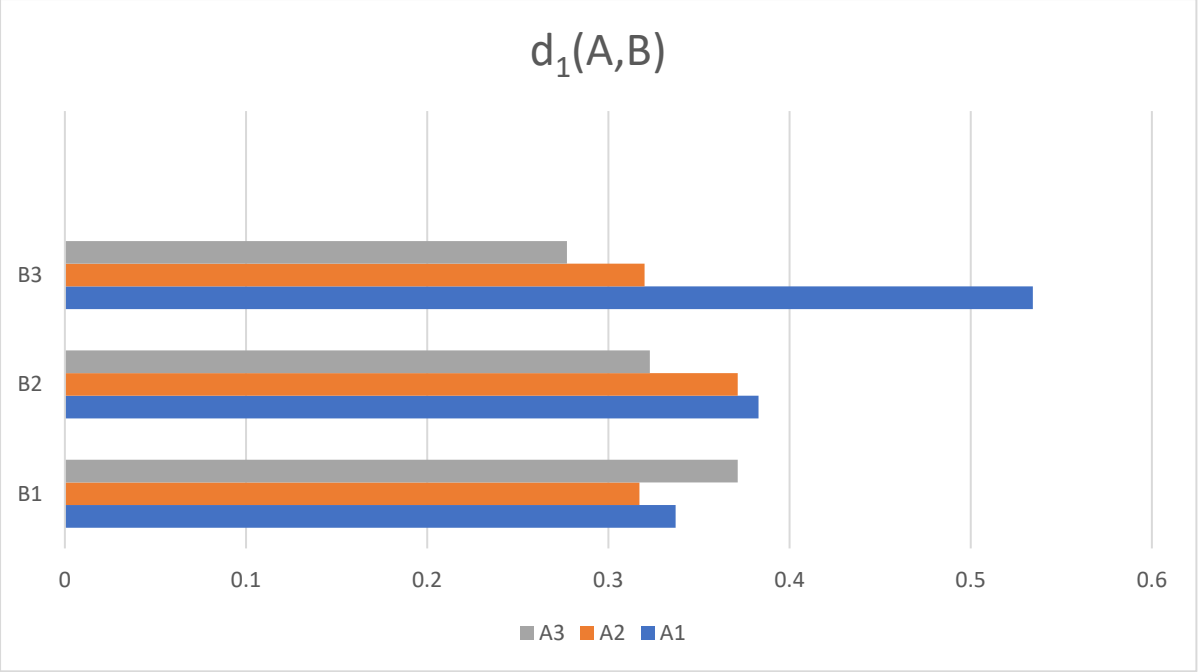


Table 8 Distance Analysis Matrix $d_2(A,B)$

	A_1	A_2	A_3	w_r
B_1	0.170255	**0.13764**	0.16777	0.158555
B_2	**0.12939**	0.17113	0.15448	0.151667
B_3	0.13905	0.14676	**0.12858**	0.13813

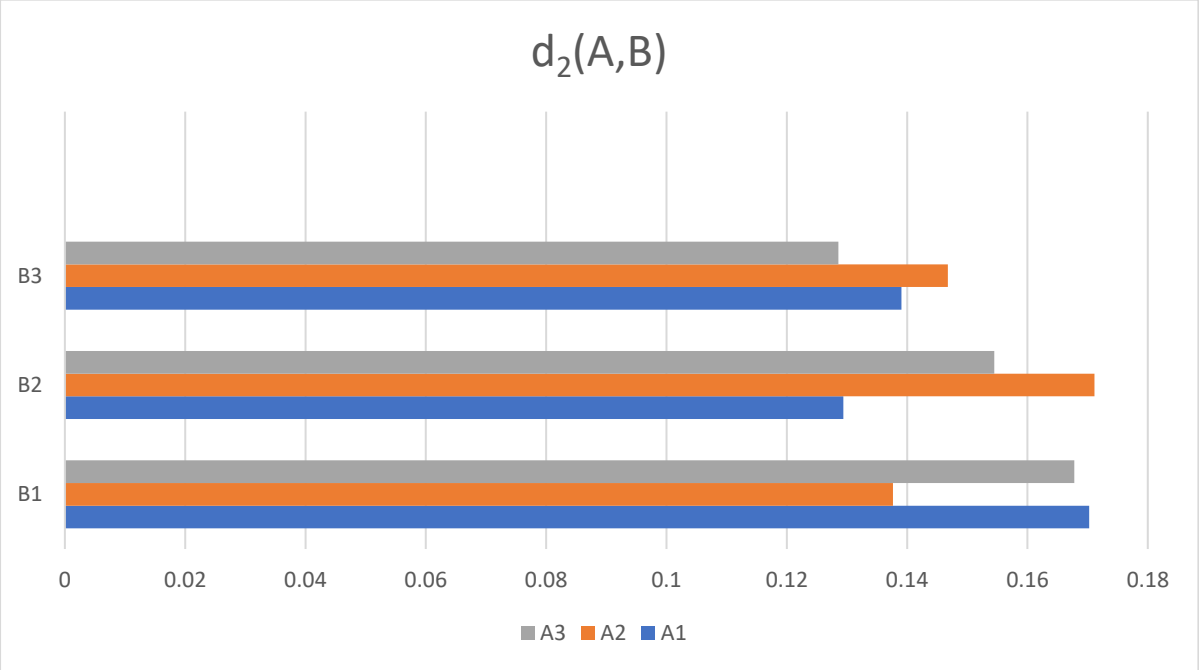


Table 9 Distance Analysis Matrix $d_3(A,B)$

	A_1	A_2	A_3	w_r
B_1	0.32914	**0.30486**	0.36914	0.33438
B_2	**0.24029**	0.34743	0.28943	0.29238
B_3	0.242	0.26	**0.22657**	0.24286

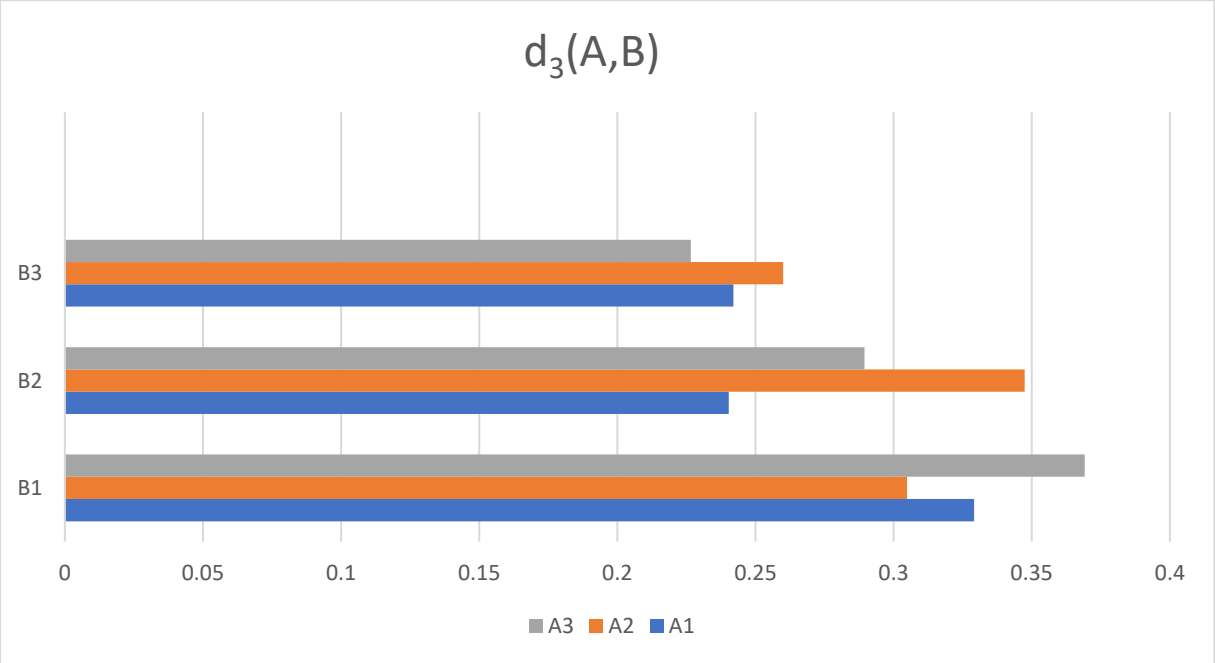


Table 10 Distance Analysis Matrix $d_4(A,B)$

	A_1	A_2	A_3	w_r
B_1	0.12286	**0.09514**	0.12057	0.11286
B_2	0.13743	**0.0826**	0.09114	0.10372
B_3	**0.09057**	0.14057	0.13	0.12038

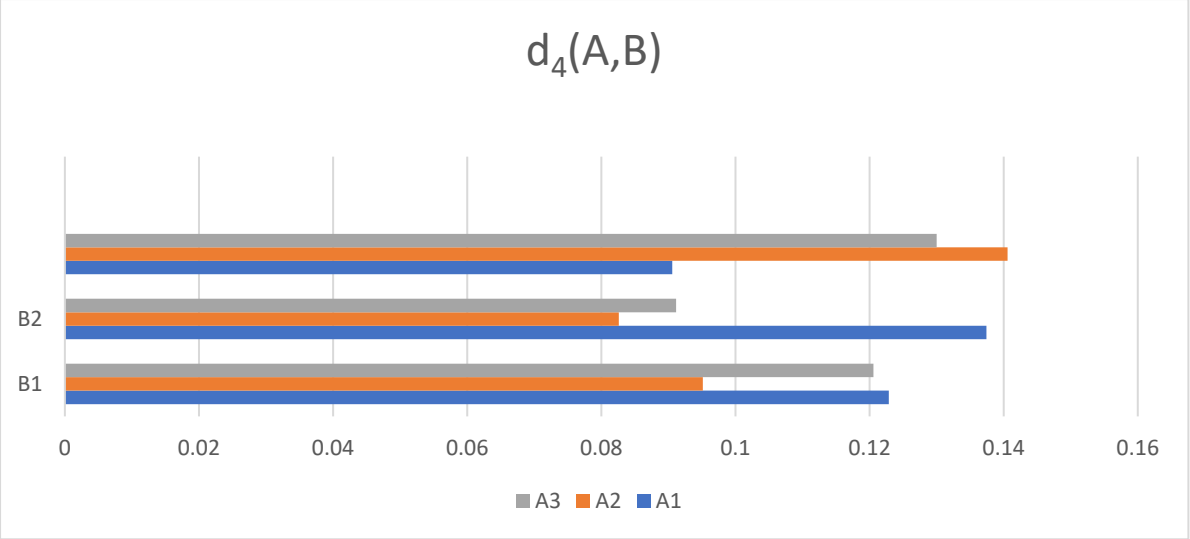


Table 11 Distance Analysis Matrix $d_5(A,B)$

	A_1	A_2	A_3	w_r
B_1	0.75671	**0.73414**	0.79347	0.76144
B_2	**0.65435**	0.79367	0.82129	0.75644
B_3	0.68690	0.71998	**0.63095**	0.67928

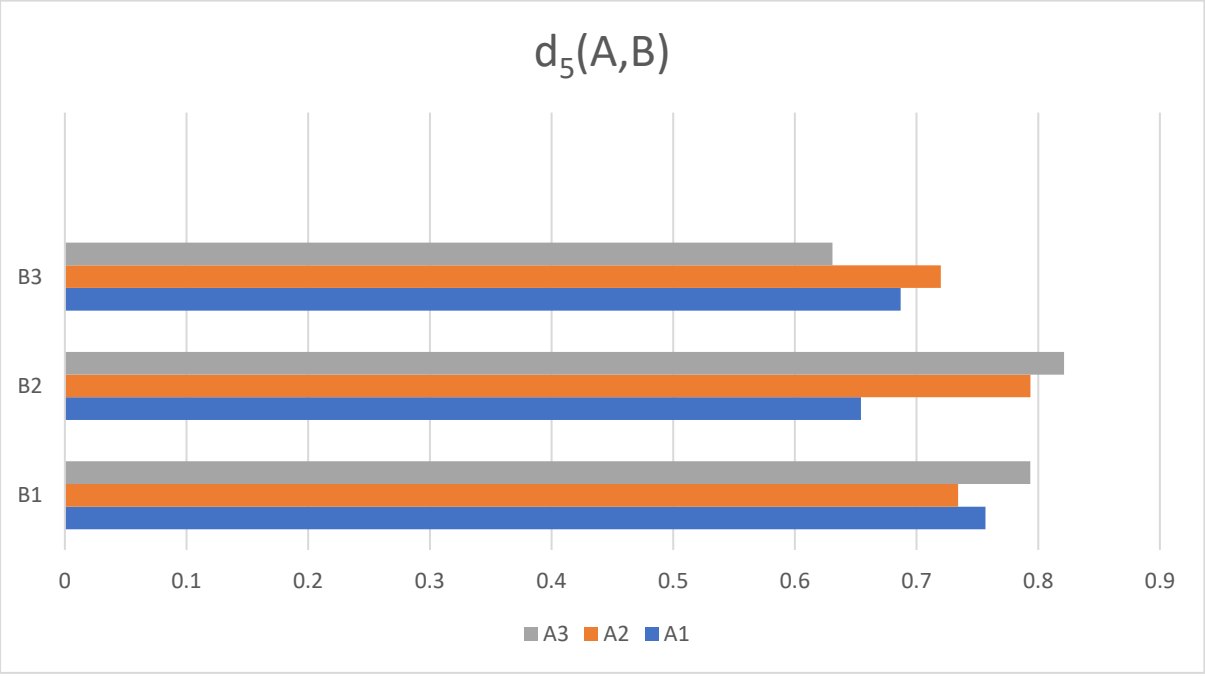


Table 12 Distance Analysis Matrix $d_6(A,B)$

	A_1	A_2	A_3	w_r
B_1	0.423434	0.12848	**0.116632**	0.2228624
B_2	**0.08335**	0.109326	0.1047452	0.0991404
B_3	0.08590	**0.07608**	0.07945	0.080477

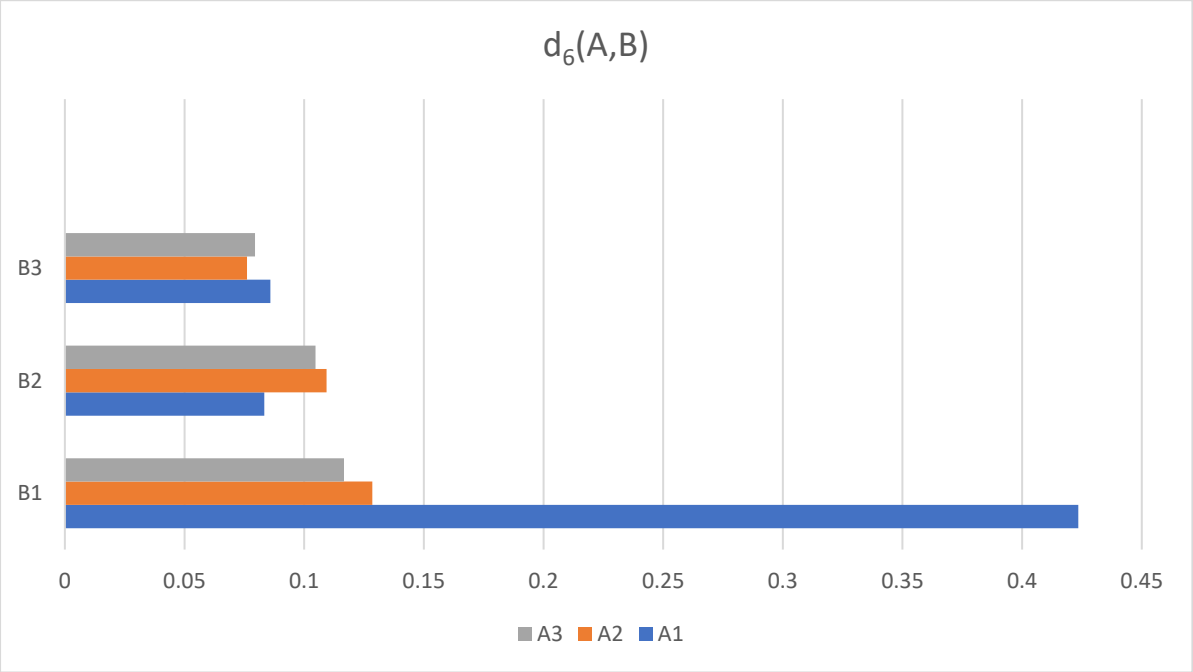


Table 13 Comparison of distance measures

	$d_1(A,B)$	$d_2(A,B)$	$d_3(A,B)$	$d_4(A,B)$	$d_5(A,B)$	$d_6(A,B)$
B_1	A_2	A_2	A_2	A_2	A_2	A_3
B_2	A_3	A_1	A_1	A_2	A_1	A_1
B_3	A_3	A_3	A_3	A_1	A_3	A_2

From the above table 13, we can observe that for B_1, decisions of $d_1(A,B), d_2(A,B), d_3(A,B), d_4(A,B), d_5(A,B)$Are the same, that is A_2 but the decision of $d_6(A,B)$ is A_3.

For B_2, the decision of $d_2(A,B)$, $d_3(A,B)$, $d_5(A,B)$, $d_6(A,B)$Re the same, that is A_1 but the decision of $d_1(A,B)$ is A_3 and $d_4(A,B)$ is A_2.

For B_3, the decision of $d_1(A,B)$, $d_2(A,B)$, $d_3(A,B)$, $d_5(A,B)$Are the same, that is A_3 but the decision of $d_4(A,B)$ is A_1 and $d_6(A,B)$ is A_2.

Thus, the final decision for the hospital is the majority decision, which is appropriate for the respective hospital is A_2.

5. Conclusion:

An expansion of the neutrosophic set, the heptapartitioned Neutrosophic set is a potent mathematical instrument for dealing with inconsistent, unclear, and incomplete data in using several factors to make decisions. In order to ensure compliance with the axioms of distance measures, this work provides a number of measures for heptapartitioned neutrosophic sets, some of which also meet metric axioms.A practical application is presented to identify the best hospital in several factors to make decision context, demonstrating the effectiveness of the proposed approach. These distance measures are pivotal in facilitating accurate decision-making across various criteria. Furthermore, the newly developed measures and methods are expected to stimulate research in heptapartitioned neutrosophic sets, with potential applications in fields such as rough topology, digital topology, and other domains within general topology.

6. References:

[1] Roliza Md Yasin, Nurul Izzah wardah Salim, Nik Fahhira Nik Ahmad Fuad, Noorzohirah Mohd Zahali Suriana Alias and Norzieha Mustapha, Average distance measure for TOPSIS-Sine Trignometric single valued Neutrosophic Weight Aggregation operator and its application in decision Making", Neutrosophic sets and systems, Vol.76, 2025.

[2] K Mondal, S Pramanik, BC Giri, "Single valued Neutrosophic hyperbolic sine similarity measure based MADM strategy", Neutrosophic sets and systems, 20, 3-11, 2018.

[3] M. Arockia Dasan, E. Bementa, Haydar Akca, V.F. Little Flower, "The Multi-decision-making problem in the recruitment process using penta-partitioned neutrosophic distance measure", Int. J. Nonlinear Anal. Appl. I press, 1-14 ISSN: 2008-6822 (Electronic).

[4] Florentin Smarandache. "Neutrosophic sets in a generalization of Intuitionistic Fuzzy set" Journal of New Theory, 2019.

[5] M. Arockia Dasan, E Bementa, Muhammad Aslam, VF Little flower, "multi-attribute decision-making problem in career determination using single valued neutrosophic distance measure", Complex and Intelligent systems, 1-15, 2024.

[6] Walid Abdullah, "A study of Neutrosophic sets and deep learning models for Breast cancer classification", Algorithm with Applications, 3, 50-59,2024.

[7] M. Abu Qamar, N. Hassan. "Entropy Measures of distance and Similarity of Q Neutrosophic soft sets and some applications", Entropy 20(9), 672, 2018.

[8] S Liu and Y Lin. (2010) "Grey system theory and application", Springer Verlag, 2010.

[9] WBV Kandasamy, F Smarandache, "Fuzzy interval Matrices Neutrosophic Interval Matrices and their Application", Infinite system (2006).

[10] Hung W.L. and Yan, M.S., "The sine distance measure of fuzzy numbers", Fuzzy sets and systems, 157(14), 1938-1949, (2006).

[11] Le Hoang Son, "Generalized picture distance measure and Applications to picture and Applications to picture fuzzy clustering", Applied soft computing 46(c), 284-295, (2016).

[12] Dutla. P, Goala S (2018) "Fuzzy decision making in medical diagnosis using an advanced distance measure on intuitionistic fuzzy sets", Open Cybernet system, J12:136-149.

[13] Shukla AK, Prakash V, Nath R. Muhuri PK (2023) "Type-2 intuitionistic fuzzy TODIM for Intelligent decision-making under uncertainty and hesitancy." Soft computing 27:13373-13390.

[14] Wengi Zing, Deging Li, Qian Yin." Distance and similarity measures between hesitant fuzzy sets and their application in pattern recognition", Pattern Recognition Letters, 84, 267-271.
[15] Weiqiong Wang, Xiaolog Xin "Distance measure between intuitionistic fuzzy sets", Pattern Recognition letters, 26, 13, 2063-2069.
[16] Ye J, Zhang Q (2014), "Single Valued neutrorophic similarity measures for multiple attribute decision-making", Neutrosophic sets and systems, 2:48-54.
[17] Wang H, Smarandache F, Zhang Y, Sunderaman R (2010), "Single valued Neutrosophic sets", multi-spaces multi-sets, 4:410-413.
[18] S Broumi and F. Smarandache, "Several similarity measures of neutrosophic sets", Neutrosophic sets system, 1(2013) no.10, 54-62.
[19] S. Broumi, R. Sundareswaran, M. Shanmugapriya, A. Bakali, and M. Talea, "Theory and application of Fermatean neutrosophic graphs", Neutrosophic sets system, 50 (2022), 248-286.
[20] R. Malick and S. Pramanik, "Pentapartitioned neutrosophic sets and its properties", Neutrosophic sets system: 36 (2020), 184-192.
[21] Jin-Shing Yao, Feng-Tre Lin," Fuzzy Critical path method based on sine distance ranking of fuzzy numbers." Systems and humans 30(1),76-82,2000.
[22] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, Haitham Rizk Fadlallah, "MADM Strategy Application of Bipolar single valued Heptapartitioned Neutrosophic set", International Journal of Neutrosophic Science, 21(4),181-194,2023.
[23] Majumdar P, Samanta SK (2014), On similarity and entropy of neutrosophic sets, J Installing Fuzzy Syst 26(3): 1245-1252.
[24] Chai JS, Selvachandran G, Smarandache F, Georgianna's VC, Hoang Son L, Bui QT, Vo B (2021), New similarity measures for single valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems, Complex Intelligent Systems, 7:703-723.

Received: Aug 30, 2024. Accepted: Nov 20, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

Neutrosophic Perspectives on the Body-Mind-Soul-Spirit Fluidity

Florentin Smarandache

University of New Mexico, Mathematics, Physics, and Natural Science Division, Gallup Campus, NM 87301, USA
https://fs.unm.edu/FlorentinSmarandache.htm
smarand@unm.edu

Abstract

This paper examines the concepts of the 'Body', 'Mind', 'Soul,' and 'Spirit' through the lens of neutrosophy. Neutrosophy challenges traditional binary logic, positing that 'truth' is not a fixed entity. Through the lens of neutrosophy, which is a philosophical framework, one seeks to address and reconcile contradictions and uncertainties in various fields, including philosophy, mathematics and science in general, as well as logic. Any phenomenon or dynamic structure can exist in different degrees (of truth, indeterminacy, and falsity). By applying this triadic approach to these four components, we propose the notion of a Neutrosophic Body-Mind-Soul-Spirit as a Quadruple Fluidity, where each of these entities influences, and in its turn is influenced, by all three others. And discuss their potential implications.

Keywords

Neutrosophy; indeterminacy; phenomenology; body; mind; soul; spirit; quadruple fluidity.

1. Introduction

Traditional models often present 'body' and 'mind' as distinct, dualistic components. However, neutrosophy—a philosophical framework developed by Smarandache [see 1-6]—offers a different approach by introducing the concept of indeterminacy. This third 'state' challenges binary logic and provides a more fluid, nuanced understanding. This paper introduces the concept of a *Neutrosophic Body-Mind-Soul-Spirit Fluidity* and discusses some of its implications. To better understand this concept, we will briefly explore the phenomenology [see 7-12], which also challenges Cartesian Dualism[1] by emphasizing the inseparability of 'body' and 'mind'.

[1] Robinson, Howard, "Dualism", The Stanford Encyclopedia of Philosophy (Spring 2023 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL: https://plato.stanford.edu/archives/spr2023/entries/dualism. Accesed: 21 May 2024.

1.1 Phenomenological Perspective

Phenomenology,[2] as explored in the works of Edmund Husserl[3] and later Maurice Merleau-Ponty,[4] emphasizes the unity of the self and its experience of the world. This philosophical framework posits that 'body' and 'mind' are not isolated or separate components. From a phenomenological perspective, human consciousness cannot be reduced solely to thoughts or bodily sensations; instead, it encompasses a lived experience that engages the entire being. This approach highlights the relational and dynamic nature of human existence, where the 'body' is not a passive entity but actively shapes the 'mind', and influences how a person perceives and interacts with the world.

Edmund Husserl's goal was to describe the structures of experience as they present themselves to consciousness, free from preconceived notions or theoretical assumptions. His method of *phenomenological reduction* sought to examine the world as directly experienced, devoid of external scientific or metaphysical frameworks. For Husserl, consciousness is inherently intentional—it is always directed towards an object, whether that object is physical or abstract.

Maurice Merleau-Ponty acknowledged Husserl's emphasis on the first-person perspective and the centrality of lived experience in understanding the world. However, Merleau-Ponty introduced significant modifications, particularly regarding his interpretation of embodiment and perception. One of the key departures in Merleau-Ponty's phenomenology is his assertion that the 'body' is central to perception and experience. While Husserl's phenomenology initially focused on the structures of consciousness and acts of intentionality—how we direct our attention to objects—Merleau-Ponty emphasized how the 'body' shapes our perception of the world. For Merleau-Ponty, the 'body' is not merely an object within the world, but rather the primary means through which we engage with and experience the world. He introduced the concept of the *lived body*, emphasizing the 'body' as a subjective entity, and rejected Cartesian 'mind'-'body' dualism.

While Husserl's focus on intentionality—how consciousness is always directed toward an object—was foundational, Merleau-Ponty extended this notion by arguing that perception is not merely a mental act but an embodied, situated experience. He asserted that perception is *pre-reflective*, occurring before any conceptualization or mental interpretation.

Whereas Husserl's phenomenology often operated within an intellectual framework that attempted to abstract away from the 'body', Merleau-Ponty's approach remains grounded in the understanding that perception is always shaped by embodied experience. For Merleau-Ponty, bodily existence precedes and structures consciousness.

---

[2] Smith, David Woodruff, "Phenomenology", *The Stanford Encyclopedia of Philosophy* (Summer 2018 Edition), Edward N. Zalta (ed.), URL: https://plato.stanford.edu/archives/sum2018/entries/phenomenology. Accesed: 23 May 2024.

[3] Beyer, Christian, "Edmund Husserl", The Stanford Encyclopedia of Philosophy (Winter 2022 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL: https://plato.stanford.edu/archives/win2022/entries/husserl. Accesed: 23 May 2024.

[4] Toadvine, Ted, "Maurice Merleau-Ponty", *The Stanford Encyclopedia of Philosophy* (Winter 2023 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL: https://plato.stanford.edu/archives/win2023/entries/merleau-ponty. Accesed: 23 May 2024.

Florentin Smarandache, Neutrosophic Perspectives on the Body-Mind-Soul-Spirit Fluidity

While Husserl, particularly in his later work, turned towards a more abstract investigation of the structures of consciousness—exploring the *essences* of experiences—Merleau-Ponty consistently focused on the lived world. For Merleau-Ponty, the world is not an external, objective entity to be studied from a detached perspective, but a world that is perceived and experienced in an embodied way. Merleau-Ponty took Husserl's analysis of intentionality and developed it into a more integrated theory of perception and existence.

Merleau-Ponty also critiqued Husserl's reliance on Cartesian distinctions between 'subject' and 'object'. While Husserl believed in the possibility of achieving pure, objective knowledge through phenomenological reduction, Merleau-Ponty was skeptical. For Merleau-Ponty, the subject is always engaged with the world through perception and embodiment, meaning that the 'mind' is never fully detached from the world.

1.2 Understanding the neutrosophic framework

Neutrosophy asserts that every proposition or phenomenon has three fundamental components: a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F). This triadic framework enables a more complex analysis of phenomena that cannot be reduced to binary categories. Neutrosophy recognizes that truth is not absolute, but can fluctuate across a spectrum, influenced by context, perception, and inherent contradictions. The principle can be applied to the 'body', 'mind,' 'soul,' and 'spirit' framework, each of which being manifested in varying degrees.

This is a model that embraces the multiple fluidity and ambiguity, and from this perspective, the 'body,' 'mind,' 'soul,' and 'spirit' are not fixed categories but interconnected and dynamic aspects of experience that can embody multiple states at once. For example, the 'body' can be seen not only as /healthy/ or /sick/ but in states of transition, and the 'spirit' may be neither fully /enlightened/ nor completely /disconnected/, but in a state of potential transformation.

2. The Neutrosophic Body-Mind-Soul-Spirit Fluidity

The 'Neutrosophic Body, Mind, Soul, Spirit' is a conceptual framework that integrates the traditional components of human existence—body, mind, soul, and spirit—through the lens of neutrosophy.

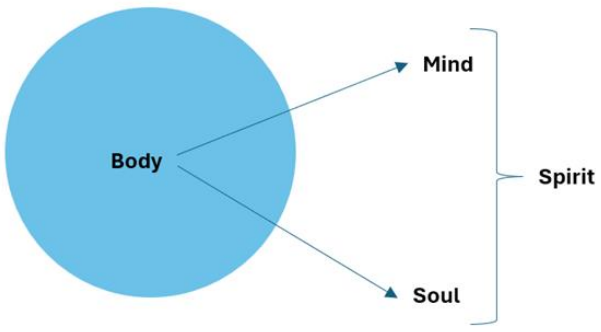


Fig. 1 - Implication Diagram

- **Body**: The body represents the physical aspect of a person, subject to biological processes and scientific study. In neutrosophic terms, the body is not simply in a state of health or illness but can also exist in a neutral state, such as transitioning between wellness and disease, growth and decay. Health, therefore, is not a fixed binary but a dynamic process with fluctuating states of balance and imbalance.
- **Mind**: The mind encompasses cognitive functions such as thinking, reasoning, and memory. In neutrosophic thought, the mind is not confined to a rational versus irrational dichotomy. Instead, it can occupy states of indeterminacy, where beliefs, ideas, and perceptions coexist in varying degrees of clarity and confusion. The mind is subject to both logical analysis and emotional influence, which creates a dynamic interplay of truth, ambiguity, and falsehood.
- **Soul**: Traditionally, the soul is seen as the essence or immaterial aspect that animates a person. In a neutrosophic framework, the soul embodies complexities that cannot be reduced to simple categories of good or evil. It may harbor both true aspects of identity (T), uncertain beliefs (I), and false societal constructs (F). The soul, therefore, can fluctuate between harmony and turmoil, light and shadow, embracing a full range of human emotional and spiritual experiences.
- **Spirit**: The spirit is often associated with transcendence or connection to the divine. Neutrosophy suggests that the spirit always exist in a transitional state, moving between moments of clarity and ambiguity, enlightenment and confusion. The spirit may embody truths about divine experiences (T), uncertainties about spiritual beliefs (I), and misconceptions about spiritual practices (F).

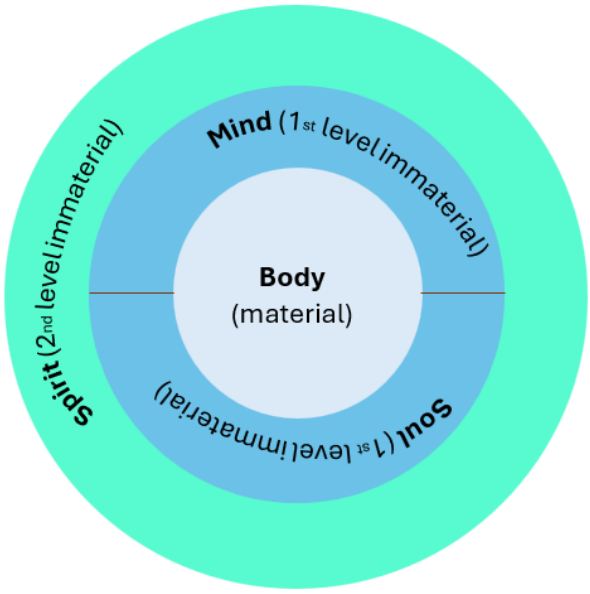


Fig. 2 - Structural Diagram

One of the key insights of neutrosophy is the recognition that these components—body, mind, soul, and spirit—do not function in isolation. Rather, they are interconnected and implicative in dynamic ways.

3. Implications and Examples (in personal development)

The neutrosophic model encourages a flexible and nuanced approach to personal development, urging individuals to recognize both certainty and uncertainty within themselves. This perspective challenges rigid identity definitions, instead promoting a dynamic understanding of self. In this context, personal growth becomes a journey of navigating these fluctuating states with openness, fostering a deeper connection to both oneself and the external world.

3.1 Physical and Mental Health

When a person feels physically depleted (truth), this can manifest as mental fog (indeterminate) and a lack of direction (indeterminate). However, after a restful night's sleep (truth), physical recovery (truth) can lead to clearer thinking (more truth) and a renewed sense of purpose (more truth). This demonstrates the interdependence of physical well-being and mental clarity, with each influencing the other in fluid, sometimes indeterminate ways.

3.2 Emotional State and Physical Health

Emotional stress (indeterminate) can have tangible effects on the body, such as fatigue (truth), which in turn may cloud mental clarity (indeterminate). Yet, practices like mindfulness (truth) can alleviate stress, improving emotional balance (truth) and physical vitality (truth). This example highlights how emotional and physical health are intertwined, constantly shifting in relation to one another.

3.3 Spiritual Growth and Cognitive Clarity

A person experiencing spiritual doubt (indeterminate) may notice a corresponding decline in mental focus (indeterminate). However, engaging in spiritual practices like meditation or prayer (truth) can restore mental clarity and deepen one's sense of purpose (truth), exemplifying how spiritual exploration impacts cognitive and emotional states. The interplay between the soul and mind is fluid, reflecting both certainty and uncertainty.

4. Integrating Neutrosophy into Human Understanding

By applying neutrosophic principles to these four components—body, mind, soul, spirit—we can explore how they interact in complex ways:

Each component does not exist in isolation; rather, they influence one another. For instance:

- The body affects mental states through health conditions.
- The mind shapes perceptions of the soul's essence.
- The soul influences spiritual beliefs which may affect mental health.

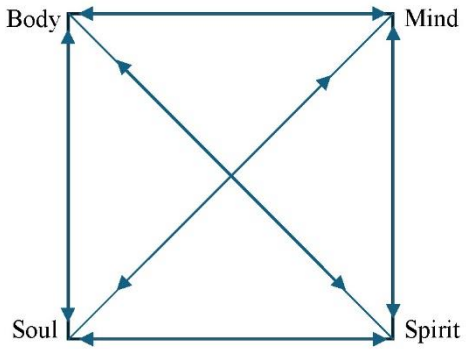


Fig. 3 - Neutrosophic Body-Mind-Soul-Spirit Fluidity [Complete Interplay]

Recognizing this interplay allows for a more holistic view of human beings that acknowledges both certainty and uncertainty in our understanding of self.

5. Implications for Personal Development

Understanding oneself through this neutrosophic lens encourages individuals to embrace complexity:

- Accepting that there are truths about oneself while also recognizing areas where one feels uncertain or misled.
- Encouraging personal growth by exploring these dimensions without rigid definitions.

This approach fosters an environment where individuals can navigate their identities with greater flexibility and openness to change.

By embracing neutrosophy, individuals can adopt a more holistic approach to life, accepting the complexity and fluidity of their physical, mental, emotional, and spiritual states. This framework fosters acceptance of change, growth, and transformation, allowing for ongoing self-exploration without the constraints of rigid definitions or static categories. It encourages a lifelong journey of development, where uncertainty is seen as a natural part of the human experience rather than something to avoid.

Conclusion

The “Neutrosophic Body Mind Soul Spirit” framework provides a comprehensive way to understand human existence beyond traditional dualistic models. It recognizes the multifaceted nature of being human while allowing for degrees of truthfulness in our self-perception.

The Neutrosophic {Body, Mind, Soul, and Spirit} framework offers a comprehensive and holistic model for understanding human existence. By applying the principles of neutrosophy, we can move beyond dualistic models of truth and engage with the complexity and nuance inherent in human experience. Ultimately, the neutrosophic perspective fosters a more flexible, open, and

evolving understanding of the self, promoting personal growth and well-being in a constantly changing world.

By applying this (truth, indeterminacy, falsehood) triadic approach to the four components, we propose the notion of a Neutrosophic Body-Mind-Soul-Spirit Fluidity as a quadruple interconnectivity, where each of these entities {Body, Mind, Soul, Spirit} influences, and in its turn is influenced, by all three others. And we discussed their potential implications.

References

I. Primary Literature

A. Neutrosophy

[1] Smarandache, F. (1999). *Neutrosophy: A New Branch of Philosophy*.

[2] Smarandache, F. (1999, 2000). *A Unifying Field in Logics: Neutrosophic Logic*. Preface by Charles Le. Rehoboth: American Research Press.

[3] Ashbacher, C. (2002). *Introduction to Neutrosophic Logic*. Rehoboth: American Research Press.

[4] Kandasamy, W. B. Vasantha; Smarandache, F. (2003). *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*.

[5] Smarandache, F. (2018). *Neutropsychic Personality. A mathematical approach to psychology.* Third updated edition. Brussels: Pons.

[6] Smarandache, F. (2019). *Introduction to Neutrosophic Sociology (Neutrosociology).* Brussels: Pons.

B. Phenomenology

[7] Husserl, Edmund (1931). *Cartesian Meditations*, trans. D. Cairns, Dordrecht: Kluwer, 1988.

[8] Husserl, Edmund (1939). *Experience and Judgement*, trans. J. S. Churchill and K. Ameriks, London: Routledge, 1973.

[9] Merleau-Ponty, Maurice (1945). *Phénoménologie de la perception.* Reprinted 2013. Paris: Gallimard.

[10] Merleau-Ponty, Maurice (1960). *Signes*. Paris: Gallimard.

[11] Merleau-Ponty, Maurice (1964). *The Primacy of Perception*. James Edie (ed.), Evanston: Northwestern University Press.

[12] Merleau-Ponty, Maurice (1996). *Sens et non-sens*. Paris: Gallimard.

II. Secondary Literature

[13] Bell, David (1990). *Husserl*. London: Routledge.

[14] Bernet, Rudolf; Kern, Iso; Marbach, Eduard (1993). *An Introduction to Husserlian Phenomenology*. Evanston: Northwestern University Press.

[15] Bleeker, C. J. (1959). The Phenomenological Method. *Numen*, 6(2), 96–111. https://doi.org/10.2307/3269308

[16] Carman, T. (1999). The Body in Husserl and Merleau-Ponty. *Philosophical Topics*, *27*(2), 205–226. http://www.jstor.org/stable/43154321

[17] Carr, D. (1970). Husserl's Problematic Concept of the Life-World. *American Philosophical Quarterly*, *7*(4), 331–339. http://www.jstor.org/stable/20009365

[18] Falque, E. (2015). Peut-on réduire le corps? *Archivio Di Filosofia*, 83(1/2), 91–107. https://www.jstor.org/stable/26572091

[19] Giovannangeli, D. (1998). Le retard de la conscience. *Revue Internationale de Philosophie*, 52(205 (3)), 367–375. http://www.jstor.org/stable/23955879

[20] Guilead, R. (1977). Le concept de monde selon Husserl. *Revue de Métaphysique et de Morale*, *82*(3), 345–364. http://www.jstor.org/stable/40901760

[21] Kidder, P. (1987). Husserl's Paradox. *Research in Phenomenology*, *17*, 227–242. http://www.jstor.org/stable/24654861

[22] Raval, R. K. (1972). An Essay on `Phenomenology'. *Philosophy and Phenomenological Research*, 33(2), 216–226. https://doi.org/10.2307/2106461

[23] Silverman, H. J. (1984). Phenomenology: From Hermeneutics To Deconstruction. *Research in Phenomenology*, 14, 19–34. http://www.jstor.org/stable/24654400

[24] Tilliette, X. (1965). Husserl et la notion de Nature: (Notes prises au cours de Maurice Merleau-Ponty). *Revue de Métaphysique et de Morale*, *70*(3), 257–269. http://www.jstor.org/stable/40902110

Received: Aug 29, 2024. Accepted: Nov 20, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Correlation and Regression Coefficients of Neutrosophic Graph Products using Minimum Spanning Tree for Enhancing Healthcare Strategies

A.Meenakshi[1], Shivangi Mishra J[2] [1,*]

[1]Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology;
meenakshiannamalai1@gmail.com

[2]Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology;
shivangimishra1427@gmail.com

[*]Correspondence: meenakshiannamalai1@gmail.com; Tel.: 9940070544

Abstract. This paper investigates the correlation and regression coefficients ($\mathbb{CRC}s$) of single-valued neutrosophic graph products ($SVNGPs$) with a focus on the minimum spanning tree (MST) and their applications in enhancing healthcare strategies. Single-valued neutrosophic Graph ($SVNG$) is an extension of traditional graph theory that uses neutrosophic logic to solve uncertainties and indeterminacy. We have introduced the modular and strong products of $SVNGs$ and calculated the $\mathbb{CRC}s$ with and without using MST to evaluate the relationships within healthcare networks. A comparative analysis has been done for $\mathbb{CRC}s$ with and without MST which helps to increase the accuracy and efficiency of $\mathbb{CRC}s$. The gained results offer valuable insights into optimizing resource allocation and improving patient care by identifying critical factors influencing healthcare outcomes. This study demonstrates the applicability of $SVNGPs$ in decision-making processes, with MST playing a pivotal role in enhancing overall healthcare efficiency and effectiveness.

Keywords: Single-valued neutrosophic graph products; Correlation coefficient; Regression coefficient; Minimum spanning tree

1. **Introduction**

A sophisticated mathematical framework known as neutrosophic fuzzy graphs manages uncertainty, imprecision, and indeterminacy in complex systems. Neutrosophic fuzzy graphs, which combine fuzzy graphs and neutrosophic logic, offer a powerful tool for modeling and analyzing situations involving inconsistent, inaccurate, or insufficient information. Neutrosophic sets suggested by Smarandache (1) and (2) as an extension of fuzzy sets and intuitionistic fuzzy

sets, are effective tools for dealing with incomplete, ambiguous, and inconsistent information in the real world. Neutrosophic sets are differentiated by the truth membership function ($\mathbb{T}$), indeterminacy membership function ($\mathbb{I}$), and falsity membership function ($\mathbb{F}$). The concept of single-valued neutrosophic sets was established by Wang et al. (3). Ye (4) introduced a correlation coefficient for single-valued neutrosophic sets applied to multiple-attribute decision-making. Broumi and Smarandache (5) and (6) proposed an improved similarity measure for neutrosophic sets using the cosine function. The many similarity metrics of neutrosophic sets were also covered, along with a complete discussion of existing methodologies and applications. Ye (7) have studied the single-valued neutrosophic graph using a minimum spanning tree and its clustering method. Sahoo and Pal (8) studied several types of products on intuitionistic fuzzy graphs and investigated various graph product methods and impacts. Banerjee et al. (9) conducted grey relational analysis (GRA) for multi-attribute decisions in a neutrosophic cubic set context. Mohanta et al.,(10) paper presents a comprehensive investigation of various forms of neutrosophic graph products, an area of growing interest due to its applicability in managing uncertainty and complexity in graph-based models. The correlation coefficient of hesitancy fuzzy graphs in decision-making processes was examined by Reddy and Basha (11). Akula and Basha (12) introduced a regression coefficient measure for intuitionistic fuzzy graphs, which they applied to soil selection for optimal paddy crop development, highlighting its agricultural decision-making relevance. Bajaj and Kumar (13) introduced a novel intuitionistic fuzzy correlation coefficient technique, demonstrating its effectiveness in complex multi-criteria decision-making. Single valued neutrosophic graphs ($SVNGs$) and their applications were thoroughly examined by Dey et al. (14). Meenakshi et al. (15) and (16) examined neutrosophic social networks with max product networks and investigated neutrosophic optimal networks using various operations. Meenakshi and Mishra (17) explored the modular product of two cubic fuzzy graph structures. Meenakshi et al. (18) conducted a comparative study of fuzzy domination and fuzzy coloring in an optimal approach. Meenakshi et al. (19) examined neutrosophic optimum networks and applied efficient domination techniques to intuitionistic fuzzy networks. Akula and Shaik (20) developed a correlation coefficient for intuitionistic fuzzy graphs to analyze financial investment strategies, demonstrating the practical use of fuzzy graph theory in financial decision-making. Ye (21) provided an improved correlation coefficient for intuitionistic fuzzy sets, representing a substantial development in fuzzy set theory. Christianto and Smarandache (22) have explored quantum communication advancements, focusing on China's Tiantong satellite and its potential to revolutionize communication through quantum entanglement. Christianto and Smarandache (23) also discussed the intersection of Ikigai and Design Thinking using a conceptual framework.

Single Valued Neutrosophic Graph (*SVNG*) expands on classical graph theory by including degrees of truth, indeterminacy, and falsity for each vertex and edge. This research addresses the necessity for a robust framework by enhancing graph theory through *SVNGs*, enabling a more thorough analysis of $\mathbb{CRC}s$ in uncertain environments. Graph products of *SVNGs*, like the modular and strong products, are an important field of study as they enable the construction of new graphs with better structural properties by combining two *SVNGs*. By utilizing *SVNGs* and exploring the relationship between $\mathbb{CRC}s$ for *SVNGPs*, this study aims to improve decision-making processes. This research carried out a sensitivity analysis to test the effectiveness of $\mathbb{CRC}s$ applied on two different types of operations of *SVNGPs* and their *MST*. The analysis results indicate that the $\mathbb{CRC}s$ have moderate to high levels of truth and falsity memberships and less indeterminacy. The modular product is more efficient in minimizing indeterminacy, making it a more reliable option in scenarios where clarity is crucial. The motivation for this research is to provide advanced mathematical models that can deal with uncertainty, especially in areas like healthcare where patient outcomes and resource management are critical. The current research lacks sufficient study on graph products in *SVNGs*, particularly in terms of analyzing correlation and regression in uncertain conditions. The present analysis draws attention to the relevance of the parameters in *SVNGPs* and their usage in decision-making processes. Specifically, the research focuses on identifying key factors that affect outcomes, such as patient care, and optimizing resource allocation, ultimately contributing to enhanced treatment protocols and patient management.

This research work is organized into nine sections. Section 1 provides a brief introduction to the research. In section 2, we have provided the preliminary definitions of the research. In section 3, we have introduced the concept of $\mathbb{CRC}s$ of the modular product of *SVNGs*. In section 4, we have introduced the $\mathbb{CRC}s$ for the modular product of *SVNGs* using *MST*. In section 5, we have analyzed the strong product of *SVNGs* and its $\mathbb{CRC}s$. The $\mathbb{CRC}s$ of the strong product of *SVNGs* using *MST* has been explored in section 6. In section 7, we present a flowchart that outlines the detailed approach used in the research. In section 8, we have discussed the applications of $\mathbb{CRC}s$ of *SVNGPs* to identify significant factors affecting patient outcomes and develop focused strategies for improving treatment protocols and patient management, ultimately improving patient care and resource allocation. Section 9 offers a comprehensive summary of the key conclusions derived from the research findings.

2. Preliminaries

This section covers the key definitions and ideas needed for the discussions in this manuscript.

Definition 2.1.(1) Let ξ represent the universe. A neutrosophic set $\mathcal{A}$ in ξ is characterized

by a truth membership function $\mathbb{T}_{\mathcal{A}}$, an indeterminacy membership function $\mathbb{I}_{\mathcal{A}}$, and a falsity membership function $\mathbb{F}_{\mathcal{A}}$, where $\mathbb{T}_{\mathcal{A}}$, $\mathbb{I}_{\mathcal{A}}$, and $\mathbb{F}_{\mathcal{A}}$ are real standard elements of $[0,1]$. It can be written as:

$$\mathcal{A} = \{\langle \mathbb{NG}, x, \mathbb{T}_{\mathcal{A}}(x), \mathbb{I}_{\mathcal{A}}(x), \mathbb{F}_{\mathcal{A}}(x)\rangle : x \in \xi\}.$$

There is no restriction on the sum of $\mathbb{T}_{\mathcal{A}}(x)$, $\mathbb{I}_{\mathcal{A}}(x)$, and $\mathbb{F}_{\mathcal{A}}(x)$, so $0 \leq \mathbb{T}_{\mathcal{A}}(x) + \mathbb{I}_{\mathcal{A}}(x) + \mathbb{F}_{\mathcal{A}}(x) \leq 3$.

Definition 2.2. (1)

Let X be a space of points (objects) with generic elements in ξ represented by x. The truth-membership function $\mathbb{T}_{\mathcal{A}}(x)$, the indeterminacy-membership function $\mathbb{I}_{\mathcal{A}}(x)$, and the falsity-membership function $\mathbb{F}_{\mathcal{A}}(x)$ define a single valued neutrosophic set $\mathcal{A}$ ($SVNS$). For each point x in ξ, $\mathbb{T}_{\mathcal{A}}(x), \mathbb{I}_{\mathcal{A}}(x), \mathbb{F}_{\mathcal{A}}(x) \in [0,1]$. The $SVNS$ $\mathcal{A}$ can be expressed as:

$$\mathcal{A} = \{\langle x, \mathbb{T}_{\mathcal{A}}(x), \mathbb{I}_{\mathcal{A}}(x), \mathbb{F}_{\mathcal{A}}(x)\rangle : x \in \xi\}$$

Definition 2.3.(11) For any two single valued neutrosophic sets ($SVNSs$) $\mathcal{A}$ and $\mathcal{B}$ in the universe of discourse $\mathcal{X} = \{t_1, t_2, \ldots, t_n\}$, the correlation coefficient $\mathcal{N}(\mathcal{A}, \mathcal{B})$ is defined by:

$$\mathcal{N}(\mathcal{A},\mathcal{B}) = \frac{\sum_{i=1}^{n} [\mathbb{T}_{\mathcal{A}}(t_i)\mathbb{T}_{\mathcal{B}}(t_i) + \mathbb{I}_{\mathcal{A}}(t_i)\mathbb{I}_{\mathcal{B}}(t_i) + \mathbb{F}_{\mathcal{A}}(t_i)\mathbb{F}_{\mathcal{B}}(t_i)]}{\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathcal{A}}(t_i)^2 + \mathbb{I}_{\mathcal{A}}(t_i)^2 + \mathbb{F}_{\mathcal{A}}(t_i)^2)} \cdot \sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathcal{B}}(t_i)^2 + \mathbb{I}_{\mathcal{B}}(t_i)^2 + \mathbb{F}_{\mathcal{B}}(t_i)^2)}} \quad (1)$$

where $\mathbb{T}_{\mathcal{A}}(t_i)$, $\mathbb{I}_{\mathcal{A}}(t_i)$, $\mathbb{F}_{\mathcal{A}}(t_i)$ and $\mathbb{T}_{\mathcal{B}}(t_i)$, $\mathbb{I}_{\mathcal{B}}(t_i)$, $\mathbb{F}_{\mathcal{B}}(t_i)$ are the truth-membership, indeterminacy-membership, and falsity-membership functions of $\mathcal{A}$ and $\mathcal{B}$ respectively at element t_i.

Definition 2.4.(11)

Assume that $\mathbb{NG} = (V, E, \mathbb{T}, \mathbb{I}, \mathbb{F})$ is a single valued Neutrosophic Fuzzy Graph ($SVNFG$); and the energy of two NFG $\mathbb{NG}_1$ and $\mathbb{NG}_2$ is defined as follows:

$$\mathcal{E}(\mathbb{NG}_1) = \sum_{i=1}^{n} \left[\mathbb{T}^2_{\mathbb{NG}_1}(t_i) + \mathbb{I}^2_{\mathbb{NG}_1}(t_i) + \mathbb{F}^2_{\mathbb{NG}_1}(t_i)\right]$$

$$\mathcal{E}(\mathbb{NG}_2) = \sum_{i=1}^{n} \left[\mathbb{T}^2_{\mathbb{NG}_2}(t_i) + \mathbb{I}^2_{\mathbb{NG}_2}(t_i) + \mathbb{F}^2_{\mathbb{NG}_2}(t_i)\right]$$

The covariance of $\mathbb{NG}_1$ and $\mathbb{NG}_2$ is defined as follows:

$$\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2) = \sum_{i=1}^{n} [\mathbb{T}_{\mathbb{NG}_1}(t_i)\mathbb{T}_{\mathbb{NG}_2}(t_i) + \mathbb{I}_{\mathbb{NG}_1}(t_i)\mathbb{I}_{\mathbb{NG}_2}(t_i) + \mathbb{F}_{\mathbb{NG}_1}(t_i)\mathbb{F}_{\mathbb{NG}_2}(t_i)]$$

The correlation coefficients of $SVNGs$ $\mathbb{NG}_1$ and $\mathbb{NG}_2$ are defined as follows:

$$\mathrm{CC}(\mathbb{NG}_1, \mathbb{NG}_2) = \frac{\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2)}{\sqrt{\mathcal{E}(\mathbb{NG}_1)\mathcal{E}(\mathbb{NG}_2)}}$$

$$= \frac{\sum_{i=1}^{n} [\mathbb{T}_{\mathbb{NG}_1}(t_i)\mathbb{T}_{\mathbb{NG}_2}(t_i) + \mathbb{I}_{\mathbb{NG}_1}(t_i)\mathbb{I}_{\mathbb{NG}_2}(t_i) + \mathbb{F}_{\mathbb{NG}_1}(t_i)\mathbb{F}_{\mathbb{NG}_2}(t_i)]}{\sqrt{\sum_{i=1}^{n} \left[\mathbb{T}^2_{\mathbb{NG}_1}(t_i) + \mathbb{I}^2_{\mathbb{NG}_1}(t_i) + \mathbb{F}^2_{\mathbb{NG}_1}(t_i)\right]} \sqrt{\sum_{i=1}^{n} \left[\mathbb{T}^2_{\mathbb{NG}_2}(t_i) + \mathbb{I}^2_{\mathbb{NG}_2}(t_i) + \mathbb{F}^2_{\mathbb{NG}_2}(t_i)\right]}}$$

Definition 2.5. (21) Xu et al. provided an alternative formula for the correlation coefficient of *NFGs* $\mathbb{NG}_1$ and $\mathbb{NG}_2$, defined as follows:

$$\mathrm{CC}(\mathbb{NG}_1, \mathbb{NG}_2) = \frac{\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2)}{\max\left(\sqrt{\mathbb{E}(\mathbb{NG}_1)}, \sqrt{\mathbb{E}(\mathbb{NG}_2)}\right)}$$

$$= \frac{\sum_{i=1}^{n} [\mathbb{T}_{\mathbb{NG}_1}(t_i)\mathbb{T}_{\mathbb{NG}_2}(t_i) + \mathbb{I}_{\mathbb{NG}_1}(t_i)\mathbb{I}_{\mathbb{NG}_2}(t_i) + \mathbb{F}_{\mathbb{NG}_1}(t_i)\mathbb{F}_{\mathbb{NG}_2}(t_i)]}{\max\left(\sqrt{\sum_{i=1}^{n} \left[\mathbb{T}^2_{\mathbb{NG}_1}(t_i) + \mathbb{I}^2_{\mathbb{NG}_1}(t_i) + \mathbb{F}^2_{\mathbb{NG}_1}(t_i)\right]}, \sqrt{\sum_{i=1}^{n} \left[\mathbb{T}^2_{\mathbb{NG}_2}(t_i) + \mathbb{I}^2_{\mathbb{NG}_2}(t_i) + \mathbb{F}^2_{\mathbb{NG}_2}(t_i)\right]}\right)}$$

The correlation coefficient function $\mathrm{CC}(\mathbb{NG}_1, \mathbb{NG}_2)$ satisfies the following conditions:

$$0 \leq \mathrm{CC}(\mathbb{NG}_1, \mathbb{NG}_2) \leq 1$$

$$\mathrm{CC}(\mathbb{NG}_1, \mathbb{NG}_2) = \mathrm{CC}(\mathbb{NG}_2, \mathbb{NG}_1)$$

$$\mathrm{CC}(\mathbb{NG}_1, \mathbb{NG}_2) = 1 \text{ if } \mathbb{NG}_1 = \mathbb{NG}_2$$

Definition 2.6. (13)
The regression coefficient of y on x is defined as:

$$\mathbb{R}_{yx} = \mathcal{K}\frac{\sigma_y}{\sigma_x} = \frac{\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2)}{\sigma_x\sigma_y}\frac{\sigma_y}{\sigma_x} = \frac{\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2)}{\sigma_x^2}$$

The regression coefficient of x on y is defined as:

$$\mathbb{R}_{xy} = \mathcal{K}\frac{\sigma_x}{\sigma_y} = \frac{\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2)}{\sigma_x\sigma_y}\frac{\sigma_x}{\sigma_y} = \frac{\mathrm{Cov}(\mathbb{NG}_1, \mathbb{NG}_2)}{\sigma_y^2}$$

Also, the relationship between the correlation and regression coefficients can be written as:

$$\mathcal{K} = \sqrt{\mathbb{R}_{yx} \times \mathbb{R}_{xy}}$$

3. $\mathbb{CRRC}s$ of Modular Product of *SVNGs*

This section examines the modular product of two *SVNGs* and the $\mathbb{CRRC}s$.

3.1. *Definition (Modular Product of SVNGs)*

Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$ and $\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of graph $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Then, the modular product $\mathbb{P}_1 \star \mathbb{P}_2 = (\sigma_1 \star \sigma_2, \mu_1 \star \mu_2)$ is defined as follows:

(1) For all $(a, b) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$\begin{aligned}
\mathbb{T}_{\sigma_1 \star \sigma_2}(a, b) &= \mathbb{T}_{\sigma_1}(a) \wedge \mathbb{T}_{\sigma_2}(b), \\
\mathbb{I}_{\sigma_1 \star \sigma_2}(a, b) &= \mathbb{I}_{\sigma_1}(a) \wedge \mathbb{I}_{\sigma_2}(b), \\
\mathbb{F}_{\sigma_1 \star \sigma_2}(a, b) &= \mathbb{F}_{\sigma_1}(a) \vee \mathbb{F}_{\sigma_2}(b).
\end{aligned}$$

(2) For all $(a_1, b_1) \in \mathbb{E}_1$ and $(a_2, b_2) \in \mathbb{E}_2$:

$$\begin{aligned}
\mathbb{T}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) &= \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2), \\
\mathbb{I}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) &= \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2), \\
\mathbb{F}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) &= \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(a_2, b_2),
\end{aligned}$$

(3) For all $(a_1, b_1) \notin \mathbb{E}_1$ and $(a_2, b_2) \notin \mathbb{E}_2$:

$$\begin{aligned}
\mathbb{T}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) &= \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2), \\
\mathbb{I}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) &= \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2), \\
\mathbb{F}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) &= \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(a_2, b_2),
\end{aligned}$$

3.2. *Example*

Consider the *SVNGs* $\mathbb{P}_1$ and $\mathbb{P}_2$ as shown in figure 1 and figure 2 and their modular product $\mathbb{P}_1 \star \mathbb{P}_2$ as shown in figure 3.

FIGURE 1. *SVNG* $\mathbb{P}_1$

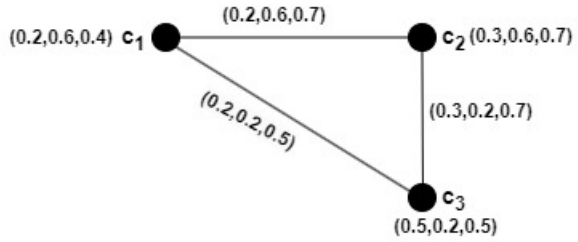


FIGURE 2. *SVNG* $\mathbb{P}_2$

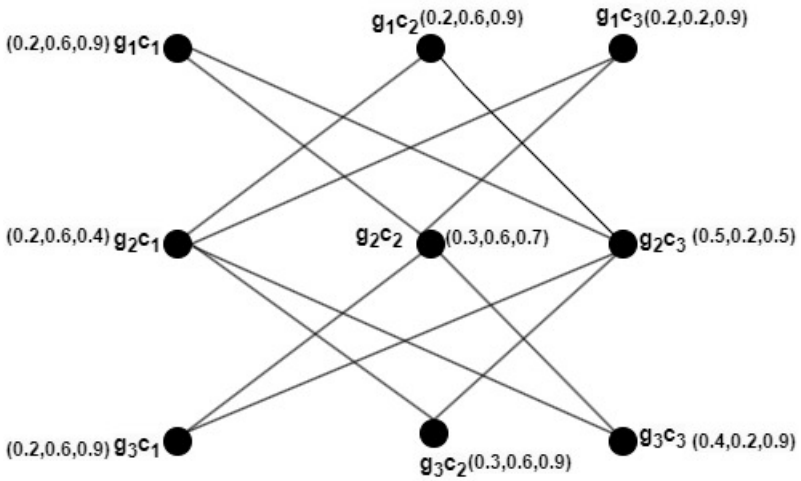


FIGURE 3. $\mathbb{P}_1 \star \mathbb{P}_2$

3.3. *Working Procedure*

Below is a working procedure to find the $\mathbb{CRC}s$ of Modular product *SVNGs*.

Step 1: Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$ and $\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a modular product of $\mathbb{P}_1$ and $\mathbb{P}_2$, $\mathbb{P}_1 \star \mathbb{P}_2$. Then, find the adjacency matrix of $\mathbb{P}_1 \star \mathbb{P}_2$.

Step 2: Compute the energy $\mathcal{E}(\mathbb{N}_i^{(r)})$ of an adjacency matrix $\mathbb{N}_i^{(r)}$ where $\mathcal{E}(\mathbb{N}_i^{(r)}) = \sum_{i=1}^{n} |\lambda_i|$

Step 3: Compute the weight scores $\mathcal{W}(\mathbb{S}_i^{(r)})$ determined by $\mathcal{E}(\mathbb{N}_i^{(r)})$

$$\mathcal{W}(\mathbb{S}_i^{(r)}) = \left(\frac{\mathcal{E}(\mathbb{N}_1^{(r)})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}_i^{(r)})}, \frac{\mathcal{E}(\mathbb{N}_2^{(r)})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}_i^{(r)})}, \frac{\mathcal{E}(\mathbb{N}_3^{(r)})}{\sum_{l=1}^{3} \mathcal{E}(\mathbb{N}_i^{(r)})} \right)$$

Step 4: Compute Correlation Coefficient $CC(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})$ between $\mathbb{N}_i^{(r)}$ and $\mathbb{S}_i^{(r)}$.

$$CC(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})) = \frac{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}_i^{(r)}}(t_i)\mathbb{T}_{\mathbb{S}_i^{(r)}}(t_i) + \mathbb{I}_{\mathbb{N}_i^{(r)}}(t_i)\mathbb{I}_{\mathbb{S}_i^{(r)}}(t_i) + \mathbb{F}_{\mathbb{N}_i^{(r)}}(t_i)\mathbb{F}_{\mathbb{S}_i^{(r)}}(t_i))}{\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}_i^{(r)}}(t_i)^2 + \mathbb{I}_{\mathbb{N}_i^{(r)}}(t_i)^2 + \mathbb{F}_{\mathbb{N}_i^{(r)}}(t_i)^2)} \sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{S}_i^{(r)}}(t_i)^2 + \mathbb{I}_{\mathbb{S}_i^{(r)}}(t_i)^2 + \mathbb{F}_{\mathbb{S}_i^{(r)}}(t_i)^2)}}$$

step 5: Find the regression coefficient $\mathbb{R}_{(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})}$ of $\mathbb{P}_1 \star \mathbb{P}_2$. The regression coefficient of $\mathbb{N}_i^{(r)}$ on $\mathbb{S}_i^{(r)}$ is defined as:

$$\mathbb{R}_{(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})} = \frac{\mathrm{Cov}(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})}{(\mathbb{N}_i^{(r)})^2}$$

The regression coefficient of $\mathbb{S}_i^{(r)}$ on $\mathbb{N}_i^{(r)}$ is defined as:

$$\mathbb{R}_{(\mathbb{S}_i^{(r)},\mathbb{N}_i^{(r)})} = \frac{\mathrm{Cov}(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})}{(\mathbb{S}_i^{(r)})^2}$$

step 6: Calculate the relationship between the $\mathbb{CRC}s$

$$\mathcal{K}(\mathbb{S}_i^{(r)},\mathbb{N}_i^{(r)}) = \sqrt{\mathbb{R}_{(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})} \times \mathbb{R}_{(\mathbb{S}_i^{(r)},\mathbb{N}_i^{(r)})}}$$

3.4. *Illustration of* $\mathbb{P}_1 \star \mathbb{P}_2$

3.4.1. *Adjacency matrix of* $\mathbb{P}_1 \star \mathbb{P}_2$

$$\mathbb{T}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0.6 & 0.2 \\ 0.6 & 0 & 0.2 & 0 & 0 & 0 & 0.6 & 0 & 0.2 \\ 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0.9 & 0.9 \\ 0.9 & 0 & 0.9 & 0 & 0 & 0 & 0.9 & 0 & 0.9 \\ 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.4.2. *Energy of* $\mathbb{P}_1 \star \mathbb{P}_2$

The energy $\mathcal{E}(\mathbb{N}_1^{(r)})$ of an adjacency matrix $\mathbb{N}_1^{(r)}$:

$$\mathcal{E}(\mathbb{N}_1^{(r)}) = \begin{bmatrix} 2.032 & 4.1854 & 4.1854 \end{bmatrix}$$

3.4.3. *Score function of* $\mathbb{P}_1 \star \mathbb{P}_2$

The weight score $\mathbb{S}_1^{(r)}$ determined by $\mathcal{E}(\mathbb{N}_1^{(r)})$:

$$\mathcal{W}(\mathbb{S}_1^{(r)}) = \begin{bmatrix} 0.1953 & 0.4023 & 0.4023 \end{bmatrix}$$

3.4.4. *Correlation Coefficient of* $\mathbb{P}_1 \star \mathbb{P}_2$

To Compute Correlation Coefficient $CC(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)})$ between $\mathbb{N}_1^{(r)}$ and $\mathbb{S}_1^{(r)}$:

$$\mathrm{CC}(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)}) = \frac{3.764}{3.763} = 1.00026$$

3.4.5. *Regression coefficients of* $\mathbb{P}_1 \star \mathbb{P}_2$

The regression coefficient measure of $\mathbb{N}_1^{(r)}$ on $\mathbb{S}_1^{(r)}$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)})} = \frac{3.764}{0.3617} = 10.406$$

The regression coefficient measure of $\mathbb{S}_1^{(r)}$) on $\mathbb{N}_1^{(r)}$ computed as:

$$\mathbb{R}_{(\mathbb{S}_1^{(r)}), \mathbb{N}_1^{(r)})} = \frac{3.764}{39.164} = 0.096$$

3.4.6. *Relationship between* $\mathbb{CRC}s$ *of* $\mathbb{P}_1 \star \mathbb{P}_2$

We defined the relationship between $\mathbb{CRC}s$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathcal{K}(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)}) = \sqrt{10.406 \times 0.096} = 0.999$$

4. $\mathbb{CRRC}s$ of $\mathbb{P}_1 \star \mathbb{P}_2$ using *MST*

The *MST* of the modular product of *SVNGs* are explored in this section. The analysis is expanded to address the relationship between the $\mathbb{CRRC}s$ of the modular product *SVNGs*.

4.1. *Working Procedure*

A working procedure to find the $\mathbb{CRRC}s$ of $\mathbb{P}_1 \star \mathbb{P}_2$ using *MST* is given below.

Step 1: Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$ and $\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a modular product of $\mathbb{P}_1$ and $\mathbb{P}_2$, $\mathbb{P}_1 \star \mathbb{P}_2$. Then, find the *MST* using the Kruskal algorithm .

Step 2: Compute the energy $\mathcal{E}(\mathbb{N}^{(r)}_{MST})$ of *MST* using its adjacency matrix $\mathbb{N}^{(r)}_{MST}$:

$$\mathcal{E}(\mathbb{N}^{(r)}_{MST}) = \sum_{i=1}^{n} |\lambda_i|$$

Step 3: Compute the weight score $\mathcal{W}(\mathbb{S}^{(r)}_{MST})$ determined by $\mathcal{E}(\mathbb{N}^{(r)}_{MST})$.

$$\mathcal{W}(\mathbb{S}^{(r)}_{MST}) = \left(\frac{\mathcal{E}(\mathbb{N}^{1}_{MST})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}^{r}_{MST})}, \frac{\mathcal{E}(\mathbb{N}^{2}_{MST})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}^{r}_{MST})}, \frac{\mathcal{E}(\mathbb{N}^{3}_{MST})}{\sum_{l=1}^{3} \mathcal{E}(\mathbb{N}^{r}_{MST})} \right)$$

Step 4: Compute Correlation Coefficient of *MST* of $\mathbb{P}_1 \star \mathbb{P}_2$ as shown in figure 4 $CC(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})$ between $\mathbb{N}^{(r)}_{MST}$ and $\mathbb{S}^{(r)}_{MST}$.

$$CC(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})) = \frac{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}^{(r)}_{MST}}(t_i)\mathbb{T}_{\mathbb{S}^{(r)}_{MST}}(t_i) + \mathbb{I}_{\mathbb{N}^{(r)}_{MST}}(t_i)\mathbb{I}_{\mathbb{S}^{(r)}_{MST}}(t_i) + \mathbb{F}_{\mathbb{N}^{(r)}_{MST}}(t_i)\mathbb{F}_{\mathbb{S}^{(r)}_{MST}}(t_i))}{\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}^{(r)}_{MST}}(t_i)^2 + \mathbb{I}_{\mathbb{N}^{(r)}_{MST}}(t_i)^2 + \mathbb{F}_{\mathbb{N}^{(r)}_{MST}}(t_i)^2)} \sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{S}^{(r)}_{MST}}(t_i)^2 + \mathbb{I}_{\mathbb{S}^{(r)}_{MST}}(t_i)^2 + \mathbb{F}_{\mathbb{S}^{(r)}_{MST}}(t_i)^2}}$$

step 5: Find the regression coefficients $\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}$ of *MST* of modular product of *SVNGs*. The regression coefficient of $\mathbb{N}^{(r)}_{MST}$ on $\mathbb{S}^{(r)}_{MST}$ is defined as:

$$\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})} = \frac{\mathrm{Cov}(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}{(\mathbb{N}^{(r)}_{MST})^2}$$

The regression coefficient of $\mathbb{S}^{(r)}_{MST}$ on $\mathbb{N}^{(r)}_{MST}$ is defined as:

$$\mathbb{R}_{(\mathbb{S}^{(r)}_{MST}, \mathbb{N}^{(r)}_{MST})} = \frac{\mathrm{Cov}(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}{(\mathbb{S}^{(r)}_{MST})^2}$$

step 6: Calculate the relationship between the $\mathbb{CRRC}s$ using *MST* of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathcal{K}(\mathbb{S}^{(r)}_{MST}, \mathbb{N}^{(r)}_{MST}) = \sqrt{\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})} \times \mathbb{R}_{(\mathbb{S}^{(r)}_{MST}, \mathbb{N}^{(r)}_{MST})}}$$

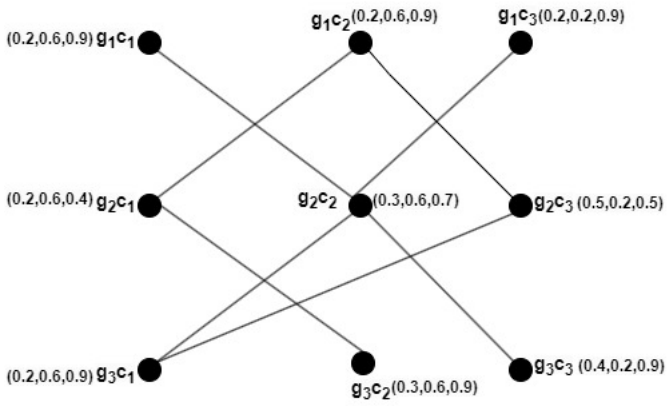


FIGURE 4. *MST* of $\mathbb{P}_1 \star \mathbb{P}_2$

4.2. *Illustration of* $\mathbb{CRC}$*s using MST of* $\mathbb{P}_1 \star \mathbb{P}_2$

Step 1: Let $\mathbb{P}_1 = (\sigma_1, \mu_1)and\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a modular product of $\mathbb{P}_1$ (figure 1) and $\mathbb{P}_2$ (figure 2), $\mathbb{P}_1 \star \mathbb{P}_2$ as shown in figure 3. Then, the find *MST* using kurskal algorithm for $\mathbb{P}_1 \star \mathbb{P}_2$.

Step 2: Adjacency matrix of *MST* of $\mathbb{P}_1 \star \mathbb{P}_2$

$$\mathbb{T}^{(1)}_{MT} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0.2 & 0 & 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(1)}_{MT} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0.6 & 0 & 0.2 & 0 & 0 & 0 & 0.6 & 0 & 0.2 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(1)}_{MT} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0.9 & 0 & 0.9 & 0 & 0 & 0 & 0.9 & 0 & 0.9 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3: Energy of the MST of $\mathbb{P}_1 \star \mathbb{P}_2$.

The energy $\mathcal{E}(\mathbb{N}^{(1)}_{MT})$ of an adjacency matrix $\mathbb{N}^{(1)}_{MT}$:

$$\mathcal{E}(\mathbb{N}^{(1)}_{MT}) = \begin{bmatrix} 1.9182 & 2.8686 & 2.8686 \end{bmatrix}$$

Step 4: Weight score of the MST of $\mathbb{P}_1 \star \mathbb{P}_2$.

The score $\mathbb{N}^{(1)}_{MT}$ determined by $\mathcal{E}(\mathbb{N}^{(1)}_{MT})$ of the energy:

$$\mathcal{W}(\mathbb{S}^{(1)}_{MT}) = \begin{bmatrix} 0.2505 & 0.3747 & 0.3747 \end{bmatrix}$$

Step 5: Compute the covariance of the MST $\mathbb{P}_1 \star \mathbb{P}_2$.

$$Cov(\mathcal{E}(\mathbb{N}^{(1)}_{MT}), \mathcal{W}(\mathbb{S}^{(1)}_{MT})) = \begin{bmatrix} 0.4805 & 1.0748 & 1.0748 \end{bmatrix} = 2.6301$$

Step 6: Calculate the correlation coefficient of the MST of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathrm{CC}(\mathcal{E}(\mathbb{N}^{(1)}_{MT}), \mathcal{W}(\mathbb{S}^{(1)}_{MT})) = \frac{2.6301}{2.6295} = 1.0002$$

Step 7: Compute the regression coefficients of the MST of $\mathbb{P}_1 \star \mathbb{P}_2$.

The regression coefficient measure of ($\mathcal{E}(\mathbb{N}^{(1)}_{MT})$ on $\mathcal{W}(\mathbb{S}^{(1)}_{MT})$) calculated as:

$$\mathbb{R}_{(\mathbb{N}^{(1)}_{MT}, \mathbb{S}^{(1)}_{MT})} = \frac{2.63}{0.343} = 7.66$$

The regression coefficient measure of $\mathcal{W}(\mathbb{S}^{(1)}_{MT})$) on ($\mathcal{E}(\mathbb{N}^{(1)}_{MT})$ computed as:

$$\mathbb{R}_{(\mathbb{S}^{(1)}_{MT}, \mathbb{N}^{(1)}_{MT})} = \frac{2.63}{20.136} = 0.1306$$

Step 8: Compute the relationship between $\mathbb{CRC}s$ of the MST of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathcal{K}(\mathbb{N}^{(1)}_{MT}, \mathbb{S}^{(1)}_{MT}) = \sqrt{7.66 \times 0.1306} = 1.00$$

5. $\mathbb{CRC}s$ of Strong Product of *SVNGs*

This section examines the strong product of two *SVNGs* and investigates the relationship between the $\mathbb{CRC}s$.

Definition: Strong Product of SVNGs

Let $\mathbb{S}_1 = (\sigma_1, \mu_1)$ and $\mathbb{S}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of graph $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. The strong product $\mathbb{S}_1 \boxtimes \mathbb{S}_2 = (\sigma_1 \boxtimes \sigma_2, \mu_1 \boxtimes \mu_2)$ is defined as follows:

(1) For all $(a, b) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$\mathbb{T}_{\sigma_1 \boxtimes \sigma_2}(a, b) = \mathbb{T}_{\sigma_1}(a) \wedge \mathbb{T}_{\sigma_2}(b),$$
$$\mathbb{I}_{\sigma_1 \boxtimes \sigma_2}(a, b) = \mathbb{I}_{\sigma_1}(a) \wedge \mathbb{I}_{\sigma_2}(b),$$
$$\mathbb{F}_{\sigma_1 \boxtimes \sigma_2}(a, b) = \mathbb{F}_{\sigma_1}(a) \vee \mathbb{F}_{\sigma_2}(b),$$

(2) For all $a_1 \in \mathbb{V}_1$ and $(b_1, b_2) \in \mathbb{E}_2$:

$$\mathbb{T}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_1, b_2)) = \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{T}_{\mu_2}(b_1, b_2),$$
$$\mathbb{I}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_1, b_2)) = \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(b_1, b_2),$$
$$\mathbb{F}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_1, b_2)) = \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(b_1, b_2),$$

(3) For all $b_1 \in \mathbb{V}_2$ and $(a_1, a_2) \in \mathbb{E}_1$:

$$\mathbb{T}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_1)) = \mathbb{T}_{\mu_1}(a_1, a_2) \wedge \mathbb{T}_{\mu_2}(b_1, b_1),$$
$$\mathbb{I}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_1)) = \mathbb{I}_{\mu_1}(a_1, a_2) \wedge \mathbb{I}_{\mu_2}(b_1, b_1),$$
$$\mathbb{F}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_1)) = \mathbb{F}_{\mu_1}(a_1, a_2) \vee \mathbb{F}_{\mu_2}(b_1, b_1),$$

(4) For all $(a_1, b_1) \in \mathbb{E}_1$ and $(a_2, b_2) \in \mathbb{E}_2$:

$$\mathbb{T}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{T}_{\mu_2}(a_2, b_2),$$
$$\mathbb{I}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2),$$
$$\mathbb{F}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(a_2, b_2),$$

5.1. *Example*

Consider the *SVNGs* $\mathbb{S}_1$ and $\mathbb{S}_2$ as shown in figure 5 and figure 6 and their strong product $\mathbb{S}_1 \boxtimes \mathbb{S}_2$ as shown in figure 7.

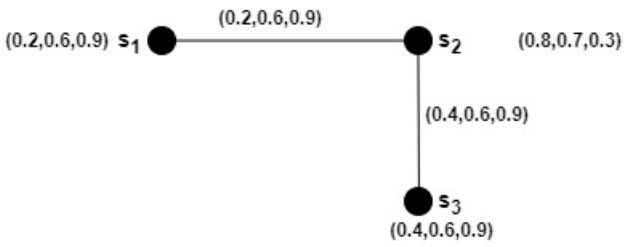


FIGURE 5. *SVNGs* $\mathbb{S}_1$

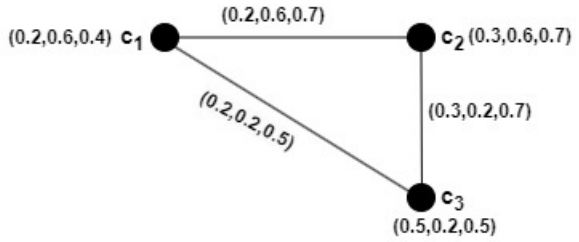


FIGURE 6. *SVNGs* $\mathbb{S}_2$

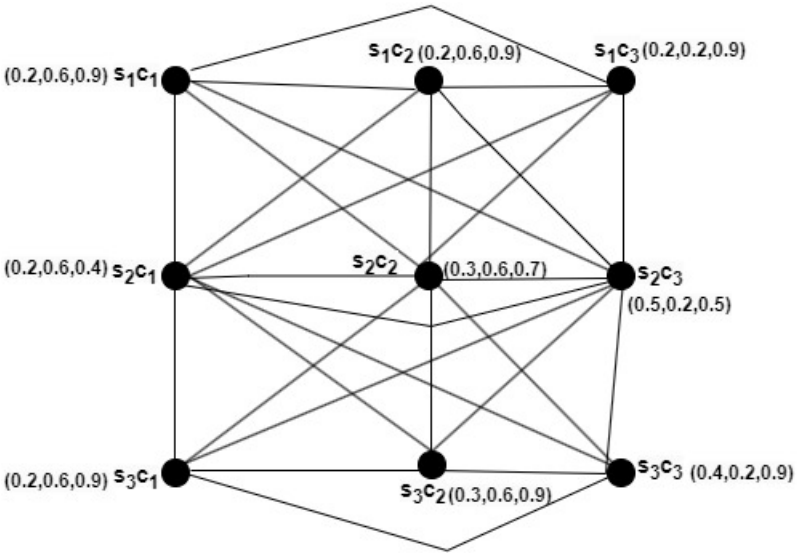


FIGURE 7. $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

5.2. *Adjacency matrix of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

$$\mathbb{T}^{(2)} = \begin{pmatrix} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0.3 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.3 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.4 & 0.2 & 0.3 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(2)} = \begin{pmatrix} 0 & 0.6 & 0.2 & 0.6 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0.6 & 0 & 0.2 & 0.6 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.6 & 0.6 & 0.2 & 0 & 0.6 & 0.2 & 0.6 & 0.6 & 0.2 \\ 0.6 & 0.6 & 0.2 & 0.6 & 0 & 0.2 & 0.6 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.6 & 0.2 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.6 & 0.2 & 0.6 & 0 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(2)} = \begin{pmatrix} 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0.9 & 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0.9 & 0.9 & 0 & 0.9 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0.9 & 0.9 & 0.9 & 0 & 0.7 & 0.5 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.7 & 0 & 0.7 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.5 & 0.7 & 0 & 0.9 & 0.9 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0 & 0.9 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0 \end{pmatrix}$$

5.3. *Energy of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

The energy $\mathcal{E}(\mathbb{N}_2^{(r)})$ of an adjacency matrix $\mathbb{N}_2^{(r)}$:

$$\mathcal{E}(\mathbb{N}_2^{(r)}) = \begin{bmatrix} 3.6266 & 6.2948 & 14.3234 \end{bmatrix}$$

5.4. *Weight Score function of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

The scores $\mathbb{S}_2^{(r)}$ determined by $\mathcal{E}(\mathbb{N}_2^{(r)})$ of the energy:

$$\mathcal{W}(\mathbb{S}_2^{(r)}) = \begin{bmatrix} 0.1495 & 0.2596 & 0.5907 \end{bmatrix}$$

5.5. *Correlation Coefficient of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

To Compute Correlation Coefficient $CC(\mathbb{N}_2^{(r)}, \mathbb{S}_2^{(r)})$ between $\mathbb{N}_2^{(r)}$ and $\mathbb{S}_2^{(r)}$:

$$\mathrm{CC}(\mathbb{N}_2^{(r)}, \mathbb{S}_2^{(r)}) = \frac{10.637}{10.633} = 1.0003$$

5.6. *Regression Coefficients of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

The regression coefficient measure of $\mathbb{N}_2^{(r)}$ on $\mathbb{S}_2^{(r)}$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_2^{(r)},\mathbb{S}_2^{(r)})} = \frac{10.63}{257.936} = 0.0412$$

The regression coefficient measure of $\mathbb{S}_2^{(r)}$) on $\mathbb{N}_2^{(r)}$ computed as:

$$\mathbb{R}_{(\mathbb{S}_2^{(r)}),\mathbb{N}_2^{(r)})} = \frac{10.63}{0.4385} = 24.2417$$

5.7. *Relationship between* $\mathbb{CRC}s$ *of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

$$\mathcal{K}(\mathbb{N}_2^{(r)}, \mathbb{S}_2^{(r)}) = \sqrt{0.0412 \times 24.2417} = 0.999$$

6. $\mathbb{CRC}s$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$ using *MST*

In this section, we have discussed the *MST* of the strong product of *SVNGs* in figure 7. The $\mathbb{CRC}s$ of *MST* of the strong product of *SVNGs* is introduced.

6.1. *The* $\mathbb{CRC}s$ *of MST of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

Step 1: Let $\mathbb{S}_1 = (\sigma_1, \mu_1)$ and $\mathbb{S}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a strong product of $\mathbb{S}_1$ (figure 5) and $\mathbb{S}_2$ (figure 6) which is denoted as $\mathbb{S}_1 \boxtimes \mathbb{S}_2$ as shown in figure 7. Then, find the *MST* using the Kruskal algorithm for $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

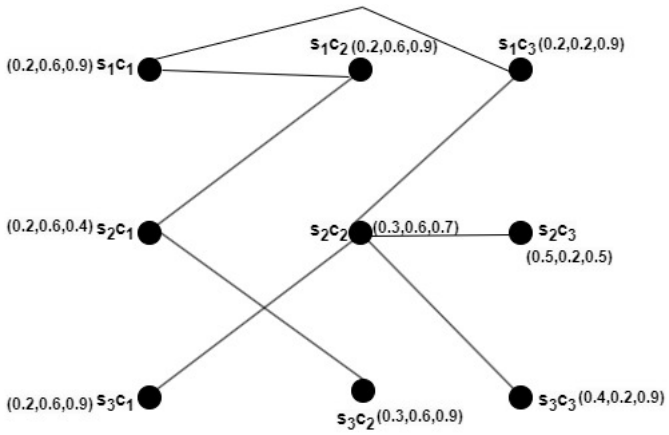


FIGURE 8. *MST* of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

Step 2: Adjacency matrix of MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

$$\mathbb{T}^{(2)}_{ST} = \begin{pmatrix} 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.3 & 0.2 & 0 & 0.3 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(2)}_{ST} = \begin{pmatrix} 0 & 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0.2 & 0 & 0.2 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(2)}_{ST} = \begin{pmatrix} 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0.7 & 0.9 & 0 & 0.9 \\ 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3: Energy of the MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

The energy $\mathcal{E}(\mathbb{N}^{(2)}_{ST})$ of an adjacency matrix $\mathbb{N}^{(2)}_{ST}$:

$$\mathcal{E}(\mathbb{N}^{(2)}_{ST}) = \begin{bmatrix} 1.8256 & 3.4306 & 3.4306 \end{bmatrix}$$

Step 4: Weight score of the MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

The score $\mathbb{N}^{(2)}_{ST}$ determined by $\mathcal{E}(\mathbb{N}^{(r)}_{ST})$ of the energy:

$$\mathcal{W}(\mathbb{S}^{(2)}_{ST}) = \begin{bmatrix} 0.2101 & 0.3949 & 0.3949 \end{bmatrix}$$

Step 5: Compute the covariance of the MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

$$Cov(\mathcal{E}(\mathbb{N}_{ST}^{(2)}), \mathcal{W}(\mathbb{S}_{ST}^{(2)})) = 3.092$$

Step 6: Calculate the correlation coefficient of the MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

$$\mathrm{CC}(\mathcal{E}(\mathbb{N}_{ST}^{(2)}), \mathcal{W}(\mathbb{S}_{ST}^{(2)})) = \frac{3.092}{3.031} = 1.02012$$

Step 7: Compute the regression coefficients of the MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.
The regression coefficient measure of $(\mathcal{E}(\mathbb{N}_{ST}^{(2)})$ on $\mathcal{W}(\mathbb{S}_{ST}^{(2)}))$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_{ST}^{(2)}, \mathbb{S}_{ST}^{(2)})} = \frac{3.092}{0.355} = 8.709$$

The regression coefficient measure of $\mathcal{W}(\mathbb{S}_{ST}^{(2)}))$ on $(\mathcal{E}(\mathbb{N}_{ST}^{(2)})$ computed as:

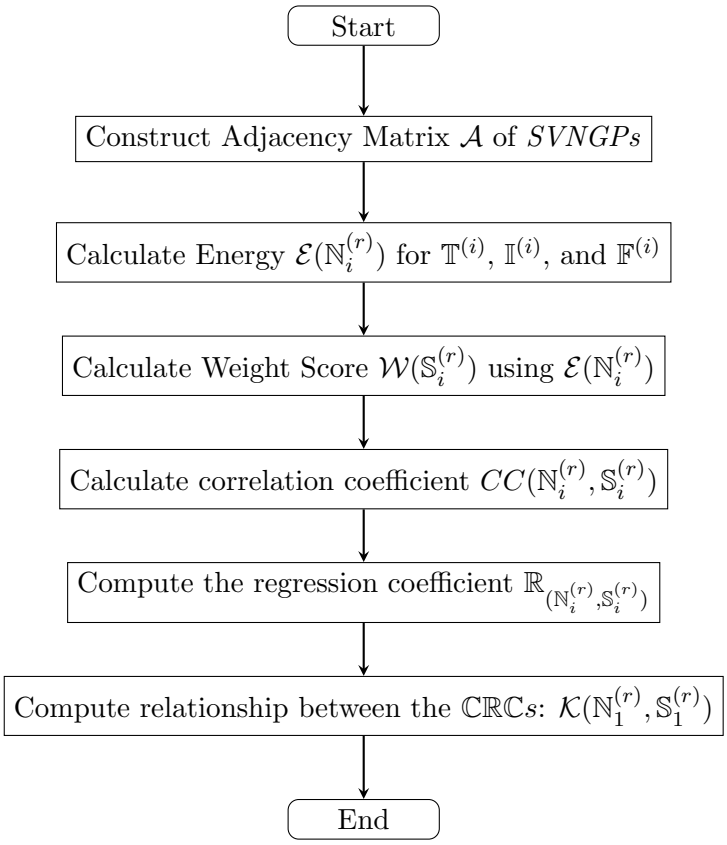
$$\mathbb{R}_{(\mathbb{S}_{ST}^{(2)}, \mathbb{N}_{ST}^{(2)})} = \frac{3.092}{25.830} = 0.119$$

Step 8: Calculate the relationship between $\mathbb{CRRC}s$ of the MST of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

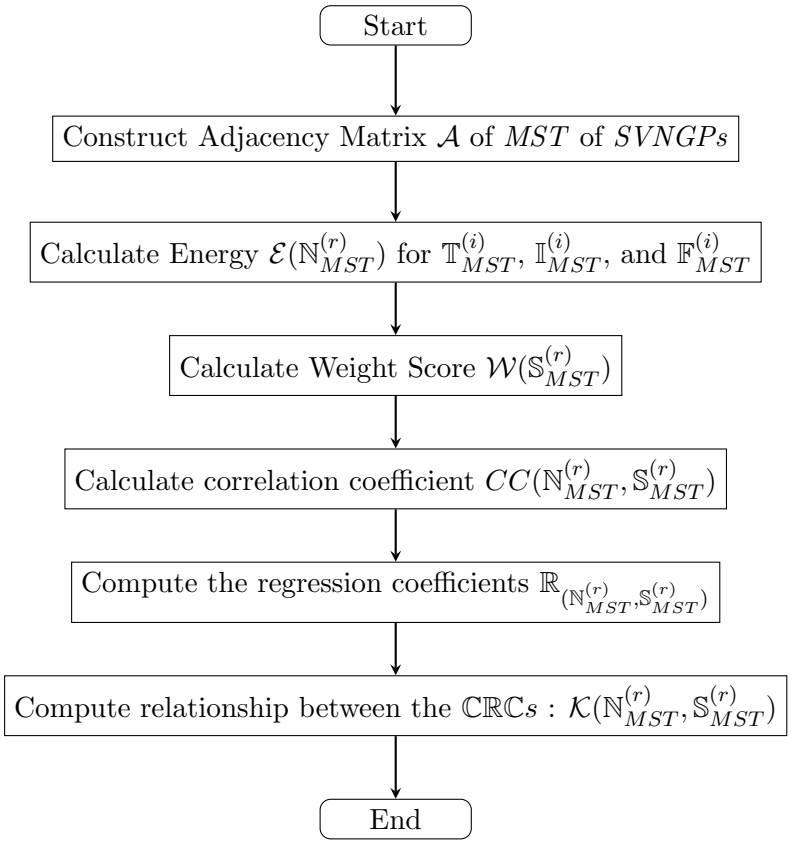
$$\mathcal{K}(\mathbb{N}_{ST}^{(2)}, \mathbb{S}_{ST}^{(2)}) = \sqrt{8.709 \times 0.119} = 1.01$$

7. **Flowchart**

7.1. *Flowchart for CRCs of SVNGPs without MST*


Start
Construct Adjacency Matrix $\mathcal{A}$ of $SVNGPs$
Calculate Energy $\mathcal{E}(\mathbb{N}_i^{(r)})$ for $\mathbb{T}^{(i)}$, $\mathbb{I}^{(i)}$, and $\mathbb{F}^{(i)}$
Calculate Weight Score $\mathcal{W}(\mathbb{S}_i^{(r)})$ using $\mathcal{E}(\mathbb{N}_i^{(r)})$
Calculate correlation coefficient $CC(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})$
Compute the regression coefficient $\mathbb{R}_{(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})}$
Compute relationship between the $\mathbb{CRRC}s$: $\mathcal{K}(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)})$
End

7.2. *Flowchart for CRCs of SVNGPs using MST*


Start
Construct Adjacency Matrix $\mathcal{A}$ of MST of $SVNGPs$
Calculate Energy $\mathcal{E}(\mathbb{N}^{(r)}_{MST})$ for $\mathbb{T}^{(i)}_{MST}$, $\mathbb{I}^{(i)}_{MST}$, and $\mathbb{F}^{(i)}_{MST}$
Calculate Weight Score $\mathcal{W}(\mathbb{S}^{(r)}_{MST})$
Calculate correlation coefficient $CC(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})$
Compute the regression coefficients $\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}$
Compute relationship between the $\mathbb{CRC}s$: $\mathcal{K}(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})$
End

8. **Applications**

In the healthcare system, it is essential to recognize the complex dynamics between patients, medical professionals, and healthcare institutions to deliver high-quality care and improve outcomes. To examine these interactions with inherent uncertainties and indeterminacies, we used the modular product of *SVNGs* and strong product *SVNGs*, as well as their minimal spanning tree (*MST*). Using $\mathbb{CRC}s$, we can identify significant interaction patterns, optimize resource allocation, and analyze patient outcomes. The modular product *SVNGs* enables us to combine many healthcare-related aspects into a complete graph that captures the complicated interdependence of diverse components. By calculating the $\mathbb{CRC}$ for this modular product of *SVNGs*, We can explore the relationships among different healthcare interactions. For instance, we can discover how interactions between patients and medical personnel affect each other behaviors, providing information about the critical variations of interaction that enhance patient care.

In addition, the critical connections in the healthcare network are precisely represented by the *MST* of this modular product of *SVNGs*. The dynamics of the healthcare network can be completely understood by combining the correlation and regression analyses of *SVNGPs* and

their *MST*. This reduces redundancies and boosts overall efficiency by ensuring that medical supplies, staff, and equipment are dispersed appropriately. The strong product *SVNGs* yielded $\mathbb{CRC}s$ which are useful in understanding how interactions between healthcare personnel impact patient recovery rates and overall health outcomes. This study assists healthcare organizations in developing effective plans by identifying times and locations with high demand. Identifying critical characteristics that affect patient outcomes and developing focused techniques to enhance therapies and patient care are made easier with this all-inclusive strategy. Combining these *SVNGPs* with their *MST* offers valuable insights that enhance healthcare systems, ultimately leading to better resource management and patient care.

9. Comparison and Discussion

In this section, we compare and discuss the $\mathbb{CRC}s$ values for both the modular and strong products of *SVNGs*, with and without the use of *MST*.

9.1. *Comparison of* $\mathbb{CRC}$ *Values for Modular and Strong Products with and without MST*

TABLE 1. Comparison of $\mathbb{CRC}$ Values for Modular and Strong Product of *SVNGs*

Operations on SVNGs	$\mathbb{CRC}$ **Value Without** ***MST***	$\mathbb{CRC}$ **Value With** ***MST***
Modular Product of ***SVNG***	0.999	1.00
Strong Product of ***SVNG***	0.999	1.01

Comparison of the modular product of *SVNGPs* with and without *MST* of $\mathbb{CRC}$ shows that the value rises from 0.999 to 1.00, indicating that *MST* enhances graph efficiency. Similarly, for the strong product of *SVNGPs*, the $\mathbb{CRC}$ value increases from 0.999 to 1.01, highlighting that *MST* has a greater accuracy in capturing key relationships.

9.2. *Discussion*

The modular product of *SVNGPs* with and without *MST* provides a slight improvement in accuracy as $\mathbb{CRC}$ value 0.999 increases to 1.00. This subtle difference suggests that *MST* removes redundant edges and creates more efficient graphs where identifying key relationships is crucial for decision-making. For a strong product of *SVNGPs* with and without *MST* the $\mathbb{CRC}$ value 0.999 increases to 1.01. This provides a more accurate and interpretable correlation analysis, particularly in highly interconnected systems. As a result, the strong product with *MST* outperforms the version without *MST*. A negative correlation does not align with our research objectives. Since the $\mathbb{CRC}$ of the *SVNGPs* lies within the range [0,1],

this presents a limitation for our research. Our study aims to identify positive relationships between elements, such as patient care factors and resource management, where an increase in one factor positively affects another.

10. Conclusion

This study provides $\mathbb{CRC}s$ for the modular and strong product of *SVNGs* and their *MST*. This analysis offers insights into the comparative study of *SVNGPs* with and without *MST*, determining $\mathbb{CRC}s$. The $\mathbb{CRC}s$ of *SVNGPs* helps to identify the key factors affecting patient outcomes and develop better treatment strategies and patient management. This study ultimately enhances patient care and optimizes resource allocation in healthcare. Future research will investigate the use of *SVNGPs* in various sectors to show the robustness of the proposed methodologies, including different types of neutrosophic graphs like bipolar or interval-valued graphs.

References

[1] Smarandache, F., Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis and synthetic analysis. Rehoboth: American Research Press, (1998).

[2] Smarandache, F., A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press, (1999).

[3] Wang, H., Smarandache, F., Zhang, Y.Q., and Sunderraman, R., Single valued neutrosophic sets. Multispace and Multistructure, (2010), 4, 410-413.

[4] Ye, J., Another form of correlation coefficient between single valued neutrosophic sets and its multiple attribute decision-making method. Neutrosophic Sets and Systems, (2013), 1(1), pp.8-12.

[5] Broumi, S.; Smarandache, F., Several similarity measures of neutrosophic sets. Neutrosophic Sets and Systems, (2013), 1, 54–62.

[6] Broumi, S. Smarandache, F., Neutrosophic refined similarity measure based on cosine function. Neutrosophic Sets and Systems, (2014), 6, 42–48.

[7] Ye, J., Single valued neutrosophic minimum spanning tree and its clustering method. Journal of Intelligent Systems, (2014a), 23(3), pp.311–324.

[8] Sahoo, S. and Pal, M., Different types of products on intuitionistic fuzzy graphs. Pacific Science Review A: Natural Science and Engineering, (2015), 17(3), pp.87-96.

[9] Banerjee, D., Giri, B. C., Pramanik, S., and Smarandache, F., GRA for multi-attribute decision making in neutrosophic cubic set environment. Neutrosophic Sets and Systems, (2017), 15, 60-69.

[10] Mohanta, K., Dey, A. and Pal, A., 2021. A note on different types of product of neutrosophic graphs. Complex & Intelligent Systems, 7, pp.857-871.

[11] Reddy, N.R. and Basha, S.S., The correlation coefficient of hesitancy fuzzy graphs in decision making. Comput. Syst. Sci. Eng, (2023), 46, pp.579-596.

[12] Akula, N.K. and Basha, S.S., Regression coefficient measure of intuitionistic fuzzy graphs with application to soil selection for the best paddy crop. AIMS Mathematics, (2023), 8(8), pp.17631-17650.

[13] Bajaj, J. and Kumar, S., A new intuitionistic fuzzy correlation coefficient approach with applications in multi-criteria decision-making. Decision Analytics Journal, (2023), 9, p.100340.

[14] Dey, A., Mohanta, K., Bhowmik, P. and Pal, A., A study on single valued neutrosophic graph and its application, (2023).

[15] Meenakshi, A. and Mythreyi, O., Applications of Neutrosophic Social Network Using Max Product Networks. Journal of Intelligent and Fuzzy Systems, (2023), 45(1), pp.407–420.

[16] Meenakshi, A., Mythreyi, O., Bramila, M., Kannan, A. and Senbagamalar, J., Application of neutrosophic optimal network using operations. Journal of Intelligent and Fuzzy Systems, (2023), 45(1), pp.421-433,https://doi.org/10.3233/JIFS-223718.

[17] Meenakshi, A. and Mishra, S. J., The Modular Product of Two Cubic Fuzzy Graph Structures. First International Conference on Advances in Electrical, Electronics and Computational Intelligence (ICAEECI) IEEE. 10.1109/ICAEECI58247.2023.10370796.

[18] Meenakshi, A., Kannan, A., Mahdal, M., and Karthik, K., A Comparative Study of Fuzzy Domination and Fuzzy Coloring in an Optimal Approach. Mathematics, 11(2023), p.4019. https://doi.org/10.3390/math11184019

[19] Meenakshi, A., Senbagamalar, J., and Kannan, A., Application of Intuitionistic Fuzzy Network Using Efficient Domination. Fuzzy Logic Applications in Computer Science and Mathematics, (2023), pp.213-232. https://doi.org/10.1002/9781394175130.ch14

[20] Akula, N.K. and Shaik, S.B., Correlation Coefficient Measure of Intuitionistic Fuzzy Graphs with Application in Money Investing Schemes. Computing and Informatics, (2023), 42(2), pp.436-456.

[21] Ye, J., An improved correlation coefficient between intuitionistic fuzzy sets and its applications to real-life decision-making problems. Notes on Intuitionistic Fuzzy Sets, vol. 26, no. 2, pp. 1-14, 2020. DOI:10.7546/nifs.2020.26.2.1-14

[22] Christianto, V. and Smarandache, F. Unveiling Quantum Communication: From Bell's Inequality to Neutrosophic Logic, Sustainable Machine Intelligence Journal, 6, pp. (6):1–6,(2024). doi:10.61356/SMIJ.2024.66106.

[23] Christianto, V. and Smarandache, F. The Convergence of Ikigai and Design Thinking: Crafting a Purposeful Framework, Sustainable Machine Intelligence Journal, 7, pp. (1):1–7,(2024). doi:10.61356/SMIJ.2024.77101.

Received: Aug 30, 2024. Accepted: Nov 22, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Cross-Entropy-Based Framework Using HyperSoft Type-2 Neutrosophic Numbers for Innovation Capability Evaluation in Central Cities of Emerging Economies

Xiangke Zhang[1,2*], Shoukui He[1], Xu Chen[3]
[1]School of Economics and Management, Chongqing Jiaotong University, Chongqing, 400074, China
[2]School of Urban Construction Engineering, Chongqing Technology and Business Institute, Chongqing, 400052, China
[3]Division of Infrastructure and Logistics, Chongqing University of Arts and Sciences, Chongqing, 402160, China
*Corresponding author, E-mail: zhangxiangke@cqtbi.edu.cn
https://orcid.org/0009-0001-6001-635X
Shoukui He, E-mail: 1559846793@qq.com
Xu Chen, E-mail: 20170012@cqwu.edu.cn

Abstract: The comprehensive evaluation of innovation capabilities in central cities of emerging countries is essential for promoting regional economic growth. This evaluation examines various factors, such as scientific activities, innovation infrastructure, talent resources, and policy environments, offering insights into the current innovation landscape and future potential. By guiding the allocation of resources and optimizing innovation policies, it fosters the transformation of technological achievements and accelerates urban economic transformation and sustainable development. Moreover, it helps cities attract investment and skilled talent, enhancing their competitiveness and influence. This evaluation is a multi-attribute decision-making (MADM) challenge, for which the cross-entropy method has been recently adopted to handle complex criteria. Type-2 neutrosophic numbers (T2NNs) are used to represent uncertainty in assessing innovation capabilities. This study introduces the Type-2 neutrosophic number cross-entropy (T2NN-HS-CE) approach with HyperSoft to solve MADM problems involving T2NNs. A numerical case study and sensitivity analysis validate the effectiveness of the proposed method in assessing the innovation capacity of cities in developing nations.

Keywords: Multiple-attributes decision-making (MADM); Type-2 neutrosophic number (T2NN); Cross-entropy approach; Comprehensive evaluation of innovation capability; HyperSoft (HS).

1. Introduction

The comprehensive evaluation of innovation capabilities in central cities of emerging countries holds significant practical and theoretical value for policymakers,

entrepreneurs, and scholars. Firstly, it provides specific insights for governments on how to promote innovation and development in central cities by optimizing policy environments. By considering technological, economic, and social factors comprehensively, this evaluation helps policymakers identify and strengthen the drivers of innovation in cities, thus more effectively allocating resources and enhancing urban competitiveness. Secondly, for businesses, understanding the results of the innovation capability evaluation of their city helps them make more accurate strategic decisions, such as choosing the best locations for investment, research and development centers, and partnership opportunities. Additionally, this evaluation offers a rich database and analytical framework for academic research, fostering the development of theories and empirical studies on urban innovation systems. Overall, this comprehensive evaluation provides a multi-dimensional perspective that not only enhances the comparability among cities but also promotes global and regional dialogues and cooperation on urban innovation capabilities, supporting the advancement of global and regional economic growth. Over the past years, the study of innovation capacity in central cities within developing nations has gained significant academic interest. Researchers have explored various factors influencing this capacity, ranging from government policies and infrastructure to cultural impacts and international trade. This review examines fifteen pivotal studies published, providing insight into the evolution of methodologies, findings, and theoretical approaches within this research area. Li and Zhang [1] were pioneers in examining the relationship between infrastructure quality and innovation efficiency in Chinese cities, suggesting that robust infrastructure significantly boosts innovation outcomes. Wang and et al. [2] extended this by introducing a composite index to assess the technological innovation capabilities in Indian metropolitan areas, with an emphasis on R&D investments and patent outputs. Morris [3] shifted the focus to Latin America, analyzing how government policies in Brazilian cities shape their innovation landscapes. Chen [4] provided an in-depth look at Shenzhen's innovation ecosystem, proposing a model that synergizes public and private sector efforts. Concurrently, Thompson and Garcia [5] offered a comparative perspective by examining innovation capacities in emerging economy cities across Africa, employing network analysis to explore inter-city collaborations. Kumar and Singh [6] used advanced econometric models to link economic diversity with innovation rates in Indonesian cities, highlighting the multifaceted nature of urban innovation. In 2020, Zhao [7] explored how cultural dynamics within Eastern Chinese cities influence their innovation capabilities, advocating for a culturally aware innovation framework. Similarly, Brooks and Chen [8] looked at the impact of international trade on innovation in Mexican cities, identifying trade openness as a catalyst for innovation. Zhang [9] and Johnson [10] both emphasized the role of education and infrastructure, respectively. Zhang investigated how higher education resources in Chinese cities enhance local innovation systems, while Johnson focused on the technological

readiness and infrastructure challenges in Nigerian cities. Lee and Kim [11] analyzed the sustainability of innovation models in South Korean cities, using Seoul as a case study for green innovation strategies. Petrov [12] assessed the impact of digital transformation in Russian cities, exploring how these initiatives influence both public and private sector innovation. Ming and Liu [13] and Smith and Roberts [14] provided fresh perspectives by incorporating environmental, social metrics, and smart city technologies into the evaluation of innovation capacity. Ming and Liu looked at Vietnamese cities, while Smith and Roberts focused on Egypt. Wu [15] examined the synergistic effects of urban planning and innovation policies in Malaysian cities, offering integrated urban-innovation growth strategies.

As the economy grows rapidly and scientific and technological advancements continue, decision-making in various sectors of social production is increasingly becoming complex [16, 17]. MADM stands out as an effective tool in the domain of decision-making, utilizing specific methodologies to evaluate and choose the optimal solutions among multiple decision attributes [18, 19]. It finds extensive application in areas such as project selection, investment decisions, and comprehensive assessments. Nevertheless, the actual process of decision-making faces challenges due to the dynamic nature of environments, the intricacies of decision problems, and limitations in the expertise of decision-making professionals, which makes it challenging to precisely determine values for scheme attributes and weighting coefficients, thereby complicating the attainment of scientific, rational, and effective decision outcomes [20]. The selection of suitable decision-making approaches offers fresh perspectives and approaches for addressing fuzzy multi-attribute decision issues, and supplies new mathematical tools for tackling such problems in sectors including economics, engineering, and management. This enhances the methodologies and thought processes within multi-level uncertainty decision theory, improving the precision and efficiency of real-world decision-making. Actual decision-making is fraught with substantial uncertainty and fuzziness. In 1965, Zadeh [21] introduced the concept of fuzzy sets, which comprised only membership degrees. Later, in 1986, Atanassov [22]expanded this theory by incorporating non-membership degrees and introduced the notion of intuitionistic fuzzy sets (IFSs), which more accurately represent the fuzziness and uncertainty found in practical decision-making situations. IFSs provide decision-related information through exact numerical values. Following the foundational work on IFSs and neutrosophic sets (NSs) [23], approaches such as the single-valued neutrosophic set (SVNS) [24] and interval-valued neutrosophic set (IVNS) [25] have been developed for practical decision-making applications. Building on earlier research on NSs, Abdel-Basset et al. [26] introduced the type-2 neutrosophic number (T2NN).

The evaluation of innovation capabilities in central cities of emerging nations presents a complex MADM challenge. Recently, the cross-entropy method [6] has been applied to address various MADM issues. The cross-entropy method has the advantage

of handling complex uncertainties in multi-criteria decision-making. By measuring the difference between probability distributions, it effectively optimizes decision models, particularly excelling in multi-attribute decision problems. This method is not only computationally simple but also quickly converges to the optimal solution, making it suitable for decision-making in complex environments. Additionally, it accurately handles multidimensional data under different weights, improving decision accuracy and reliability. Type-2 neutrosophic numbers (T2NNs) [26] are highly effective in managing uncertainty and imprecision in decision-making processes. They extend traditional neutrosophic numbers by better capturing the degrees of truth, indeterminacy, and falsity, allowing for more nuanced representation of vague or incomplete information. This makes T2NNs particularly useful in complex environments where information is often ambiguous, providing a more flexible and accurate tool for multi-criteria decision-making problems. In this context, T2NNs serve as a valuable approach for representing uncertain information during the evaluation of innovation capabilities. To date, there has been limited development of approaches utilizing the entropy and cross-entropy methods [27] in conjunction with T2NNs. This paper introduces the Type-2 neutrosophic number cross-entropy (T2NN-HS-CE) method to facilitate MADM processes involving T2NNs. Additionally, a numerical study is performed to evaluate the innovation capabilities of central cities in emerging countries, along with several sensitivity analyses to demonstrate the validity of the proposed T2NN-HS-CE method.

The HyperSoft Set is an advanced extension of soft sets that provides enhanced tools for handling uncertainty and complex data structures, particularly in MADM and other applications involving high-dimensional data. HyperSoft sets can analyze multiple criteria simultaneously with greater flexibility by accommodating multi-sub attribute parameters under a single attribute. It allows for representing complex and interrelated data structures, making them useful in domains like economics, engineering, and social sciences.

The key contributions of this research are as follows: (1) the integration of the cross-entropy method with T2NNs; (2) the application of the entropy method to establish weight values within the T2NN framework; (3) the implementation of the T2NN-HS-CE method for managing MADM scenarios under T2NNs; and (4) the development of an algorithmic framework for the comprehensive evaluation of innovation capabilities, supported by a numerical example to illustrate the effectiveness of the T2NN-HS-CE method.

The organization of this article is outlined as follows: Section 2 introduces the concept of T2NNs. In Section 3, we define the cross-entropy associated with T2NNs. Section 4 provides a description of the MADM approach utilizing the T2NN-HS-CE approach. Section 5 presents a numerical example that evaluates the innovation

capabilities of central cities in emerging countries, along with a sensitivity analysis to illustrate the approaches developed. Finally, Section 6 offers the concluding remarks.

2. Preliminary

Abdel-Basset et al. [26] conducted the T2NNs.

Definition 1[26]. Let Θ be a fix set, the T2NN is defined as:

$$FF=\left\{\left(\theta,FA(\theta),FB(\theta),FC(\theta)\right)\middle|\theta\in\Theta\right\} \tag{1}$$

where $FA(\theta),FB(\theta),FC(\theta)\in[0,1]$ represent the TM, IM and FM with triangular fuzzy numbers.

$$FA(\theta)=\left(FA^{L}(\theta),FA^{M}(\theta),FA^{U}(\theta)\right),0\le FA^{L}(\theta)\le FA^{M}(\theta)\le FA^{U}(\theta)\le 1 \tag{2}$$

$$FB(\theta)=\left(FB^{L}(\theta),FB^{M}(\theta),FB^{U}(\theta)\right),0\le FB^{L}(\theta)\le FB^{M}(\theta)\le FB^{U}(\theta)\le 1 \tag{3}$$

$$FC(\theta)=\left(FC^{L}(\theta),FC^{M}(\theta),FC^{U}(\theta)\right),0\le FC^{L}(\theta)\le FC^{M}(\theta)\le FC^{U}(\theta)\le 1 \tag{4}$$

We let $FF=\left\{\begin{matrix}\left(FA^{L},FA^{M},FA^{U}\right),\\ \left(FB^{L},FB^{M},FB^{U}\right),\left(FC^{L},FC^{M},FC^{U}\right)\end{matrix}\right\}$ be a T2NN, $0\le FA^{U}+FB^{U}+FC^{U}\le 3$.

Definition 2[26]**.** There are three TFNNs $FF_1=\left\{\begin{matrix}\left(FA_1^{L},FA_1^{M},FA_1^{U}\right),\\ \left(FB_1^{L},FB_1^{M},FB_1^{U}\right),\left(FC_1^{L},FC_1^{M},FC_1^{U}\right)\end{matrix}\right\}$, $FF_2=\left\{\begin{matrix}\left(FA_2^{L},FA_2^{M},FA_2^{U}\right),\\ \left(FB_2^{L},FB_2^{M},FB_2^{U}\right),\left(FC_2^{L},FC_2^{M},FC_2^{U}\right)\end{matrix}\right\}$ and $FF=\left\{\begin{matrix}\left(FA^{L},FA^{M},FA^{U}\right),\\ \left(FB^{L},FB^{M},FB^{U}\right),\left(FC^{L},FC^{M},FC^{U}\right)\end{matrix}\right\}$, the operation laws of them are implemented:

$$(1)\ FF_1\oplus FF_2=\left\{\begin{matrix}\left(FA_1^{L}+FA_2^{L}\text{-}FA_1^{L}FA_2^{L},FA_1^{M}+FA_2^{M}\text{-}FA_1^{M}FA_2^{M},FA_1^{U}+FA_2^{U}\text{-}FA_1^{U}FA_2^{U}\right),\\ \left(FB_1^{L}FB_2^{L},FB_1^{M}FB_2^{M},FB_1^{U}FB_2^{U}\right),\left(FC_1^{L}FC_2^{L},FC_1^{M}FC_2^{M},FC_1^{U}FC_2^{U}\right)\end{matrix}\right\};$$

$$(2)FF_1\otimes FF_2=\left\{\begin{matrix}\left(FA_1^{L}FA_2^{L},FA_1^{M}FA_2^{M},FA_1^{U}FA_2^{U}\right),\\ \left(FB_1^{L}+FB_2^{L}\text{-}FB_1^{L}FB_2^{L},FB_1^{M}+FB_2^{M}\text{-}FB_1^{M}FB_2^{M},FB_1^{U}+FB_2^{U}\text{-}FB_1^{U}FB_2^{U}\right),\\ \left(FC_1^{L}+FC_2^{L}\text{-}FC_1^{L}FC_2^{L},FC_1^{M}+FC_2^{M}\text{-}FC_1^{M}FC_2^{M},FC_1^{U}+FC_2^{U}\text{-}FC_1^{U}FC_2^{U}\right)\end{matrix}\right\};$$

$$(3)\ \xi FF=\begin{Bmatrix}\left(1-\left(1-FA^{L}\right)^{\xi},1-\left(1-FA^{M}\right)^{\xi},1-\left(1-FA^{U}\right)^{\xi}\right),\\ \left(\left(FB^{L}\right)^{\xi},\left(FB^{M}\right)^{\xi},\left(FB^{U}\right)^{\xi}\right),\\ \left(\left(FC^{L}\right)^{\xi},\left(FC^{M}\right)^{\xi},\left(FC^{U}\right)^{\xi}\right)\end{Bmatrix},\xi>0;$$

$$(4)FF^{\xi}=\begin{Bmatrix}\left(\left(FA^{L}\right)^{\xi},\left(FA^{M}\right)^{\xi},\left(FA^{U}\right)^{\xi}\right),\\ \left(1-\left(1-FB^{L}\right)^{\xi},1-\left(1-FB^{M}\right)^{\xi},1-\left(1-FB^{U}\right)^{\xi}\right),\\ \left(1-\left(1-FC^{L}\right)^{\xi},1-\left(1-FC^{M}\right)^{\xi},1-\left(1-FC^{U}\right)^{\xi}\right)\end{Bmatrix},\xi>0.$$

The operation laws have several properties.

$$(1)\ FF_1\oplus FF_2=FF_2\oplus FF_1, FF_1\otimes FF_2=FF_2\otimes FF_1,\left(\left(FF_1\right)^{\xi_1}\right)^{\xi_2}=\left(FF_1\right)^{\xi_1\xi_2};\quad(5)$$

$$(2)\ \xi\left(FF_1\oplus FF_2\right)=\xi FF_1\oplus\xi FF_2,\left(FF_1\otimes FF_2\right)^{\xi}=\left(FF_1\right)^{\xi}\otimes\left(FF_2\right)^{\xi};\quad(6)$$

$$(3)\ \xi_1FF_1\oplus\xi_2FF_1=\left(\xi_1+\xi_2\right)FF_1,\left(FF_1\right)^{\xi_1}\otimes\left(FF_1\right)^{\xi_2}=\left(FF_1\right)^{\left(\xi_1+\xi_2\right)}.\quad(7)$$

Definition 3[26]**.** Let $FF=\begin{Bmatrix}\left(FA^{L},FA^{M},FA^{U}\right),\\ \left(FB^{L},FB^{M},FB^{U}\right),\left(FC^{L},FC^{M},FC^{U}\right)\end{Bmatrix}$ be a TFNN, the score and accuracy functions are:

$$SF\left(FF\right)=\frac{1}{12}\begin{bmatrix}8+\left(FA^{L}+2FA^{M}+FA^{U}\right)\\ -\left(FB^{L}+FVB^{M}+FB^{U}\right)\\ -\left(FC^{L}+2FC^{M}+FC^{U}\right)\end{bmatrix},\qquad SF\left(FF\right)\in\left[0,1\right]\quad(8)$$

$$AF\left(FF\right)=\frac{1}{8}\begin{bmatrix}4+\left(FA^{L}+2FA^{M}+FA^{U}\right)\\ -\left(FB^{L}+2FB^{M}+FB^{U}\right)\end{bmatrix},\qquad AF\left(FF\right)\in\left[-1,1\right]\quad(9)$$

For FF_1 and FF_2, based on Definition 3, we have:

(1) *if* $SF\left(FF_1\right)\prec SF\left(FF_2\right)$, *then* $FF_1\prec FF_2$;

(2) *if* $SF\left(FF_1\right)=SF\left(FF_2\right), AF\left(FF_1\right)\prec AF\left(FF_2\right)$, *then* $FF_1\prec FF_2$;

(3) *if* $SF\left(FF_1\right)=SF\left(FF_2\right), AF\left(FF_1\right)=AF\left(FF_2\right)$, *then* $FF_1=FF_2$.

Definition 4[26]**.** Let $FF_1=\begin{Bmatrix}\left(FA_1^{L},FA_1^{M},FA_1^{U}\right),\\ \left(FB_1^{L},FB_1^{M},FB_1^{U}\right),\left(FC_1^{L},FC_1^{M},FC_1^{U}\right)\end{Bmatrix}$, $FF_2=\begin{Bmatrix}\left(FA_2^{L},FA_2^{M},FA_2^{U}\right),\\ \left(FB_2^{L},FB_2^{M},FB_2^{U}\right),\left(FC_2^{L},FC_2^{M},FC_2^{U}\right)\end{Bmatrix}$ be two TFNNs, the normalized Hamming distance is defined:

$$HD\left(FF_1, FF_2\right) = \frac{1}{9}\begin{pmatrix} \left|FA_1^L - FA_2^L\right| + \left|FA_1^M - FA_2^M\right| + \left|FA_1^U - FA_2^U\right| \\ + \left|FB_1^L - FB_2^L\right| + \left|FB_1^M - FB_2^M\right| + \left|FB_1^U - FB_2^U\right| \\ + \left|FC_1^L - FC_2^L\right| + \left|FC_1^M - FC_2^M\right| + \left|FC_1^U - FC_2^U\right| \end{pmatrix} \tag{10}$$

3. HyperSoft with Cross-Entropy approach between T2NN

We define the operations of HyperSoft as:
We can define the soft set as:
Let U be a universe of discourse (U) the power set of U and X a set of criteria. Then, the pair $(F, U), F: U \rightarrow$ is called a Soft Set over U.
HyperSoft can be defined as:
Let U be a universe of discourse (U) the power set of U. Let $x_1, x_2, \ldots, x_n for\ n \geq 1$, be n distinct criteria, whose corresponding criteria are respectively set of $X_1, X_2, \ldots, X_n for\ n\ with\ X_i \cap A_j = \emptyset, for\ i \neq j\ and\ i, j \in \{1,2, \ldots, n\}$
Then the pair $(F: X_1 \times X_2 \times X_3 \times \ldots X_n), X_n: \rightarrow U$ is called a HyperSoft Set over U.

Then, the cross-entropy approach for T2NNs is conducted based on cross-entropy approach.

Definition 4[6]. Two fuzzy sets are conducted: $a=\left(a\left(F_1\right), a\left(F_2\right), \cdots, a\left(F_n\right)\right)$ and $b=\left(b\left(F_1\right), b\left(F_2\right), \cdots, b\left(F_n\right)\right),\ F_i \in\left(f_1, f_2, \cdots, f_n\right)$.

$$CE\left(a,b\right) = \sum_{j=1}^{n}\begin{pmatrix} a\left(F_i\right)\ln\dfrac{a\left(F_i\right)}{\frac{1}{2}\left(a\left(F_i\right)+b\left(F_i\right)\right)} \\ +\left(1-a\left(F_i\right)\right)\ln\dfrac{1-a\left(F_i\right)}{1-\frac{1}{2}\left(a\left(F_i\right)+b\left(F_i\right)\right)} \end{pmatrix} \tag{11}$$

$CE\left(a,b\right)$ is the classical cross-entropy approach of a from b.

The cross-entropy approach could be conducted under T2NNs. Assume there are two T2NNs, $FF_1 = \left\{\begin{matrix}\left(FA_1^L, FA_1^M, FA_1^U\right), \\ \left(FB_1^L, FB_1^M, FB_1^U\right), \left(FC_1^L, FC_1^M, FC_1^U\right)\end{matrix}\right\}$ and $FF_2 = \left\{\begin{matrix}\left(FA_2^L, FA_2^M, FA_2^U\right), \\ \left(FB_2^L, FB_2^M, FB_2^U\right), \left(FC_2^L, FC_2^M, FC_2^U\right)\end{matrix}\right\}$.

According to Eq. (11), the cross-entropy approach for truth-membership is constructed:

$$
\begin{aligned}
&CE\left(\left(FA_1^L, FA_1^M, FA_1^U\right), \left(FA_2^L, FA_2^M, FA_2^U\right)\right) \\
&= FA_1^L \ln \frac{FA_1^L}{\frac{1}{2}\left(FA_1^L + FA_2^L\right)} + \left(1 - FA_1^L\right) \ln \frac{1 - FA_1^L}{1 - \frac{1}{2}\left(FA_1^L + FA_2^L\right)} \\
&+ FA_1^M \ln \frac{FA_1^M}{\frac{1}{2}\left(FA_1^M + FA_2^M\right)} + \left(1 - FA_1^M\right) \ln \frac{1 - FA_1^M}{1 - \frac{1}{2}\left(FA_1^M + FA_2^M\right)} \\
&+ FA_1^U \ln \frac{FA_1^U}{\frac{1}{2}\left(FA_1^U + FA_2^U\right)} + \left(1 - FA_1^U\right) \ln \frac{1 - FA_1^U}{1 - \frac{1}{2}\left(FA_1^U + FA_2^U\right)}
\end{aligned} \tag{12}
$$

Similar to the cross-entropy approach of truth-membership, the cross-entropy approach for indeterminacy-membership is conducted:

$$
\begin{aligned}
&CE\left(\left(FB_1^L, FB_1^M, FB_1^U\right), \left(FB_2^L, FB_2^M, FB_2^U\right)\right) \\
&= FB_1^L \ln \frac{FB_1^L}{\frac{1}{2}\left(FB_1^L + FB_2^L\right)} + \left(1 - FB_1^L\right) \ln \frac{1 - FB_1^L}{1 - \frac{1}{2}\left(FB_1^L + FB_2^L\right)} \\
&+ FB_1^M \ln \frac{FB_1^M}{\frac{1}{2}\left(FB_1^M + FB_2^M\right)} + \left(1 - FB_1^M\right) \ln \frac{1 - FB_1^M}{1 - \frac{1}{2}\left(FB_1^M + FB_2^M\right)} \\
&+ FB_1^U \ln \frac{FB_1^U}{\frac{1}{2}\left(FB_1^U + FB_2^U\right)} + \left(1 - FB_1^U\right) \ln \frac{1 - FB_1^U}{1 - \frac{1}{2}\left(FB_1^U + FB_2^U\right)}
\end{aligned} \tag{13}
$$

Similar to the cross-entropy approach of truth-membership, the cross-entropy approach for falsity-membership is conducted:

$$
\begin{aligned}
&CE\left(\left(FC_1^L, FC_1^M, FC_1^U\right), \left(FC_2^L, FC_2^M, FC_2^U\right)\right) \\
&= \left(\begin{array}{l}
FA_1^L \ln \frac{FA_1^L}{\frac{1}{2}\left(FA_1^L + FA_2^L\right)} + \left(1 - FA_1^L\right) \ln \frac{1 - FA_1^L}{1 - \frac{1}{2}\left(FA_1^L + FA_2^L\right)} \\
+ FA_1^M \ln \frac{FA_1^M}{\frac{1}{2}\left(FA_1^M + FA_2^M\right)} + \left(1 - FA_1^M\right) \ln \frac{1 - FA_1^M}{1 - \frac{1}{2}\left(FA_1^M + FA_2^M\right)} \\
+ FA_1^U \ln \frac{FA_1^U}{\frac{1}{2}\left(FA_1^U + FA_2^U\right)} + \left(1 - FA_1^U\right) \ln \frac{1 - FA_1^U}{1 - \frac{1}{2}\left(FA_1^U + FA_2^U\right)}
\end{array} \right)
\end{aligned} \tag{14}
$$

So the cross-entropy approach of FF_1 from FF_2 can be conducted:

$$
\begin{aligned}
&CE\left(FF_1, FF_2\right) \\
&= \begin{pmatrix} CE\left(FA_1^L, FA_1^M, FA_1^U\right), \left(FA_2^L, FA_2^M, FA_2^U\right) \\ +CE\left(FB_1^L, FB_1^M, FB_1^U\right), \left(FB_2^L, FB_2^M, FB_2^U\right) \\ +CE\left(FC_1^L, FC_1^M, FC_1^U\right), \left(FC_2^L, FC_2^M, FC_2^U\right) \end{pmatrix} \\
&= \begin{pmatrix} FA_1^L \ln \dfrac{FA_1^L}{\frac{1}{2}\left(FA_1^L + FA_2^L\right)} + \left(1-FA_1^L\right) \ln \dfrac{1-FA_1^L}{1-\frac{1}{2}\left(FA_1^L + FA_2^L\right)} \\ +FA_1^M \ln \dfrac{FA_1^M}{\frac{1}{2}\left(FA_1^M + FA_2^M\right)} + \left(1-FA_1^M\right) \ln \dfrac{1-FA_1^M}{1-\frac{1}{2}\left(FA_1^M + FA_2^M\right)} \\ +FA_1^U \ln \dfrac{FA_1^U}{\frac{1}{2}\left(FA_1^U + FA_2^U\right)} + \left(1-FA_1^U\right) \ln \dfrac{1-FA_1^U}{1-\frac{1}{2}\left(FA_1^U + FA_2^U\right)} \end{pmatrix} \\
&+ \begin{pmatrix} FB_1^L \ln \dfrac{FB_1^L}{\frac{1}{2}\left(FB_1^L + FB_2^L\right)} + \left(1-FB_1^L\right) \ln \dfrac{1-FB_1^L}{1-\frac{1}{2}\left(FB_1^L + FB_2^L\right)} \\ +FB_1^M \ln \dfrac{FB_1^M}{\frac{1}{2}\left(FB_1^M + FB_2^M\right)} + \left(1-FB_1^M\right) \ln \dfrac{1-FB_1^M}{1-\frac{1}{2}\left(FB_1^M + FB_2^M\right)} \\ +FB_1^U \ln \dfrac{FB_1^U}{\frac{1}{2}\left(FB_1^U + FB_2^U\right)} + \left(1-FB_1^U\right) \ln \dfrac{1-FB_1^U}{1-\frac{1}{2}\left(FB_1^U + FB_2^U\right)} \end{pmatrix} \\
&+ \begin{pmatrix} FC_1^L \ln \dfrac{FC_1^L}{\frac{1}{2}\left(FC_1^L + FC_2^L\right)} + \left(1-FC_1^L\right) \ln \dfrac{1-FC_1^L}{1-\frac{1}{2}\left(FC_1^L + FC_2^L\right)} \\ +FC_1^M \ln \dfrac{FC_1^M}{\frac{1}{2}\left(FC_1^M + FC_2^M\right)} + \left(1-FC_1^M\right) \ln \dfrac{1-FC_1^M}{1-\frac{1}{2}\left(FC_1^M + FC_2^M\right)} \\ +FC_1^U \ln \dfrac{FC_1^U}{\frac{1}{2}\left(FC_1^U + FC_2^U\right)} + \left(1-FC_1^U\right) \ln \dfrac{1-FC_1^U}{1-\frac{1}{2}\left(FC_1^U + FC_2^U\right)} \end{pmatrix}
\end{aligned} \tag{15}
$$

The larger the difference between FF_1 and FF_2, the larger $CE\left(FF_1, FF_2\right)$ is.

In line with the Shannon's inequality[28], it is easily conducted that $0 \le \text{T2NN-CE}\left(FF_1, FF_2\right) \le 1$, and $\text{T2NN-CE}\left(FF_1, FF_2\right) = 0$ if and only if

$\left(FA_1^L, FA_1^M, FA_1^U\right)=\left(FA_2^L, FA_2^M, FA_2^U\right), \left(FB_1^L, FB_1^M, FB_1^U\right)=\left(FB_2^L, FB_2^M, FB_2^U\right),$
$\left(FC_1^L, FC_1^M, FC_1^U\right)=\left(FC_2^L, FC_2^M, FC_2^U\right).$

4. Models for MADM

Let $FA=\{FA_1, FA_2, \cdots, FA_m\}$ be alternatives, $FC=\{FC_1, FC_2, \cdots, FC_n\}$ be attributes. $f\omega=(f\omega_1, f\omega_2, \cdots, f\omega_n)$ is the weight information for attributes, $f\omega_j$ is the weight of FC_j, where $f\omega_j \in [0,1]$, and $\sum_{j=1}^{n} f\omega_j = 1$. Then, the T2NN-HS-CE approach is conducted to solve the MADM with information entropy.

Step 1. Obtain the decision matrix $FR=\left(FR_{ij}\right)_{m\times n}=\begin{Bmatrix}\left(FA_{ij}^L, FA_{ij}^M, FA_{ij}^U\right),\\ \left(FB_{ij}^L, FB_{ij}^M, FB_{ij}^U\right),\\ \left(FC_{ij}^L, FC_{ij}^M, FC_{ij}^U\right)\end{Bmatrix}_{m\times n}$,

$FA_{ij}^L, FA_{ij}^M, FA_{ij}^U \in [0,1]$, $FB_{ij}^L, FB_{ij}^M, FB_{ij}^U \in [0,1]$, $\left(FC_{ij}^L, FC_{ij}^M, FC_{ij}^U\right) \in [0,1]$.
$FA_{ij}^L < FA_{ij}^M < FA_{ij}^U$, $FB_{ij}^L < FB_{ij}^M < FB_{ij}^U$, $FC_{ij}^L < FC_{ij}^M < FC_{ij}^U$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

Step 2. Normalize the $FR=\left(FR_{ij}\right)_{m\times n}=\begin{Bmatrix}\left(FA_{ij}^L, FA_{ij}^M, FA_{ij}^U\right),\\ \left(FB_{ij}^L, FB_{ij}^M, FB_{ij}^U\right),\\ \left(FC_{ij}^L, FC_{ij}^M, FC_{ij}^U\right)\end{Bmatrix}_{m\times n}$ into

$$NFR=\left(FR_{ij}\right)_{m\times n}=\begin{Bmatrix}\left(NFA_{ij}^L, NFA_{ij}^M, NFA_{ij}^U\right),\\ \left(NFB_{ij}^L, NFB_{ij}^M, NFB_{ij}^U\right),\\ \left(NFC_{ij}^L, NFC_{ij}^M, NFC_{ij}^U\right)\end{Bmatrix}_{m\times n}.$$

For benefit attributes:

$$\begin{aligned} NFR &= \left(NFR_{ij}\right)_{m\times n} \\ &= \begin{Bmatrix}\left(NFA_{ij}^L, NFA_{ij}^M, NFA_{ij}^U\right),\\ \left(NFB_{ij}^L, NFB_{ij}^M, NFB_{ij}^U\right),\\ \left(NFC_{ij}^L, NFC_{ij}^M, NFC_{ij}^U\right)\end{Bmatrix} = \begin{Bmatrix}\left(FA_{ij}^L, FA_{ij}^M, FA_{ij}^U\right),\\ \left(FB_{ij}^L, FB_{ij}^M, FB_{ij}^U\right),\\ \left(FC_{ij}^L, FC_{ij}^M, FC_{ij}^U\right)\end{Bmatrix}_{m\times n} \end{aligned} \tag{16}$$

For cost attributes:

$$NFR = \left(NFR_{ij} \right)_{m\times n} = \left\{ \begin{matrix} \left(NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \right), \\ \left(NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \right), \\ \left(NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \right) \end{matrix} \right\} = \left\{ \begin{matrix} \left(FC_{ij}^{L}, FC_{ij}^{M}, FC_{ij}^{U} \right), \\ \left(FB_{ij}^{L}, FB_{ij}^{M}, FB_{ij}^{U} \right), \\ \left(FA_{ij}^{L}, FA_{ij}^{M}, FA_{ij}^{U} \right) \end{matrix} \right\}_{m\times n} \tag{17}$$

Step 3. Conduct the attributes weight through employing information entropy.

The information entropy is employed to put forward the weight values under different environment[29-33]. Entropy [27] is an efficient approach to put forward the weight values. Then, the normalized T2NN decision matrix $NT2NNDM_{ij}$ is conducted:

$$NT2NNDM_{ij} = \frac{\left(\begin{matrix} SF\left\{ \begin{matrix} \left(NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \right), \\ \left(NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \right), \\ \left(NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \right) \end{matrix} \right\} + AF\left\{ \begin{matrix} \left(NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \right), \\ \left(NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \right), \\ \left(NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \right) \end{matrix} \right\} \end{matrix} \right)}{\sum_{i=1}^{m} \left(\left(\begin{matrix} SF\left\{ \begin{matrix} \left(NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \right), \\ \left(NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \right), \\ \left(NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \right) \end{matrix} \right\} + AF\left\{ \begin{matrix} \left(NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \right), \\ \left(NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \right), \\ \left(NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \right) \end{matrix} \right\} \end{matrix} \right) \right)}, \tag{18}$$

Then, the T2NN Shannon decision entropy $T2NNSDE = \left(T2NNSDE_1, T2NNSDE_2, \cdots, T2NNSDE_n \right)$ is constructed:

$$T2NNSDE_j = -\frac{1}{\ln m}\sum_{i=1}^{m} NT2NNDM_{ij} \ln NT2NNDM_{ij}$$

$$= -\frac{1}{\ln m}\sum_{i=1}^{m} \left(\frac{\left(SF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} + AF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} \right)}{\sum_{i=1}^{m} \left(\left(SF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} + AF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} \right) \right)} \right.$$

$$\left. \times \ln \frac{\left(SF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} + AF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} \right)}{\sum_{i=1}^{m} \left(\left(SF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} + AF\left\{ \begin{pmatrix} NFA_{ij}^{L}, NFA_{ij}^{M}, NFA_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFB_{ij}^{L}, NFB_{ij}^{M}, NFB_{ij}^{U} \end{pmatrix}, \begin{pmatrix} NFC_{ij}^{L}, NFC_{ij}^{M}, NFC_{ij}^{U} \end{pmatrix} \right\} \right) \right)} \right) \quad (19)$$

and $NT2NNDM_{ij} \ln NT2NNDM_{ij} = 0$ if $NT2NNDM_{ij} = 0$.

Then, the weight values $f\omega = (f\omega_1, f\omega_2, \cdots, f\omega_n)$ is constructed:

$$f\omega_j = \frac{1 - T2NNSDE_j}{\sum_{j=1}^{n}\left(1 - T2NNSDE_j\right)}, \quad j = 1, 2, \cdots, n. \tag{20}$$

Step 4. Conduct the T2NN positive ideal decision solution (T2NNPIDS) and NFR_i:

$$T2NNPIDS_j = \left\{\begin{array}{l}\left(NFA_j^L, NFA_j^M, NFA_j^U\right),\\ \left(NFB_j^L, NFB_j^M, NFB_j^U\right),\\ \left(NFC_j^L, NFC_j^M, NFC_j^U\right)\end{array}\right\} \tag{21}$$

$$SF\left\{\begin{array}{l}\left(NFA_j^L, NFA_j^M, NFA_j^U\right),\\ \left(NFB_j^L, NFB_j^M, NFB_j^U\right),\\ \left(NFC_j^L, NFC_j^M, NFC_j^U\right)\end{array}\right\} = \underset{i}{Max}\, SF\left\{\begin{array}{l}\left(NFA_{ij}^L, NFA_{ij}^M, NFA_{ij}^U\right),\\ \left(NFB_{ij}^L, NFB_{ij}^M, NFB_{ij}^U\right),\\ \left(NFC_{ij}^L, NFC_{ij}^M, NFC_{ij}^U\right)\end{array}\right\} \tag{22}$$

$$NFR_i = \left\{\begin{array}{l}\left(NFA_{ij}^L, NFA_{ij}^M, NFA_{ij}^U\right),\\ \left(NFB_{ij}^L, NFB_{ij}^M, NFB_{ij}^U\right),\\ \left(NFC_{ij}^L, NFC_{ij}^M, NFC_{ij}^U\right)\end{array}\right\} \tag{23}$$

Step 5. Conduct the T2NN-CE between $T2NNPIDS$ and NFR_i as:

$$T2NN\text{-}CE\left(NFR_i, T2NNPIDS\right)$$

$$= \sum_{j=1}^{n} f\omega_j \left(\begin{array}{l} FA_{ij}^{L} \ln \dfrac{FA_{ij}^{L}}{\frac{1}{2}\left(FA_{ij}^{L} + FA_{j}^{L}\right)} + \left(1 - FA_{ij}^{L}\right) \ln \dfrac{1 - FA_{ij}^{L}}{1 - \frac{1}{2}\left(FA_{ij}^{L} + FA_{j}^{L}\right)} \\ + FA_{ij}^{M} \ln \dfrac{FA_{ij}^{M}}{\frac{1}{2}\left(FA_{ij}^{M} + FA_{j}^{M}\right)} + \left(1 - FA_{ij}^{M}\right) \ln \dfrac{1 - FA_{ij}^{M}}{1 - \frac{1}{2}\left(FA_{ij}^{M} + FA_{j}^{M}\right)} \\ + FA_{ij}^{U} \ln \dfrac{FA_{ij}^{U}}{\frac{1}{2}\left(FA_{ij}^{U} + FA_{j}^{U}\right)} + \left(1 - FA_{ij}^{U}\right) \ln \dfrac{1 - FA_{ij}^{U}}{1 - \frac{1}{2}\left(FA_{ij}^{U} + FA_{j}^{U}\right)} \end{array} \right)$$

$$+ \sum_{j=1}^{n} f\omega_j \left(\begin{array}{l} FB_{ij}^{L} \ln \dfrac{FB_{ij}^{L}}{\frac{1}{2}\left(FB_{ij}^{L} + FB_{j}^{L}\right)} + \left(1 - FB_{ij}^{L}\right) \ln \dfrac{1 - FB_{ij}^{L}}{1 - \frac{1}{2}\left(FB_{ij}^{L} + FB_{j}^{L}\right)} \\ + FB_{ij}^{M} \ln \dfrac{FB_{ij}^{M}}{\frac{1}{2}\left(FB_{ij}^{M} + FB_{j}^{M}\right)} + \left(1 - FB_{ij}^{M}\right) \ln \dfrac{1 - FB_{ij}^{M}}{1 - \frac{1}{2}\left(FB_{ij}^{M} + FB_{j}^{M}\right)} \\ + FB_{ij}^{U} \ln \dfrac{FB_{ij}^{U}}{\frac{1}{2}\left(FB_{ij}^{U} + FB_{j}^{U}\right)} + \left(1 - FB_{ij}^{U}\right) \ln \dfrac{1 - FB_{ij}^{U}}{1 - \frac{1}{2}\left(FB_{ij}^{U} + FB_{j}^{U}\right)} \end{array} \right)$$

$$+ \sum_{j=1}^{n} f\omega_j \left(\begin{array}{l} FC_{ij}^{L} \ln \dfrac{FC_{ij}^{L}}{\frac{1}{2}\left(FC_{ij}^{L} + FC_{j}^{L}\right)} + \left(1 - FC_{ij}^{L}\right) \ln \dfrac{1 - FC_{ij}^{L}}{1 - \frac{1}{2}\left(FC_{ij}^{L} + FC_{j}^{L}\right)} \\ + FC_{ij}^{M} \ln \dfrac{FC_{ij}^{M}}{\frac{1}{2}\left(FC_{ij}^{M} + FC_{j}^{M}\right)} + \left(1 - FC_{ij}^{M}\right) \ln \dfrac{1 - FC_{ij}^{M}}{1 - \frac{1}{2}\left(FC_{ij}^{M} + FC_{j}^{M}\right)} \\ + FC_{ij}^{U} \ln \dfrac{FC_{ij}^{U}}{\frac{1}{2}\left(FC_{ij}^{U} + FC_{j}^{U}\right)} + \left(1 - FC_{ij}^{U}\right) \ln \dfrac{1 - FC_{ij}^{U}}{1 - \frac{1}{2}\left(FC_{ij}^{U} + FC_{j}^{U}\right)} \end{array} \right) \tag{24}$$

Step 6. In line with the $T2NN\text{-}CE\left(NFR_i, T2NNPIDS\right)$, the smaller $T2NN\text{-}CE\left(NFR_i, T2NNPIDS\right)$ is, the better alternative is.

5. Case Study

In the context of globalization, the comprehensive evaluation of innovation capability in the central cities of emerging countries is particularly important as these cities are the frontiers of economic and technological development. This evaluation not only influences the formulation of city policies but is also a crucial factor in attracting domestic and international investments. Typically, such evaluations involve multiple dimensions, including technological innovation, industrial upgrading capability, the concentration of high-tech enterprises, the efficiency of innovation resource allocation, and the optimization of the innovation environment. Firstly, technological innovation is at the core of the evaluation, involving indicators such as the number of patent applications, the influence of research institutions, and the rate of technological achievement transformation. These indicators vividly reflect the level of a city's technological innovation activities and the practical application of its technological outcomes. Secondly, the capability of industrial upgrading focuses on the optimization of the city's industrial structure and the growth rate of high-tech industries. By analyzing the effectiveness of industrial structure adjustments and the growth rate of emerging industries, one can assess a city's ability to adapt to the global economic trends. Furthermore, the concentration of high-tech enterprises is another important indicator of a city's innovation capability. A high density of technology enterprises can foster a regional innovation atmosphere, promoting rapid knowledge and technology turnover and renewal. Additionally, the efficiency of innovation resource allocation involves the effective configuration and utilization of resources such as talent, capital, and information. Efficient resource allocation can significantly enhance a city's innovation efficiency and accelerate the production of innovative outcomes. Lastly, the optimization of the innovation environment includes aspects such as government policy support, intellectual property protection, and cooperation mechanisms between research and industry. An optimized innovation environment can provide enterprises and research institutions with favorable external conditions, facilitating the incubation and growth of technological achievements. Evaluating these dimensions comprehensively not only provides a thorough understanding of a central city's innovation capability but also offers decision support for policymakers, helping them to identify strengths and weaknesses and formulate corresponding enhancement strategies. The comprehensive evaluation of innovation capability in central cities of emerging countries can be seen as a MADM problem. In this section, a numerical example is conducted for comprehensive evaluation of innovation capability in central cities of emerging countries through employing the T2NN-HS-CE approach. Assume that seven central cities of emerging countries are selected with 8 attributes.

5.1 Identify a Set of Criteria and Alternatives

Choice of the best alternatives based on a set of criteria. We defined 8 criteria and seven alternatives with several values as shown in Table 1.

Table 1. The set of criteria.

	Criteria	Values	Weights
C_1	Tech Transfer Efficiency	100, 250, and 500projects/year	0.126719
C_2	Environmental Sustainability: Green Innovation	10%, 25%, and 50%	0.069933
C_3	Governance and Policy: Innovation-Friendly Policies	10, 25, and 50 policies	0.104498
C_4	Infrastructure and Connectivity: Urban Mobility	5,10, and 20	0.064706
C_5	Economic: Startup Density	0–50 startups and 10,000 residents	0.198644
C_6	Patent Output	100, 250, and 500 patents	0.147166
C_7	Technological: Digital Infrastructure	10 Mbps, 100, and 500 Mbps	0.162071
C_8	Human Capital and Social: Educational Attainment	10%, 50%, and 70%.	0.126263

5.2 HyperSoft with Cross Entropy Results

Let $C = C_1 \times C_2 \times C_3 \times C_4 \times C_5 \times C_6 \times C_7 \times C_8$ *and the values of crietria are* $(V_1, V_2, \ldots.)$ *we select the* 8 *values of crietria*

We bult the decision matrix using the T2NNs as shown in Table 2.

Then we compute the criteria by using the entropy method as shown in Table 1.

Then we normalize the decision matrix.

Then we conduct the T2NNPIDS.

Then we Conduct the T2NN-CE$\left(NFR_i, TFNNPIDS\right)$.

Then we rank the alternatives as shown in Figure 1.

Table 2. The T2NNs with criteria and alternatives.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
A_2	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
A_3	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))
A_4	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))
A_5	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))
A_6	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))
A_7	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
A_2	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
A_3	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))
A_4	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))
A_5	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))
A_6	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
A_7	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))

A_2	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
A_3	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
A_4	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
A_5	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
A_6	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
A_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))

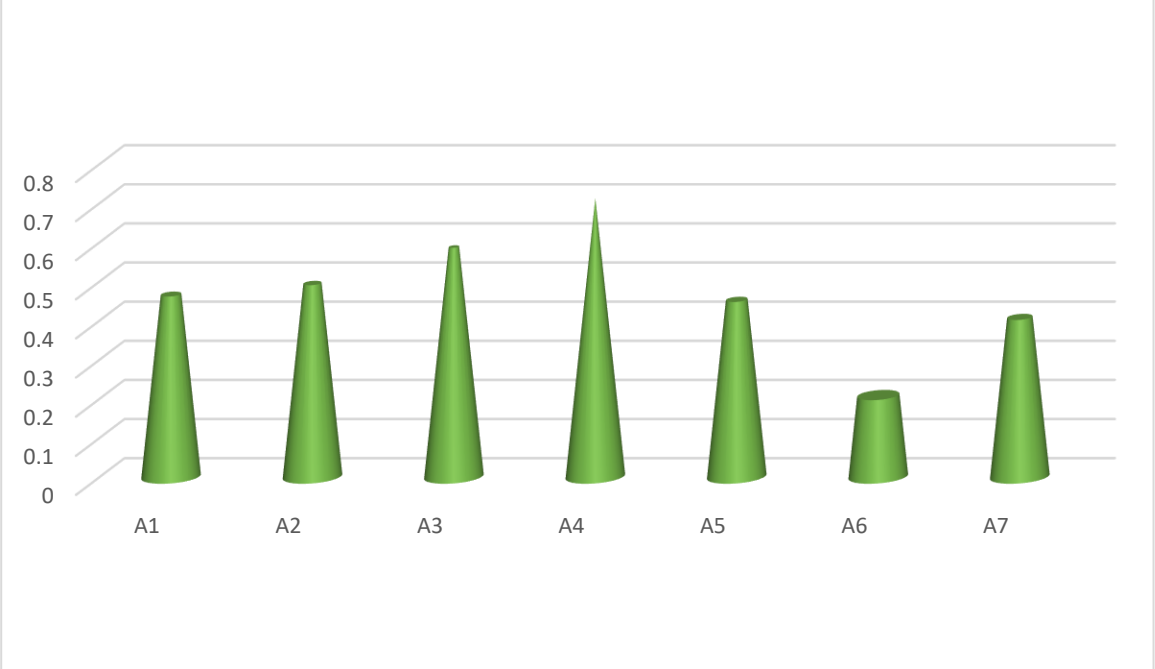


Figure 1. The rank of alternatives.

5.3 Sensitivity Analysis

This section shows the rank of alternatives under different weights of criteria. We change the criteria weights by 9 cases as shown in Figure 2. The first case show the equal weight of criteria. The second case shows the first attribute has 0.2 weights and other attributes are equal weights. Figure 3 shows the rank of alternatives under different weights. We show the alternative 4 is the best under different weights.

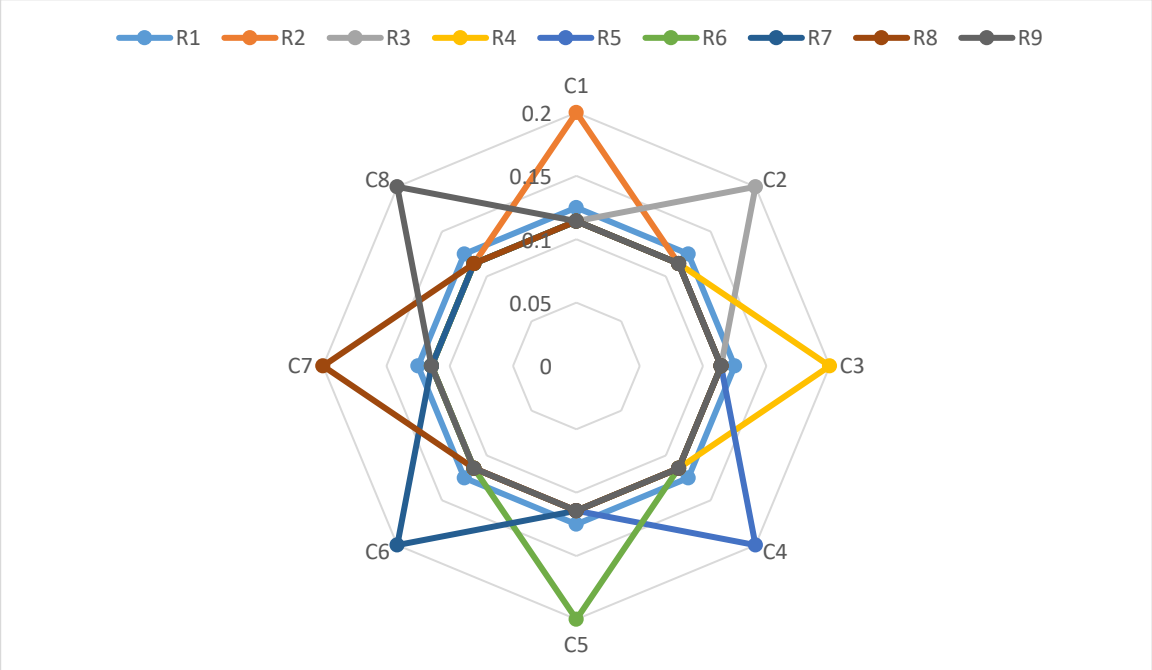


Figure 2. The different weights of criteria.

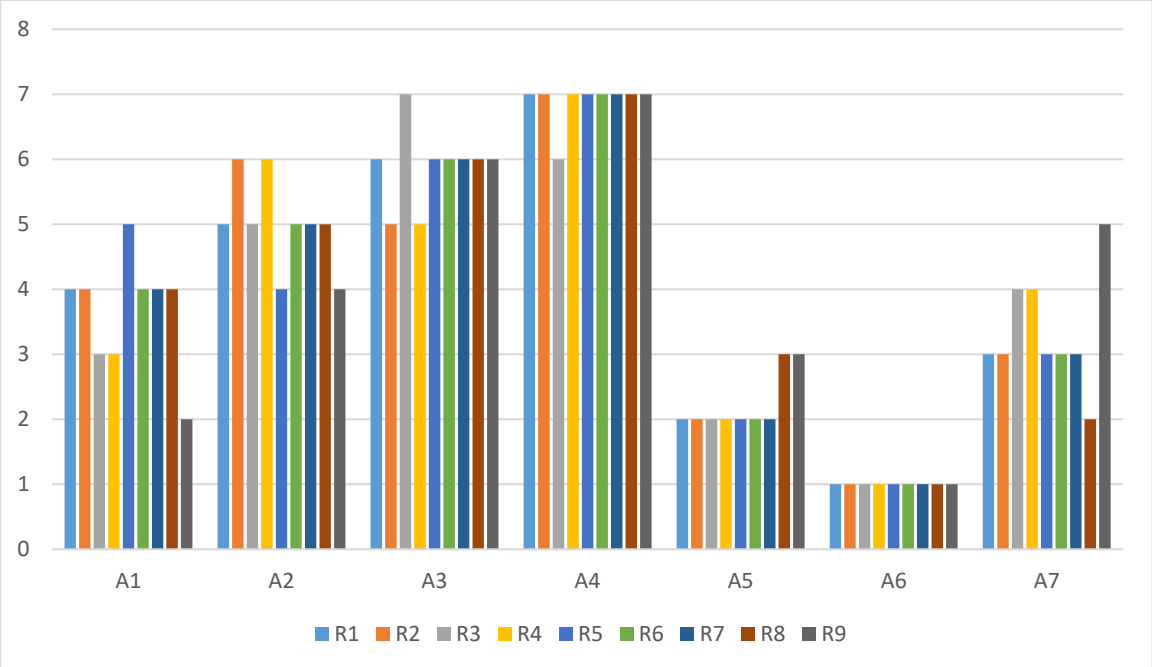


Figure 3. The rank of alternatives under different weights.

6. Conclusion

The comprehensive evaluation of innovation capacity in central cities of growing countries is a critical analytical and decision-making tool designed to measure and compare the innovation capabilities and potential of different central cities in innovation-driven development. This evaluation integrates multiple key indicators—such as scientific activities, enterprise innovation, policy support, and talent aggregation—to build a comprehensive model of innovation capacity assessment. Its purpose is to identify strengths and weaknesses

in each city's efforts to promote technological progress and the development of high-tech industries, thereby providing data support for policymakers to optimize resource allocation and develop more effective regional innovation promotion strategies. Through this evaluation, cities can better position themselves within the national and global innovation networks, promote the optimization and upgrading of economic structures, and accelerate the transition from "growth-oriented" to "innovation-oriented" cities. The comprehensive evaluation of innovation capability in central cities of emerging countries is a MADM problem. Lately, the cross-entropy method has been adapted to address such MADM challenges. T2NNs have been utilized effectively to represent uncertain data throughout the evaluation process. This study introduces the application of the T2NN-HS-CE method to advance MADM analysis using T2NNs with HyperSoft. The HyperSoft is used to deal with complex decision making process. Additionally, this research includes a numerical study and multiple sensitivity analyses to assess and confirm the effectiveness of the proposed T2NN-HS-CE method. The seminal contributions of this research are enumerated as follows: (1) Implementation of the cross-entropy method alongside T2NNs with HyperSoft; (2) Application of the entropy method to establish weight values when using T2NNs and HyperSoft; (3) Development and application of the T2NN-HS-CE method for handling MADM problems characterized by T2NNs; (4) Creation of an algorithmic framework for a comprehensive evaluation of innovation capabilities in central cities of emerging countries, supported by a numerical example and sensitivity analyses to demonstrate the utility of the T2NN-HS-CE method.

The paper introduces a new MADM framework using T2NNs and T2NN-HS-CE method to evaluate the innovation capabilities of central cities in emerging countries. **While this research is innovative and potentially very practical, it has some shortcomings that require further improvements and exploration.** Firstly, the complexity of the T2NN-HS-CE method could pose a barrier to its practical application, especially in policymaking and urban planning settings. The high data requirements and complex processing involved may limit its popularity among non-specialists. Therefore, simplifying and optimizing this methodology to make it more understandable and operable is an important direction for future research. Secondly, this study primarily targets central cities in emerging nations, and its universality and adaptability have not been fully tested across different cultural and economic contexts. **Future research could expand the scope to include more countries and cities, exploring the effectiveness and flexibility of the model in varying environments.** A deeper analysis of data applicability and sensitivity would also help assess the potential for global application of the model. Lastly, integrating the T2NN-HS-CE method with other tools and indicators for assessing urban innovation capabilities, such as the Global Innovation Index, could offer a more comprehensive evaluation perspective. Developing a multi-level assessment framework that combines quantitative and qualitative analyses would provide a more thorough and in-depth evaluation of innovation capabilities. These suggested research directions can further refine and extend the methods and applications of this study, providing

more effective tools and strategies for assessing the innovation capabilities of central cities in emerging countries.

Conflict of interest

The authors declare that they have no conflict of interest.

Funding Information

This work was supported by the project "An Investigation into the Mechanisms Influencing the Green Transformation and Development of the Construction Industry Driven by the Digital Economy", part of the Science and Technology Research Program of the Chongqing Education Commission of China (Project No. KJQN202404011).

Reference

[1] H. Li, Y. Zhang, Infrastructure quality and innovation efficiency in chinese central cities, 15 (2016) 321-334.

[2] B. Wang, et al., Assessing technological innovation capabilities in indian metro cities, 16 (2017) 210-225.

[3] R. Morris, The role of government policy in innovation landscapes of brazilian central cities, 16 (2017) 120-135.

[4] S. Chen, Innovation ecosystems in shenzhen: A model of public-private integration, 17 (2018) 298-312.

[5] J. Thompson, E. Garcia, Comparative analysis of innovation capacities in emerging economy cities, 17 (2018) 540-555.

[6] P. Kumar, M. Singh, Econometric models for measuring innovation potential in indonesian central cities, 18 (2019) 410-426.

[7] W. Zhao, Cultural factors and urban innovation in eastern china, 19 (2020) 190-207.

[8] N. Brooks, X. Chen, International trade and urban innovation in mexican cities, 19 (2020) 501-519.

[9] Q. Zhang, Higher education resources and local innovation systems in chinese cities, 20 (2021) 102-118.

[10] D. Johnson, Technological readiness and infrastructure in nigerian central cities, 20 (2021) 305-320.

[11] J. Lee, H. Kim, Sustainability of innovation models in south korean central cities, 21 (2022) 421-435.

[12] A. Petrov, Digital transformation in russian central cities, 21 (2022) 180-198.

[13] X. Ming, Y. Liu, A multidimensional approach to evaluating innovation capacity in vietnamese cities, 22 (2023) 110-129.

[14] T. Smith, L. Roberts, Challenges and opportunities of smart city technologies in egyptian central cities, 22 (2023) 315-330.

[15] C. Wu, Urban planning and innovation policies in malaysian central cities, 22 (2023) 440-455.

[16] W.J. Zuo, D.F. Li, G.F. Yu, L.P. Zhang, A large group decision-making method and its application to the evaluation of property perceived service quality, Journal of Intelligent & Fuzzy Systems, 37 (2019) 1513-1527.

[17] X.X. Zheng, D.F. Li, Z. Liu, F. Jia, J.B. Sheu, Coordinating a closed-loop supply chain with fairness concerns through variable-weighted shapley values, Transportation Research Part E-Logistics and Transportation Review, 126 (2019) 227-253.

[18] J. Reig-Mullor, A. Garcia-Bernabeu, D. Pla-Santamaria, M. Vercher-Ferrandiz, Evaluating esg corporate performance using a new neutrosophic ahp-topsis based approach, Technological and Economic Development of Economy, 28 (2022) 1242-1266.

[19] S. Bhat, An enhanced ahp group decision-making model employing neutrosophic trapezoidal numbers, Journal of Operational and Strategic Analytics, 1 (2023) 81-89.

[20] K. Pandey, A. Mishra, P. Rani, J. Ali, R. Chakrabortty, Selecting features by utilizing intuitionistic fuzzy entropy method, Decision Making: Applications in Management and Engineering, 6 (2023) 111-133.

[21] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[22] K.T. Atanassov, More on intuitionistic fuzzy-sets, Fuzzy Sets and Systems, 33 (1989) 37-45.

[23] F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, in, American Research Press, Rehoboth, 1999.

[24] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single valued neutrosophic sets, Infinite study, 2010.

[25] H. Wang, F. Smarandache, R. Sunderraman, Y.-Q. Zhang, Interval neutrosophic sets and logic: Theory and applications in computing: Theory and applications in computing, Infinite Study, 2005.

[26] M. Abdel-Basset, M. Saleh, A. Gamal, F. Smarandache, An approach of topsis technique for developing supplier selection with group decision making under type-2 neutrosophic number, Applied Soft Computing, 77 (2019) 438-452.

[27] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[28] J. Lin, Divergence measures based on the shannon entropy, IEEE Transactions on Information theory, 37 (1991) 145-151.

[29] Tehreem, A. Hussain, A. Alsanad, M.A.A. Mosleh, Spherical cubic fuzzy extended topsis method and its application in multicriteria decision-making, Mathematical Problems in Engineering, 2021 (2021) 14.

[30] R.P. Tan, W.D. Zhang, S.Q. Chen, Decision-making method based on grey relation analysis and trapezoidal fuzzy neutrosophic numbers under double incomplete information and its application in typhoon disaster assessment, Ieee Access, 8 (2020) 3606-3628.

[31] J.H. Kim, B.S. Ahn, The hierarchical vikor method with incomplete information: Supplier selection problem, Sustainability, 12 (2020) 15.

[32] M.S.A. Khan, F. Khan, J. Lemley, S. Abdullah, F. Hussain, Extended topsis method based on pythagorean cubic fuzzy multi-criteria decision making with incomplete weight information, Journal of Intelligent & Fuzzy Systems, 38 (2020) 2285-2296.

[33] P.D. Liu, W.Q. Liu, Multiple-attribute group decision-making method of linguistic q-rung orthopair fuzzy power muirhead mean operators based on entropy weight, International Journal of Intelligent Systems, 34 (2019) 1755-1794.

Received: Aug 27, 2024. Accepted: Nov 24, 2024

Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

SuperHyperSoft-Driven Evaluation of Smart Transportation in Centroidous-Moosra: Real-World Insights for the UAV Era

Mona Mohamed[1], Ahmed M. AbdelMouty [2] , Khalid Mohamed[2] , Florentin Smarandache[3]
1Higher Technological Institute, 10th of Ramadan City 44629, Egypt;Mona.fouad@hti.edu.eg;
[2]Information System Department, Faculty of Computers and Information, Zagazig University,
a_abdelmouty@zu.edu.eg;Khaled.zaitoun1@gmail.com;
3University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA;
smarand@unm.edu.

* Correspondence:

Abstract: Over recent decades unmanned aerial vehicles (UAVs) have significantly impacted many areas and applications that affect our daily lives. Such as transportation, healthcare, and agricultural surveillance and management. Along with intelligently digitizing these sectors. Hence, this study focuses on exhibiting UAVs' contributions to transportation systems to be smart. An intelligent decision-maker framework is constructed to evaluate smart transportation systems (STSs) that leverage UAVSs in their operations. Preferencing candidates of STSs conduct the evaluation process based on a set of criteria and attributes. Moreover, the new multi-criteria decision-making (MCDM) of centroidous to obtain criteria attribute weights. As well as Multi-objective optimization on the basis of simple ratio analysis (Moosra) leverages the generated weights to rank STSs and recommend optimal STS. These MCDM techniques collaborated with the uncertainty theory of Single Value Neutrosophic Sets (SVNSs) to enhance decisions in ambiguous situations. Along with Moosra-SVNSs are integrating in the ranking process under the dominance of SuperHyperSoft (SHS) environment which depends on a set of hypersoftsets formed into a set of possibilities. Hence, we applied six possibilities in our constructed framework.

Keywords: Unmanned aerial vehicles (UAVs); smart transportation systems (STSs); centroidous; Moosra; SuperHyperSoft (SHS)

1. Introduction

Recently, an abundance of scholars has been eager to make various services and industries smarter and more sustainable by integrating cutting-edge technologies like Internet of Things (IoTs), Cyber-physical systems (CPS), digital twin (DT), deep learning (DL) as a portion of machine learning (ML)…etc. In this regard smart cities [1]strives to be an innovative concept in modern society, rendering social services more accessible while being beneficial, fostering sustainability, and

making life easier for residents. One sector wherein stakeholders must implement contemporary technologies in the realm of smart cities is the transportation sector to be smart.

1.1 The problem at hand in the study

Herein, we exhibit the study's problems through a set of dimensions. Wherein each dimension exhibits a certain problem. Hence, it considers the motivation for conducting the study to solve it.

1.1.1 First dimension-D1: Ecological and societal problems

Transportation services are mandatory for people's everyday existence [2].Regretfully, this sector faces numerous obstacles. As noted in [3]Vehicle development will continue to accelerate globally due to the expanding urban population, which will harm the environment and make issues like traffic congestion, noise pollution, and accidents worse. Hence, it harms society and the economy owing to [4] the growing number of vehicles and inadequate road capacity cause traffic congestion to worsen every year. Debated that [5] Behavioral lapses, such as excessive speeding, driving while inebriated and losing concentration, and ignoring safety equipment like seat belts, all contribute to catastrophes and emergencies.

Generally, these issues are jeopardy for the intentions of the Sustainable Development Agenda (SDA). Ecologically[6]nearly 25% of worldwide energy-related emissions are caused by transportation, making it a major source of greenhouse gas emissions. This presents a significant obstacle in the prevailing against climate change and the pursuit of sustainable objectives. Socially[7] environmental risks brought on by fossil fuels are associated with respiratory and cardiovascular ailments as well as early mortality. Increased stress and a lower quality of life have been associated with long commutes and traffic [8].Economically [9] Transportation and its associated infrastructure impact economic growth through various pathways, including employment, industrial activity, productivity, competitiveness, property values, and tax revenues.

- **Proposition-P1: Innovative Solutions**

Numerous studies [2] proclaimed that increased use of information technology (IT) is essential to enhancing the operational efficacy of transportation services. In line with [10] deploying cutting-edge sensors, technological devices, and communication technologies in tandem with management techniques to increase transportation services' efficiency and safety. These technologies contribute to churning out smart transportation systems (STSs) [11] to achieve sustainable transportation systems (STSs). Due to [12] intelligent traffic signal control, vehicle surveillance, road condition updates, handling traffic in real time, and autonomous vehicle system deployment. Fig 1 exhibits the role of ICT in STS.

1.1.2 Second dimension-D2: Appraising Smart Transportation System

According to Figure 1, harnessing cutting-edge technologies in transportation services, operations, and planning has a positive implication. To achieve digitizing services of transportation accompanied by its sustainability. From the perspectives of scholars as [13]Evaluating the extent of transportation's efficacy and sustainability is imperative. The evaluation process has been conducted based on a set of criteria related to deployed technologies and sustainability pillars.

Accordingly, multi-criteria decision-making (MCDM) [14] is regarded as an intricacy method for balancing a multitude of opposing criteria and objectives. In line with [15] sustainability of

transportation systems has been evaluated through entropy to obtain weights for criteria related to pillars of sustainability and a Combined Compromise Solution (CoCoSo) for recommending the optimal alternative of transportation systems.

Despite the MCDM approaches' ability to take into account the disproportionate and conflicting effects of actions, the solutions offered contradict several objectives, which means that the optimal point is not reached because of the nature of the issue [16].

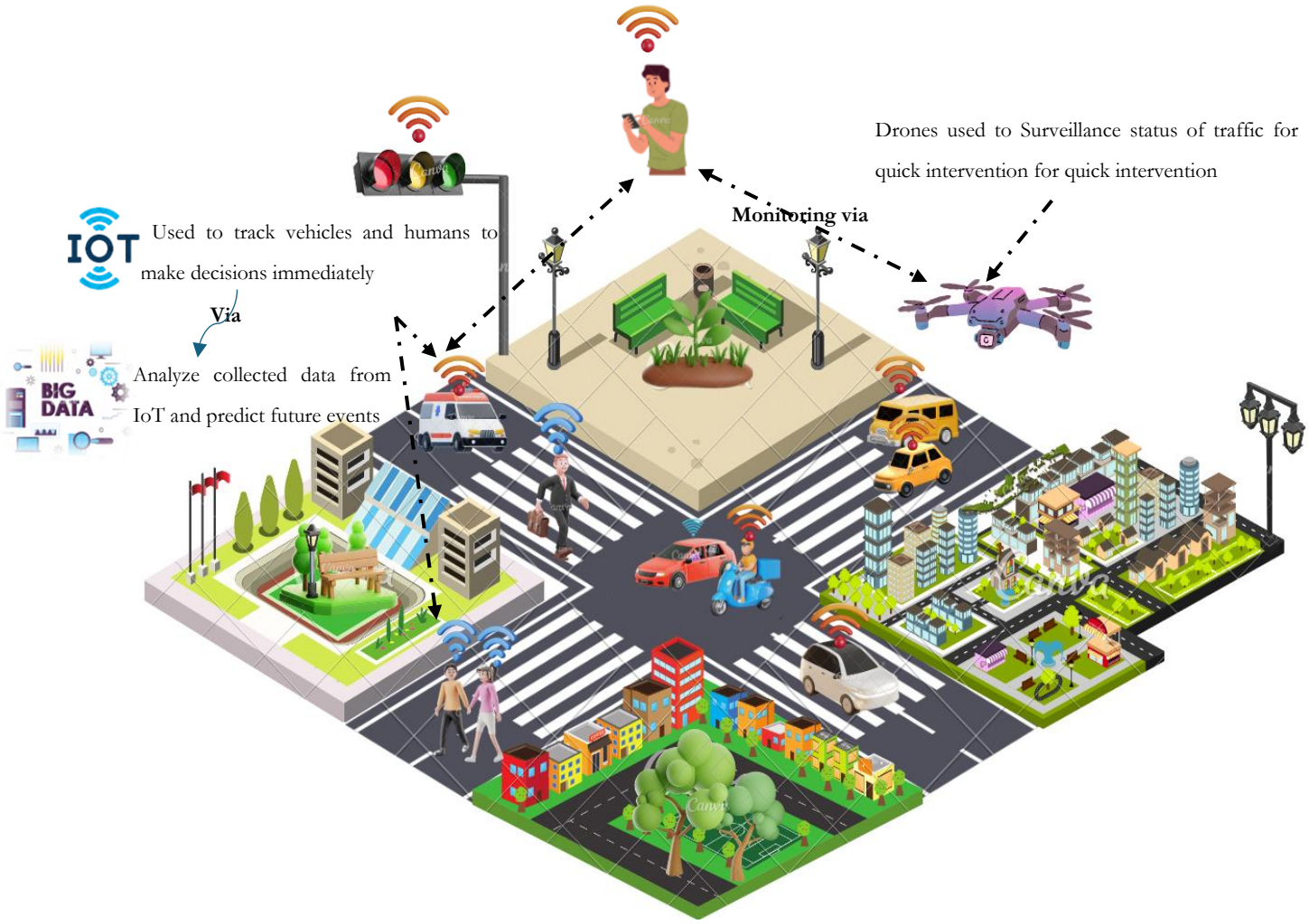

Drones used to Surveillance status of traffic for quick intervention for quick intervention
Monitoring via
IOT
Used to track vehicles and humans to make decisions immediately
Via
BIG DATA
Analyze collected data from IoT and predict future events

Figure 1. Deploying cutting-edge technologies toward smart transport systems

- **Proposition-P2: Leveraging Members Family of Uncertainty Theory**

To combat the issue mentioned in [16], the scholars integrated MCDM with techniques for handling uncertainty and vagueness information[17]; a thorough evaluation of alternatives may be performed by using uncertainty techniques with MCDM, which may effectively assess alternatives based on several conflicting criteria[18]. Hence[13] they have enacted a new integrated method to evaluate sustainable vehicles according to the expert-identified criteria. Compromise solution (MARCOS) techniques under Spherical Fuzzy Set (SFS) which belongs to uncertainty theory are used for ranking alternatives by leveraging weights of criteria obtained from SFS-based Stepwise weight assessment ratio analysis (SWARA). In the same vein [19] integrated type 2 neutrosophic sets (T2NSs)

as a branch of uncertainty theory with entropy and Complex Proportional Assessment (COPRAS) of MCDM to evaluate digital twin applications.

1.2 Contributions of Study

Based on prior scholars' perceptions of the importance of leveraging cutting-edge technologies in transportation to make its services smart and gain customer satisfaction. This study discusses novelty perceptions in solving transportation problems as a set of contributions.

Firstly: Although [20] discussed the importance of IoT in transportation through automated fee collection, vehicle surveillance, and the effectiveness of logistics; IoT in [21] suffers from some challenges such as interoperability, data security and privacy, and the requirement for a strong infrastructure to accommodate an extensive number of connected devices. Hence, we exploit the capabilities of Unmanned Aerial Vehicles (UAV) that entailed gathering information to monitor and control traffic in real-time and congestion patterns[22]. Herein, we discuss the importance of UAV in transportation to be STS.

Second: As mentioned previously, recommending the most sustainable and smart transportation is imperative. This process is the result of the evaluation process for STSs based on a set of criteria. The evaluation is conducted based on data collected but this data may be incomplete or inaccurate which has negative effects on judgments and decisions. To eliminate doubt in judgments, we are implementing Neutrosophic theory and superhypersoft (SHS) as members of the family of uncertainty theory where [23] implemented for the first time to evaluate blockchain applications. As well as we also deploy SHS in our problem of evaluating STSs which harness UAVs to manage traffic for mitigating congestion. Generally, we are constructing a robust intelligent decision-maker by integrating MCDM with Neutrosophic theory and SHS to evaluate transportation systems.

Third: the efficacy and validation of constructed intelligent decision-makers to make accurate decisions and recommendations is vital. Hence, we applied this framework to real scenarios to verify it.

1.3 Simple Recap of Our Study: Objectives

Herein we exhibit the objective of the study and the procedures to which we adhere to achieve the determined objectives. As well as we showcase the procedures' results in the form of action as illustrated in Figure 2.

2. Decision Making Methodological

Various mathematical techniques have been collaborated to construct intelligent decision-makers for evaluation process to achieve the study's objectives. Each technique plays a vital role in this process, which will be showcased in the procedures of constructing intelligent decision-making to evaluate STSs that deploy and leverage UAVs' capabilities to manage traffic and mitigate overcrowding through traffic and road surveillance. Moreover, the following procedures describe how the evaluation process is conducted.

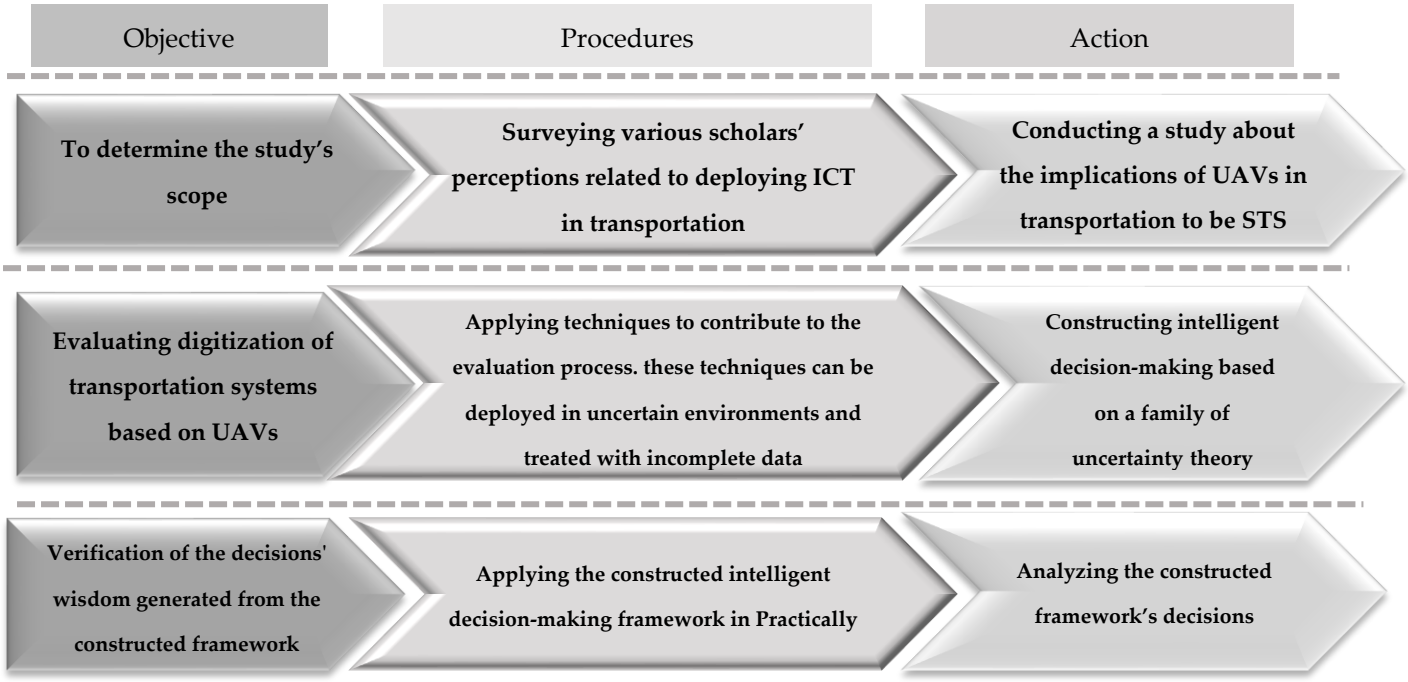

Objective
Procedures
Action
To determine the study's scope
Surveying various scholars' perceptions related to deploying ICT in transportation
Conducting a study about the implications of UAVs in transportation to be STS
Evaluating digitization of transportation systems based on UAVs
Applying techniques to contribute to the evaluation process. these techniques can be deployed in uncertain environments and treated with incomplete data
Constructing intelligent decision-making based on a family of uncertainty theory
Verification of the decisions' wisdom generated from the constructed framework
Applying the constructed intelligent decision-making framework in Practically
Analyzing the constructed framework's decisions

Figure 2. Study Summarization

2.1 Outlining the Essential Aspects of the Evaluation Procedure

1. Surveying STSs is currently in existence and deploying UAVs in their management and operations. Accordingly, these STSs have been volunteered in this process as alternatives.
2. Evaluating the volunteered STSs is conducted based on a set of criteria and their attributes. Hence, determining the criteria and their attributes related to adopting UAVs as automated surveillance.
3. Another important factor in the evaluation process is forming an expert panel to contribute based on criteria and their attributes.

2.2 Finding out criteria and their attributes' weights: SVNSs-Centroidous

Herein, the centroidous technique of MCDM is proposed by Zinkevič [24] for evaluating each criterion's significance concerning the group's overall center. This technique is used to generate criteria weights. Hence, as the following steps exhibit, we implement centroidous under the sovereignty of uncertainty theory of single-value Neutrostomic sets (SVNSs) to bolster the centroids in ambiguous situations.

4. STSs have been evaluated by a formed panel based on determined criteria. Linguistic scale in [25] have been used by the panel in evaluation. Hence, the decision matrices for panel evaluation have been constructed.

5. Linguistics decision matrices are converted into the corresponding neutrosophic scale to construct neutrosophic decision matrices.
6. The neutrosophic matrices are converted into crisp values by the score function in equation (1) .

$$\boldsymbol{s}(\sigma_{ij}) = \frac{(2+\alpha-\beta-\theta)}{3} \quad (1)$$

Where α, β, θ refers to truth, false, and indeterminacy respectively.

7. The crisp matrices are integrated into an aggregated matrix based on equation (2).

$$\wp_{ij} = \frac{(\sum_{j=1}^{N} \sigma_{ij})}{N} \quad (2)$$

Where σ_{ij} refers to the value of the criterion in the matrix, and N refers to the number of decision-makers.

8. Normalizing the aggregated matrix by equation (3).

$$\widetilde{\delta_{ij}} = \frac{\wp_{ij}}{\sum_{j=1}^{m} \wp_{ij}} \quad (3)$$

9. Utilizing a normalized $\widetilde{\wp_{ij}}$, we determine the center of gravity of a distinct set of criteria based on equation (4).

$$c_j = \frac{1}{l}\sum_{i=1}^{n} \widetilde{\delta_{ij}} \quad (4)$$

10. The following formula is used to get the Euclidean distance between each criterion and the group center as in equation (5).

$$d_i = \sqrt{\sum_{j=1}^{m}\left(\widetilde{\delta_{ij}} - c_j\right)^2} \quad (5)$$

11. The criteria weights are generated through equation (6).

$$\widetilde{d_i} = \frac{\min_i(d_i)}{d_i} \quad (6)$$

12. The criteria weights are calculated based on equation (7)

$$w_i = \frac{\widetilde{d_i}}{\sum_{i=1}^{n} \widetilde{d_i}} \quad (7)$$

2.3 Discovering Optimal STS: Ranking Process

In this process, we integrated various techniques to implement together wherein Multi-objective optimization on the basis of simple ratio analysis (Moosra) collaborates with the uncertainty theory of SVNSs to rank candidates of STS. Alongside, these techniques are working under the sovereignty of SHS environment to examine the framework's decision based on the possibilities generated from SHS with MOOSRA-SVNSs.

2.3.1 SuperHyperSoft Environment

SHS is proposed by Smarandache [26] comprising many HyperSoft Sets, this method is seen as an extension of HyperSoft. In this study, SHS was used as the technique to represent the established criteria and their attributes to choose the best alternative based on the criteria and attributes that were chosen.

- Suppose the universe set $\mathfrak{R}$= {STS 1, STS 2… STS n}. Moreover, P ($\mathfrak{R}$) the powerset of $\mathfrak{R}$. As well as C1, C2, C3 are utilized attributes where STS platforms have been evaluated over these criteria. Hence, P(C1), P(C2), and P(C3) are powersets of C1, C2, C3 .

- Let F: P(C1) × P(C2) × P(C3) → P(ℜ), where × indicates to Cartesian product. Hence, this is called SHs over ℜ.
- For instance, Cartesian product for criteria and attributes formed as

$$P(C_1) \times P(C_2) \times P(C_3) = \Big\{\{\{A_{11}\},\{A_{12}\},\{A_{11},A_{12}\}\} \times \{\{A_{21}\},\{A_{22}\},\{A_{21},A_{22}\},\} \times$$

$$\{\{A_{31}\},\{A_{32}\},\{A_{33}\},\{A_{31},A_{32}\},\{A_{31},A_{33}\},\{A_{32},A_{33}\},\{A_{31},A_{32},A_{33}\}\}\Big\}.$$

- Moreover, equation (8)used to express $P(A_1) \times P(A_2) \times P(A_3) =$

$$\left\{\begin{array}{c} \widetilde{\boldsymbol{\partial_1}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\boldsymbol{A_{31}});\ \widetilde{\boldsymbol{\partial_2}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\boldsymbol{A_{32}});\ \widetilde{\boldsymbol{\partial_3}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\boldsymbol{A_{33}});\\ \widetilde{\boldsymbol{\partial_4}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}}\});\ \widetilde{\boldsymbol{\partial_5}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\{\boldsymbol{A_{31}},\boldsymbol{A_{33}}\});\ \widetilde{\boldsymbol{\partial_6}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\{\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\\ \widetilde{\boldsymbol{\partial_7}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{21}},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\ \widetilde{\boldsymbol{\partial_8}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{22}},\boldsymbol{A_{31}});\ \ldots\ \widetilde{\boldsymbol{\partial_{14}}}\ (\boldsymbol{A_{11}},\boldsymbol{A_{22}},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\\ \widetilde{\boldsymbol{\partial_{15}}}\ (\boldsymbol{A_{11}},\{\boldsymbol{A_{21}},\boldsymbol{A_{22}}\},\boldsymbol{A_{31}});\ \ldots\ \widetilde{\boldsymbol{\partial_{21}}}\ (\boldsymbol{A_{11}},\{\boldsymbol{A_{21}},\boldsymbol{A_{22}}\},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\ \widetilde{\boldsymbol{\partial_{22}}}\ (\boldsymbol{A_{12}},\boldsymbol{A_{21}},\boldsymbol{A_{31}});\cdots\\ \widetilde{\boldsymbol{\partial_{28}}}\ (\boldsymbol{A_{12}},\boldsymbol{A_{21}},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\ \widetilde{\boldsymbol{\partial_{29}}}\ (\boldsymbol{A_{12}},\boldsymbol{A_{22}},\boldsymbol{A_{31}});\cdots\ \widetilde{\boldsymbol{\partial_{35}}}\ (\boldsymbol{A_{12}},\boldsymbol{A_{22}},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\\ \widetilde{\boldsymbol{\partial_{36}}}\ (\boldsymbol{A_{12}},\{\boldsymbol{A_{21}},\boldsymbol{A_{22}}\},\boldsymbol{A_{31}});\ \cdots\ \widetilde{\boldsymbol{s_{42}}}\ (\boldsymbol{A_{12}},\{\boldsymbol{A_{21}},\boldsymbol{A_{22}}\},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\ \widetilde{\boldsymbol{\partial_{43}}}\ (\{\boldsymbol{A_{11}},\boldsymbol{A_{12}}\},\boldsymbol{A_{21}},\boldsymbol{A_{31}});\cdots\\ \widetilde{\boldsymbol{\partial_{49}}}\ (\{\boldsymbol{A_{11}},\boldsymbol{A_{12}}\},\boldsymbol{A_{21}},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\});\ \widetilde{\boldsymbol{\partial_{50}}}\ (\{\boldsymbol{A_{11}},\boldsymbol{A_{12}}\},\boldsymbol{A_{22}},\boldsymbol{A_{31}});\cdots\\ \widetilde{\boldsymbol{\partial s_{63}}}\ (\{\boldsymbol{A_{11}},\boldsymbol{A_{12}}\},\{\boldsymbol{A_{21}},\boldsymbol{A_{22}}\},\{\boldsymbol{A_{31}},\boldsymbol{A_{32}},\boldsymbol{A_{33}}\}). \end{array}\right\} \quad (8)$$

2.3.2 Optimal STS: SVNSs-MOOSRA

This procedure is the sinew in the ranking process because the final decision depends on the results generated from this procedure. Overall, the ranking procedures have been conducted as follows:

1. Reapply the panel's evaluation from the previous procedure for attributes to construct Neutrosophic decision matrices.
2. These matrices have been constructed based on SHS possibilities where the attributes contributed to the evaluation of STSs and were selected based on SHS's possibilities.
3. The constructed matrices are transformed into de-neutrosophic decision matrices by using equation (1).
4. The crisp matrices are aggregated into an aggregated matrix based on equation (2).
5. The aggregated matrix is normalized based on equation (9).

$$\mathrm{Nor}_{ij} = \frac{\wp_{ij}}{\left[\sum_j^m \wp_{ij}\right]^{1/2}} \quad (9)$$

6. Compute a weighted decision matrix based on equation (10).

i.$\mathfrak{H}_{ij} = \mathrm{Nor}_{ij} * w_i$ (10)

7. Calculating ratio as in equation (11) to obtain the final rank for alternatives.

$$\mathrm{Ratio} = \frac{\sum_{j=1}^{B} \mathfrak{H}_{ij}}{\sum_{j=1}^{NB} \mathfrak{H}_{ij}} \quad (11)$$

3. Deploying Intelligent Decision Maker Framework in Reality: Case Study

To verify the efficiency of the constructed framework, we implemented it in a real case study. We communicated with five transportation systems as a result of surveys conducted for numerous transportation systems that deploy contemporary technologies such as UAVs to be smart transportation systems. Hence, we are applying the constructed framework to these systems as follows.

3.1 Ascertaining Initial Aspects

- First Aspect is alternatives: herein, five STSs have been leveraged to be candidates for our study. These alternatives are formed as $\mathrm{STS_n} = \{\mathrm{STS_1}, \mathrm{STS_2}, \ldots., \mathrm{STS_5}\}$.
- Second Aspect is criteria: we have determined a set of criteria related to the technology of UAVs and formed as $\mathrm{C_n} = \{\mathrm{C_1}, \mathrm{C_2}, \mathrm{C_3}\}$.
- Third Aspect is attributes related to determined criteria and formed as $\mathrm{A_{mn}} = \{\mathrm{A_{11}}, \mathrm{A_{12}}, \mathrm{A_{21}}, \mathrm{A_{22}}, \mathrm{A_{31}}, \mathrm{A_{32}}\}$. Table 1 also exhibits the description of the criteria and attributes utilized.

3.2 Valuation weights: SVNSs-Centroidous

- Three Neutrosophic decision matrices are constructed and transformed into de-neutrosophic (crisp) matrices by using equation (1).
- Aggregation matrix is constructed by deploying equation (2) into crisp matrices as listed in Table 2.
- Equation (3) applies in an aggregated matrix formed in Table 2 to generate a normalized matrix as mentioned in Table 3.
- Utilizing equation (4) to compute the vector of the center of gravity of a distinct set of criteria formed in Table 4.
- Euclidean distance between each criterion and the group center is calculated based on equation (5) and generated in Table 5.
- Equation (7) is utilized for generating criteria weights that are formed in Figure 3, where C1 has the highest value. Otherwise, C2 is the worst with the lowest value.

Table 1. Description of Utilized Criteria and Attributes

Technical Performance (C1)	
Reliability (A11)	Efficiency of transportation system operation based on UAV in a range of circumstances
Collision Rate (A12)	Rate of accidents caused by the system that adopted UAV
Organizational Efficacy (C2)	
Real-Time Processing (A21)	Ability to make decisions quickly by analyzing data as it is gathered.
Energy Consumption Metrics(A22)	analysis of the energy consumption for each flight or delivery distance.
Performance (C3)	
Speed of Data Collection (A31)	Time spent on data collection in comparison to conventional techniques.
Flight Time Analysis (A32)	Examination of average flight intervals for various delivery circumstances.

Table 2. An aggregated matrix

	C1	**C2**	**C3**
STS1	0.604444444	0.5722222	0.42666667
STS2	0.816666667	0.5777778	0.71666667
STS3	0.538888889	0.38	0.5
STS4	0.705555556	0.8055556	0.65
STS5	0.644444444	0.7111111	0.81666667

Table 3. Normalized Matrix

	C1	**C2**	**C3**
STS1	0.182611615	0.18781911	0.137191854
STS2	0.24672709	0.189642597	0.230439443
STS3	0.162806311	0.124726477	0.160771704
STS4	0.213158778	0.264405543	0.209003215
STS5	0.194696207	0.233406273	0.262593783

Table 4. Vector of center

Cj	0.169207526	0.2222697	0.14943483	0.228855846	0.230232088

Table 5. Vector of Euclidean distance

di	0.049657087	0.057404519	0.05159146

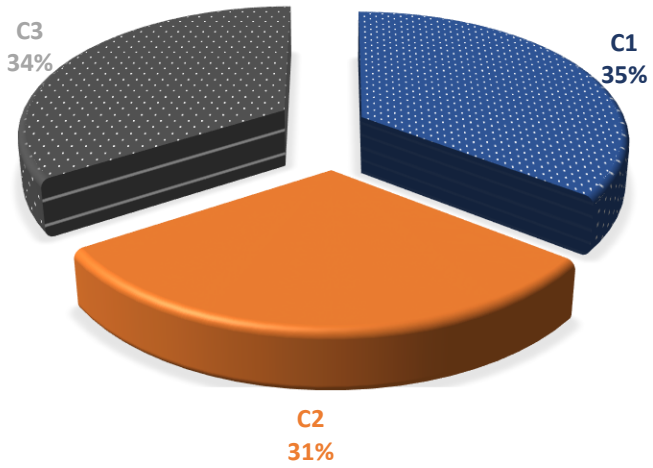


Figure 3. Criteria weights based on SVNSs-Centroidous

3.3 Recommending Optimal STSs

In this procedure, we generate optimal STS. Hence, three techniques have been leveraged to achieve the procedure's objective. SVNSs-MOOSR are collaborating in the environment of SHS. Wherein there are eight possibilities for attributes generated based on SHS and expressed as:

$$P(C_1) \times P(C_2) \times P(C_3) = \left\{\{\{A_{11}\},\{A_{12}\},\{A_{11},A_{12}\}\} \times \{\{A_{21}\},\{A_{22}\},\{A_{21},A_{22}\},\} \times \right.$$

$$\left.\{\{A_{31}\},\{A_{32}\},\{A_{33}\},\{A_{31},A_{32}\},\{A_{31},A_{33}\},\{A_{32},A_{33}\},\{A_{31},A_{32},A_{33}\}\}\right\}.$$

These possibilities contribute to constructing Neutrosophic decision matrices by deploying the previous rating for panel that are used in finding criteria and attributes weights.

- Let F:$P(C_1) \times P(C_2) \times P(C_3)$, hence the eight possibilities formed as:

  Possible 1: P1:{A11,A21,A31}.

  P2:{A11,A21,A32}.

  P3:{A11,A22,A31}.

  P4:{A11,A22,A32}.

  P5:{A12,A21,A31}.

  P6:{A12,A21,A32}.

  P7:{A12,A22,A31}.

  P8:{A12,A22,A32}.

3.3.1 According to P1:

- To construct decision matrices, we collect the previous panel's rate that was conducted for attribute weights. After that equation (1) was utilized for deneutrosophic matrices.
- Aggregate these matrices into an aggregated matrix based on equation (2) as in Table 6.

- normalized matrix is generated in Table 7 according to equation (9).
- weighted normalized matrix is generated based on equation (10) (see Table 8).
- The final ranking for STSs is exhibited in Figure 4, which indicates that STS2 is the optimal transportation system.
-

Table 6. Aggregated Matrix

	A11	**A21**	**A31**
STS1	0.644444	0.644444	0.578889
STS2	0.498889	0.498889	0.32
STS3	0.42	0.42	0.42
STS4	0.598889	0.598889	0.598889
STS5	0.565556	0.666667	0.5

Table 7. Normalized Matrix

	A11	**A21**	**A31**
STS1	0.522865	0.502658	0.523548
STS2	0.40477	0.389127	0.289409
STS3	0.340764	0.327594	0.379849
STS4	0.485904	0.467125	0.541636
STS5	0.458859	0.519991	0.452201

Table 8. Weighted Decision Matrix

	A11	**A21**	**A31**
STS1	0.148122626	0.130715021	0.145157608
STS2	0.114667343	0.101191456	0.080240674
STS3	0.096535091	0.085190135	0.105315885
STS4	0.137651889	0.121474822	0.15017265
STS5	0.129990374	0.135222436	0.125376053

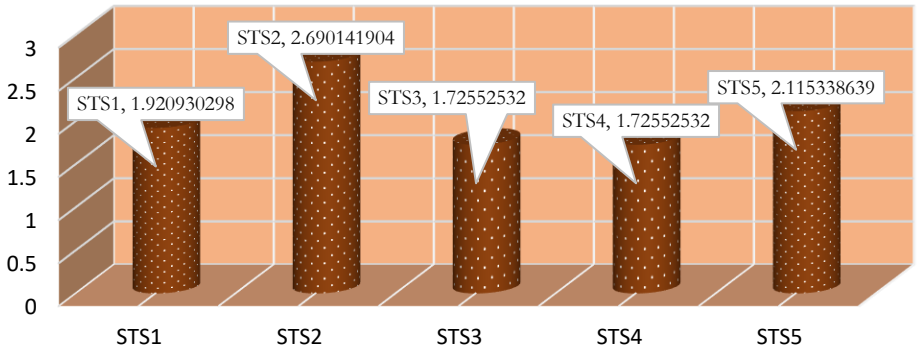
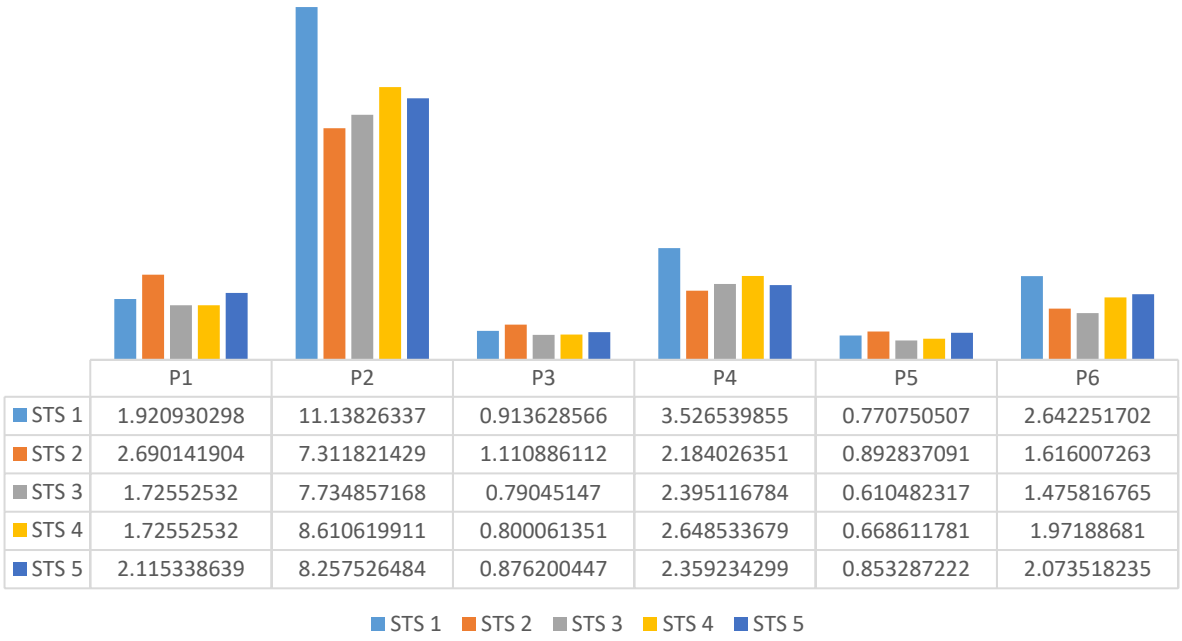


Figure 4. Ranking of STSs

3.3.2 Deploying Various SHS Possibilities

We applied five SHS possibilities to obtain the optimal STS among a set of alternatives, following the same steps mentioned in the previous section. Fig 5 exhibits the ranking of STSs based on implementing all possibilities in this study. According to Figure 5 STS 2 is optimal in possibilities of P1, P3, and P5 while STS1 is optimal in possibilities of P2, P4, and P6.



	P1	P2	P3	P4	P5	P6
STS 1	1.920930298	11.13826337	0.913628566	3.526539855	0.770750507	2.642251702
STS 2	2.690141904	7.311821429	1.110886112	2.184026351	0.892837091	1.616007263
STS 3	1.72552532	7.734857168	0.79045147	2.395116784	0.610482317	1.475816765
STS 4	1.72552532	8.610619911	0.800061351	2.648533679	0.668611781	1.97188681
STS 5	2.115338639	8.257526484	0.876200447	2.359234299	0.853287222	2.073518235

Figure 5. Ranking of STSs based on six possibilities

4. Conclusion

Herein, we surveyed the main technologies currently contributing to transforming transportation systems into smart transportation systems, providing a thorough overview of smart transportation systems and applications. One of these technologies is UAV, which has an immense impact on transportation automation. Due to its ability to conduct traffic surveillance and control through deploying advanced sensors and cameras for collecting real-time data related to traffic. As well as in the era of smart cities where IoT devices may collaborate with UAVs to provide a complete data ecosystem that improves urban transportation decision-making.

Moreover, this study seeks to exhibit UAVs' contributions to transportation systems to be smart. Accordingly, we constructed an intelligent decision-maker to evaluate STSs that deploy UAVs in their operations and management. The evaluation for STSs is conducted based on a set of criteria and their attributes. Hence, we harnessed centroidous as the new MCDM technique that was utilized for the first time in this problem and collaborated with SVNSs to obtain criteria and attribute weights. As well, the attributes' weights are leveraged in Moosra based on SVNSs to recommend the optimal STS. In the ranking process, Moosra based on SVNSs implemented under the dominance of SHS environment which depends on a set of hypersoftsets is formed into a set of possibilities. Hence, we applied six possibilities in our constructed framework.
The findings of the constructed framework indicated that STS 2 is optimal in the possibilities of P1, P3, and P5 while STS1 is optimal in the possibilities of P2, P4, and P6.

5. Future Contributions and Perceptions

The objective of this section is to showcase the authors' future perceptions that can be exploited in future research. Such as the challenges that obstruct the implementation of UAVs in transportation systems to be smart. Wherein the implementation of UAVs in smart transportation is fraught with difficulties, notwithstanding its potential advantages as ecological obstacles. Particularly susceptible to harsh weather conditions including strong winds, torrential rain, and extremely hot or low temperatures are UAVs. These elements may have an impact on battery performance and operational security. Another challenge is operational obstacles, such as barriers to communication. Hence, analyzing and evaluating these challenges is imperative to determine the most influential obstacle through deploying our constructed framework in this problem.

References

1. A. A. Musa, S. I. Malami, F. Alanazi, W. Ounaies, M. Alshammari, and S. I. Haruna, "Sustainable Traffic Management for Smart Cities Using Internet-of-Things-Oriented Intelligent Transportation Systems (ITS): Challenges and Recommendations," Sustainability, 2023, vol. 15, no. 13, p. 9859.
2. D. Oladimeji, K. Gupta, N. A. Kose, K. Gundogan, L. Ge, and F. Liang, "Smart Transportation: An Overview of Technologies and Applications," Sensors, 2023, vol. 23, no. 8, pp. 1–32, doi: 10.3390/s23083880.
3. Y. Kandogan and S. D. Johnson, "Role of economic and political freedom in the emergence of global middle class," Int. Bus. Rev.,2016, vol. 25, no. 3, pp. 711–725.
4. T. Afrin and N. Yodo, "A probabilistic estimation of traffic congestion using Bayesian network," Measurement, 2021, vol. 174, p. 109051.

5. M. Mohamed, A. Elsayed, and M. Sharawi, "Modeling Metaverse Perceptions for Bolstering Traffic Safety using Novel TrSS-Based OWCM-RAM MCDM Techniques: Purposes and Strategies," Neutrosophic Syst. with Appl., 2024, vol. 16, pp. 12–23. doi: 10.61356/j.nswa.2024.16204.
6. A. L. C. Ferrer and A. M. T. Thomé, "Carbon Emissions in Transportation: A Synthesis Framework," Sustain., 2023, vol. 15, no. 11. doi: 10.3390/su15118475.
7. A. Hassan, S. Z. Ilyas, A. Jalil, and Z. Ullah, "Monetization of the environmental damage caused by fossil fuels," Environ. Sci. Pollut. Res., 2021, vol. 28, pp. 21204–21211.
8. M. C. Liya, S. K. Rajan, and A. Kenath, "Psychological experiences and travel Adversities: A Mixed-Method study of the regular commuters in traffic congestion," Transp. Res. part F traffic Psychol. Behav.,2024, vol. 101, pp. 130–141.
9. R. P. Pradhan, M. S. Nair, J. H. Hall, and S. E. Bennett, "Planetary health issues in the developing world: Dynamics between transportation systems, sustainable economic development, and CO2 emissions," J. Clean. Prod., 2024, vol. 449, p. 140842.
10. J. Yu, W. Li, Z. Song, S. Wang, J. Ma, and B. Wang, "The role of attitudinal features on shared autonomous vehicles," Res. Transp. Bus. Manag., 2023, vol. 50, p. 101032.
11. C. Zhao, "The power of technology innovation: can smart transportation technology innovation accelerate green transportation efficiency?", Smart Resilient Transp., 2024, doi: 10.1108/srt-12-2023-0015.
12. L. Yang, H. Cai, and W. Y. Szeto, "Environmental implications of emerging transportation technologies," Transp. Res. Part D Transp. Environ.2023.
13. S. J. Ghoushchi et al., "Assessing Sustainable Passenger Transportation Systems to Address Climate Change Based on MCDM Methods in an Uncertain Environment," Sustain.,2023, vol. 15, no. 4, doi: 10.3390/su15043558.
14. F. M. Donais, I. Abi-Zeid, E. O. D. Waygood, and R. Lavoie, "Municipal decision-making for sustainable transportation: Towards improving current practices for street rejuvenation in Canada," Transp. Res. part A policy Pract., 2022, vol. 156, pp. 152–170.
15. C. N. Wang, T. Q. Le, K. H. Chang, and T. T. Dang, "Measuring Road Transport Sustainability Using MCDM-Based Entropy Objective Weighting Method," Symmetry (Basel).,2022, vol. 14, no. 5, pp. 1–19, doi: 10.3390/sym14051033.
16. H. Taherdoost and M. Madanchian, "Multi-criteria decision making (MCDM) methods and concepts," Encyclopedia, 2023, vol. 3, no. 1, pp. 77–87.
17. G. S. Hussein, A. N. H. Zaied, and M. Mohamed, "ADM: Appraiser Decision Model for Empowering Industry 5.0-Driven Manufacturers toward Sustainability and Optimization: A Case Study," Neutrosophic Syst. with Appl., 2023, vol. 11, pp. 22–30.
18. A. A. Al Mohamed, S. Al Mohamed, and M. Zino, "Application of fuzzy multicriteria decision-making model in selecting pandemic hospital site," Futur. Bus. J., 2023, vol. 9, no. 1, doi: 10.1186/s43093-023-00185-5.
19. M. Mohamed, A. A. Metwaly, M. Ibrahim, F. Smarandache, and M. Voskoglou, "Partnership of Lean Six Sigma and Digital Twin under Type 2 Neutrosophic Mystery Toward Virtual Manufacturing Environment: Real Scenario Application," Sustain. Mach. Intell. J., , 2024., vol. 8, pp. 6–99.
20. S. Chandrappa, M. S. Guruprasad, H. N. N. Kumar, K. Raju, and D. K. S. Kumar, "An IOT-Based Automotive and Intelligent Toll Gate Using RFID," SN Comput. Sci., , 2023, vol. 4, no. 2, p. 154.
21. K. K. Dewangan, V. Panda, S. Ojha, A. Shahapure, and S. R. Jahagirdar, "Cyber Threats and Its Mitigation

to Intelligent Transportation System," SAE Technical Paper, 2024.

22. A. Saboor, S. Coene, E. Vinogradov, E. Tanghe, W. Joseph, and S. Pollin, "Elevating the future of mobility: UAV-enabled Intelligent Transportation Systems,"2021, pp. 1–7, [Online]. Available: http://arxiv.org/abs/2110.09934
23. M. Mohamed, A. Elmor, F. Smarandache, and A. A. Metwaly, "An efficient superhypersoft framework for evaluating llms-based secure blockchain platforms," Neutrosophic Sets Syst., 2024, vol. 72, pp. 1–21
24. I. Vinogradova-Zinkevič, "Centroidous Method for Determining Objective Weights," Mathematics, 2024, vol. 12, no. 14,doi: 10.3390/math12142269.
25. I. El-Henawy, S. El-Amir, M. Mohamed, and F. Smarandache, "Modeling Influenced Criteria in Classifiers' Imbalanced Challenges Based on TrSS Bolstered by The Vague Nature of Neutrosophic Theory," Neutrosophic Sets Syst., 2024, vol. 65, pp. 183–198,
26. F. Smarandache, "Foundation of the SuperHyperSoft Set and the Fuzzy Extension SuperHyperSoft Set: A New Vision," Neutrosophic Syst. with Appl., , 2023, vol. 11, pp. 48–51.

Received: Month Day, Year. Accepted: Month Day, Year

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

Utilizing CoCoSo and Interval Neutrosophic MAGDM with SuperHyperSoft for Competitiveness Assessment in Rural Tourism of Surrounding Cities

Aiping Cui, Zhiyao Liu*
Hebei Vocational University of Industry and Technology, Shijiazhuang, 050091, Hebei, China
*Corresponding author, E-mail: 13933152617@163.com; https://orcid.org/0009-0004-2664-061X
Aiping Cui, E-mail: 15032791167@163.com

Abstract: With rising living standards, rural tourism near cities has rapidly developed, playing a key role in the sustainable development of urban economies and societies. Factors like ecological environment, historical and cultural resources, and landscape significantly influence the competitiveness of rural tourism. However, current research on this topic is still in its early stages, with a focus on tourism benefits to cities, often overlooking other impacts. Rural tourism boosts employment, promotes rural industries, and increases farmers' incomes, accelerating rural urbanization and economic growth. Therefore, evaluating the competitiveness of rural tourism near cities is essential for its development. The evaluation of rural tourism competitiveness in the vicinity of cities is multiple-attribute group decision-making (MAGDM) problem. Recently, the Combined Compromise Solution (CoCoSo) approach has been used to address MAGDM. Interval neutrosophic sets (INSs) are employed as approach to characterize uncertain data in the evaluation of rural tourism competitiveness in areas surrounding cities. In this research, the CoCoSo approach is structured for MAGDM with INSs. Then, the interval neutrosophic numbers CoCoSo with SuperHyperSoft (INN-SHS-CoCoSo) approach, based on Hamming distance with the SuperHyperSoft, is developed for MAGDM. SuperHyperSoft is used to teat a set of criteria and sub criteria. Finally, a practical numerical example for the evaluation of rural tourism competitiveness in areas surrounding cities is developed to demonstrate the INN-SHS-CoCoSo approach.

Keywords: MAGDM; INSs; Combined Compromise Solution (CoCoSo); rural tourism competitiveness evaluation; SuperHyperSoft.

1. Introduction

The traditional attraction tourism model, with the ticket economy as the link, majorly by the tourism authorities planning and management of single attractions (scenic spots) construction and operation, in general, has a closed and self-circulating characteristics and the development of tourism industry is highly dependent on the attractions (scenic spots) of the ticket revenue, the lack

of " tourism+"[1]. The development of tourism industry is highly dependent on the ticket income of attractions (scenic spots), lacking the idea of "tourism+" integrated development, failing to coordinate the construction of tourism infrastructure and public services, and having a single mechanism for tourism growth. Promoting "regional tourism" is a far-reaching change in thinking and strategic adjustment for the modernization and development of China's tourism[2]. Earlier, "regional tourism" appeared in the local tourism sector in the work of planning and summarizing the report, just a conceptual term, but has attracted the attention of the academic community [3]. The so-called regional tourism can be understood as a new concept and mode of integrating natural resources and social resources in a specific geographic space, developing regional characteristic industries with tourism industry as a tractor, and promoting sustainable and coordinated economic and social development of the region with the support of relevant policies and institutional mechanisms[4]. The idea of regional tourism helps to give full play to the economic and social development advantages of a specific geographic space, realizing the cross-border integration of regional industries, especially the optimization of the whole region in the development of natural resources, common construction of transportation and communication, ecological environmental protection, the construction of a unified market, and the sharing of public services, etc., which can enhance the effectiveness of the integration of the tourism industry as a regional characteristic industry[5]. With the socio-economic development, local economic restructuring, and the promotion of metropolitan areas and regional integration development strategies, the formation and development of regional tourism has a natural resource base and socio-economic conditions. It is an innovative model to implement the five development concepts, fully reflecting the strategic position and social value of tourism. Some commentators have suggested that the key issue for the development of regional tourism lies in the deeper integration of industrial development, new urbanization construction, business development, cultural tourism development, etc., so as to realize the integration of "industry, city, business, tourism and culture". Therefore, regional culture has a multiplier effect, which can significantly enhance the value of the tourism industry chain. "Regional Culture + Regional Tourism" is a new business model for the development of the tourism industry, whose core winning element is regional culture, the platform for the operation of the model is regional tourism, and the products sold are tourism service products with additional cultural connotations, which participate in the competition in the tourism market.

Fuzzy decision-making is a prominent area of research, significantly impacting both production and daily life [6-8]. Despite its relatively recent emergence, fuzzy MAGDM has garnered substantial interest from scholars globally, resulting in numerous research achievements. Evaluating rural tourism competitiveness in nearby cities exemplifies MAGDM [9]. Recently, the CoCoSo approach [10-14] and information entropy approach [15] have been employed to address MAGDM challenges. Interval neutrosophic sets (INSs) [16] serve as a means to characterize uncertain information in evaluating rural tourism competitiveness in adjacent cities. To date, few techniques have applied the CoCoSo approach using Hamming distance within the context of INSs. Consequently, an interval neutrosophic numbers CoCoSo (INN-SHS-CoCoSo) approach, Hamming distance, has been developed for MAGDM under INSs. An example illustrating rural tourism competitiveness evaluation in nearby cities, underscores the effectiveness and reliability of the INN-SHS-CoCoSo technique. This paper introduces the INN-SHS-CoCoSo approach, utilizing Hamming distance to

tackle MAGDM under SVNSs. Ultimately, a numerical case study focused on rural tourism competitiveness evaluation in nearby cities is presented to validate the proposed approach.

The main objectives of this paper are constructed:

(1) Development of a novel MAGDM approach: The paper aims to introduce a new Multiple Attribute Group Decision-Making (MAGDM) method by integrating the Combined Compromise Solution (CoCoSo) approach with Interval Neutrosophic Sets (INSs). This novel approach is designed to manage complex decision-making scenarios where uncertainty and imprecision play a significant role.

(2) Application of the INN-SHS-CoCoSo method in rural tourism evaluation: The paper introduces a new MAGDM model using the INN-SHS-CoCoSo technique with SuperHyperSoft to evaluate the competitiveness of rural tourism in nearby cities. This approach allows for a comprehensive assessment of various tourism attributes, providing a more robust and accurate analysis of competitiveness in the rural tourism sector.

The framework of the paper is structured as follows:

Section 2: Introduction to INSs: This section introduces INSs as a mathematical tool for handling uncertain and imprecise information, which is key to evaluating rural tourism competitiveness and other decision-making scenarios.

Section 3: Construction of INN-SHS-CoCoSo approach under INSs: Here, the authors detail the development of INN-SHS-CoCoSo approach, which integrates the Hamming distance. This approach is designed to solve MAGDM in environments with uncertainty.

Section 4: Illustrative Case for User Experience Evaluation of Virtual Reality Products: This section provides a real-world example to demonstrate how the INN-SHS-CoCoSo approach can be applied beyond rural tourism. Specifically, it applies the method to evaluate user experiences with virtual reality products, showcasing the method's versatility.

Section 5: Final Remarks: Concluding observations about the method's performance, strengths, and potential improvements are discussed in this section.

Section 6: Research Limitations and Future Directions: This section outlines the limitations of the current research, acknowledging areas for improvement, and suggests potential avenues for future research, such as refining the method or applying it to other domains.

This structured framework ensures that the paper provides a comprehensive and detailed examination of the INN-SHS-CoCoSo method, from its theoretical underpinnings to practical applications and future considerations.

2. Basic knowledge

Wang et al. [17] assembled the SVNSs

Definition 1 [17]. The SVNSs WA in Φ is assembled:

$$WA=\left\{\left(\phi,WT_A(\phi),WI_A(\phi),WF_A(\phi)\right)\middle|\phi\in\Phi\right\} \tag{1}$$

where $WT_A(\phi),WI_A(\phi),WF_A(\phi)\in[0,1]$ is truth-membership (TM), indeterminacy-membership (IM) and falsity-membership (FM), $WT_A(\phi),WI_A(\phi),WF_A(\phi)\in[0,1]$ and satisfies $0\le WT_A(\phi)+WI_A(\phi)+WF_A(\phi)\le 3.$

Wang et al.[16] assembled the INSs.

Definition 2[16]. The INSs A in Φ is assembled:

$$W\tilde{A}=\left\{\left(\phi,WT_{\tilde{A}}(\phi),WI_{\tilde{A}}(\phi),WF_{\tilde{A}}(\phi)\right)\middle|\phi\in\Phi\right\} \tag{2}$$

where the $WT_{\tilde{A}}(\phi),WI_{\tilde{A}}(\phi),WF_{\tilde{A}}(\phi)$ depicts the TM, IM and FM, $WT_{\tilde{A}}(\phi),WI_{\tilde{A}}(\phi),WF_{\tilde{A}}(\phi)\subseteq[0,1]$ and satisfies $0\le \sup WT_{\tilde{A}}(\phi)+\sup WI_{\tilde{A}}(\phi)+\sup WF_{\tilde{A}}(\phi)\le 3.$

The (INN) is expressed as $W\tilde{A}=\left(WT_{\tilde{A}},WI_{\tilde{A}},WF_{\tilde{A}}\right)=\left(\left[WTL_{\tilde{A}},WTR_{\tilde{A}}\right],\left[WIL_{\tilde{A}},WIR_{\tilde{A}}\right],\left[WFL_{\tilde{A}},WFR_{\tilde{A}}\right]\right)$, where $WT_{\tilde{A}},WI_{\tilde{A}},WF_{\tilde{A}}\subseteq[0,1]$, and $0\le WTR_{\tilde{A}}+WIR_{\tilde{A}}+WFR_{\tilde{A}}\le 3$.

Definition 3 [18]. Let $W\tilde{A}=\left(\left[WTL_{\tilde{A}},WTR_{\tilde{A}}\right],\left[WIL_{\tilde{A}},WIR_{\tilde{A}}\right],\left[WFL_{\tilde{A}},WFR_{\tilde{A}}\right]\right)$ be INN, a score value is assembled:

$$SV\left(W\tilde{A}\right)=\frac{\left(2+WTL_{\tilde{A}}-WIL_{\tilde{A}}-WFL_{\tilde{A}}\right)+\left(2+WTR_{\tilde{A}}-WIR_{\tilde{A}}-WFR_{\tilde{A}}\right)}{6},\ S\left(W\tilde{A}\right)\in[0,1]. \tag{3}$$

Definition 4[18]. Let $W\tilde{A}=\left(\left[WTL_{\tilde{A}},WTR_{\tilde{A}}\right],\left[WIL_{\tilde{A}},WIR_{\tilde{A}}\right],\left[WFL_{\tilde{A}},WFR_{\tilde{A}}\right]\right)$ be an INN, an accuracy value is assembled:

$$AV\left(W\tilde{A}\right)=\frac{\left(WTL_{\tilde{A}}+WTR_{\tilde{A}}\right)-\left(WFL_{\tilde{A}}+WFR_{\tilde{A}}\right)}{2},\ AV\left(W\tilde{A}\right)\in[-1,1]\ . \tag{4}$$

Huang et al. [19] assembled the order between INNs.

Definition 5[18]. Let $W\tilde{A}=\left(\left[WTL_{\tilde{A}},WTR_{\tilde{A}}\right],\left[WIL_{\tilde{A}},WIR_{\tilde{A}}\right],\left[WFL_{\tilde{A}},WFR_{\tilde{A}}\right]\right)$ and $W\tilde{B}=\left(\left[WTL_{\tilde{B}},WTR_{\tilde{B}}\right],\left[WIL_{\tilde{B}},WIR_{\tilde{B}}\right],\left[WFL_{\tilde{B}},WFR_{\tilde{B}}\right]\right)$ be two INNs,

$SV\left(W\tilde{A}\right)=\frac{\left(2+WTL_{\tilde{A}}-WIL_{\tilde{A}}-WFL_{\tilde{A}}\right)+\left(2+WTR_{\tilde{A}}-WIR_{\tilde{A}}-WFR_{\tilde{A}}\right)}{6}$ and

$SV\left(W\tilde{B}\right)=\frac{\left(2+WTL_{\tilde{B}}-WIL_{\tilde{B}}-WFL_{\tilde{B}}\right)+\left(2+WTR_{\tilde{B}}-WIR_{\tilde{B}}-WFR_{\tilde{B}}\right)}{6}$, and

$AV\left(W\tilde{A}\right)=\frac{\left(WTL_{\tilde{A}}+WTR_{\tilde{A}}\right)-\left(WFL_{\tilde{A}}+WFR_{\tilde{A}}\right)}{2}$ and

$AV\left(W\tilde{B}\right)=\frac{\left(WTL_{\tilde{B}}+WTR_{\tilde{B}}\right)-\left(WFL_{\tilde{B}}+WFR_{\tilde{B}}\right)}{2}$, if $SV\left(W\tilde{A}\right)<SV\left(W\tilde{B}\right)$, $W\tilde{A}<W\tilde{B}$; if $SV\left(W\tilde{A}\right)=SV\left(W\tilde{B}\right)$, (1) if $AV\left(W\tilde{A}\right)=AV\left(W\tilde{B}\right)$, $W\tilde{A}=W\tilde{B}$; (2) if $SV\left(W\tilde{A}\right)<SV\left(W\tilde{B}\right)$, $W\tilde{A}<W\tilde{B}$.

Definition 6[20]. Let $W\tilde{A}=\left(\left[WTL_{\tilde{A}},WTR_{\tilde{A}}\right],\left[WIL_{\tilde{A}},WIR_{\tilde{A}}\right],\left[WFL_{\tilde{A}},WFR_{\tilde{A}}\right]\right)$ and $W\tilde{B}=\left(\left[WTL_{\tilde{B}},WTR_{\tilde{B}}\right],\left[WIL_{\tilde{B}},WIR_{\tilde{B}}\right],\left[WFL_{\tilde{B}},WFR_{\tilde{B}}\right]\right)$ be two INNs, the operations are presented:

$$(1)W\tilde{A}\oplus W\tilde{B}=\begin{pmatrix}\left(WTL_{\tilde{A}}+WTL_{\tilde{B}}-WTL_{\tilde{A}}WTL_{\tilde{B}},WTR_{\tilde{A}}+WTR_{\tilde{B}}-WTR_{\tilde{A}}WTR_{\tilde{B}}\right),\\ \left[VIL_{\tilde{A}}VIL_{\tilde{B}},VIR_{\tilde{A}}VIR_{\tilde{B}}\right],\left[VFL_{\tilde{A}}VFL_{\tilde{B}},VFR_{\tilde{A}}VFR_{\tilde{B}}\right]\end{pmatrix};$$

$$(2)\ W\tilde{A}\otimes W\tilde{B}=\begin{pmatrix}\left[WTL_{\tilde{A}}WTL_{\tilde{B}},WTR_{\tilde{A}}WTR_{\tilde{B}}\right],\\ \left[WIL_{\tilde{A}}+WIL_{\tilde{B}}-WIL_{\tilde{A}}WIL_{\tilde{B}},WIR_{\tilde{A}}+WIR_{\tilde{B}}-WIR_{\tilde{A}}WIR_{\tilde{B}}\right],\\ \left[WFL_{\tilde{A}}+WFL_{\tilde{B}}-WFL_{\tilde{A}}WFL_{\tilde{B}},WFR_{\tilde{A}}+WFR_{\tilde{B}}-WFR_{\tilde{A}}WFR_{\tilde{B}}\right]\end{pmatrix};$$

$$(3)\ \lambda W\tilde{A}=\begin{pmatrix}\left[1-\left(1-WTL_{\tilde{A}}\right)^{\lambda},1-\left(1-WTR_{\tilde{A}}\right)^{\lambda}\right],\\ \left[\left(WIL_{\tilde{A}}\right)^{\lambda},\left(WIR_{\tilde{A}}\right)^{\lambda}\right],\left[\left(WFL_{\tilde{A}}\right)^{\lambda},\left(WFR_{\tilde{A}}\right)^{\lambda}\right]\end{pmatrix},\lambda>0;$$

$$(4)\left(V\tilde{A}\right)^{\lambda}=\begin{pmatrix}\left[\left(WTL_{\tilde{A}}\right)^{\lambda},\left(WTR_{\tilde{A}}\right)^{\lambda}\right],\left[\left(WIL_{\tilde{A}}\right)^{\lambda},\left(WIR_{\tilde{A}}\right)^{\lambda}\right],\\ \left[1-\left(1-WFL_{\tilde{A}}\right)^{\lambda},1-\left(1-WFR_{\tilde{A}}\right)^{\lambda}\right]\end{pmatrix},\lambda>0.$$

Definition 7[21]. Let $W\tilde{A}=\left(\left[WTL_{\tilde{A}},WTR_{\tilde{A}}\right],\left[WIL_{\tilde{A}},WIR_{\tilde{A}}\right],\left[WFL_{\tilde{A}},WFR_{\tilde{A}}\right]\right)$ and $W\tilde{B}=\left(\left[WTL_{\tilde{B}},WTR_{\tilde{B}}\right],\left[WIL_{\tilde{B}},WIR_{\tilde{B}}\right],\left[WFL_{\tilde{B}},WFR_{\tilde{B}}\right]\right)$, then INN Hamming distance (INNHD) is assembled:

$$INNHD\left(W\tilde{A},W\tilde{B}\right)=\frac{1}{6}\left(\begin{array}{l}\left|WTL_{\tilde{A}}-WTL_{\tilde{B}}\right|+\left|WTR_{\tilde{A}}-WTR_{\tilde{B}}\right|+\left|WIL_{\tilde{A}}-WIL_{\tilde{B}}\right|+\\ \left|WIR_{\tilde{A}}-WIR_{\tilde{B}}\right|+\left|WFL_{\tilde{A}}-WFL_{\tilde{B}}\right|+\left|WFR_{\tilde{A}}-WFR_{\tilde{B}}\right|\end{array}\right) \quad (5)$$

The INNWA operator [20] are assembled:

Definition 8[20]. Let $V\tilde{A}_j=\left(\left[VTL_j,VTR_j\right],\left[VIL_j,VIR_j\right],\left[VFL_j,VFR_j\right]\right)$ be INNs, the INNWG operator is assembled:

$$\begin{aligned}&\mathrm{INNWG}\left(W\tilde{A}_1,W\tilde{A}_2,\ldots,W\tilde{A}_n\right)\\&=\left(W\tilde{A}_1\right)^{w_1}\otimes\left(W\tilde{A}_2\right)^{w_2},\ldots\otimes\left(W\tilde{A}_n\right)^{w_n}=\overset{n}{\underset{j=1}{\otimes}}\left(W\tilde{A}_j\right)^{w_j}\\&=\left(\begin{array}{l}\left[\prod_{j=1}^{n}\left(WTL_{ij}\right)^{w_j},\prod_{j=1}^{n}\left(WTR_{ij}\right)^{w_j}\right],\\ \left[1-\prod_{k=1}^{l}\left(1-WFL_{ij}^{k}\right)^{w_j},1-\prod_{k=1}^{l}\left(1-WFR_{ij}^{k}\right)^{w_j}\right],\\ \left[1-\prod_{k=1}^{l}\left(1-WTL_{ij}^{k}\right)^{w_j},1-\prod_{k=1}^{l}\left(1-WTR_{ij}^{k}\right)^{w_j}\right]\end{array}\right)\end{aligned} \quad (6)$$

where $w=\left(w_1,w_2,...,w_n\right)^T$ be weight of $W\tilde{A}_j$, $w_j>0,\sum_{j=1}^{n}w_j=1.$

3. Method

INN-SHS-CoCoSo approach is assembled for MAGDM. Let $WA=\left\{WA_1,WA_2,\cdots,WA_m\right\}$ be alternatives, and attributes $WG=\left\{WG_1,WG_2,\cdots,WG_n\right\}$ with weight ω, where $\omega_j\in\left[0,1\right]$, $\sum_{j=1}^{n}\omega_j=1$ and experts $WE=\left\{WE_1,WE_2,\cdots,WE_q\right\}$ with weight be $\left\{w_1,w_2,\cdots,w_t\right\}$.

Suppose the universe set $D=\{C_1,C_2,\ldots,C_n\}$ and the $P(D)$ is the power set of D. As well $Y_1,Y_2,\ldots,Y_n$ are criteria. $P(Y_1),P(Y_2),and\ P(Y_3)$ are the power sets of $Y_1,Y_2,\ldots,Y_n$.

Let $F:P(Y_1)\times P(Y_2)\times P(Y_3)\to P(D)$ and this is called SuperHyperSoft.

$$\begin{aligned}&P(Y_1)\times P(Y_2)\times P(Y_3)\\&=\left\{\{Y_{11}\},\{Y_{12}\},\{Y_{11},Y_{12}\}\right\}\times\left\{\{Y_{21}\},\{Y_{22}\},\{Y_{21},Y_{22}\}\right\}\\&\times\left\{\{Y_{31}\},\{Y_{32}\},\{Y_{33}\},\{Y_{31},Y_{32}\},\{Y_{31},Y_{33}\},\{Y_{32},Y_{33}\},\{Y_{31},Y_{32},Y_{33}\}\right\}\end{aligned}$$

The INN-SHS-CoCoSo approach is assembled for MAGDM:

Step 1. Build the INN-matrix $WR^t=\left[WR_{ij}^t\right]_{m\times n}=\left(\left[WTL_{ij}^t,WTR_{ij}^t\right],\left[WIL_{ij}^t,WIR_{ij}^t\right],\left[WFL_{ij}^t,WFR_{ij}^t\right]\right)_{m\times n}$ and average matrix $WR=\left[WR_{ij}\right]_{m\times n}$:

$$WR^t = \left[WR_{ij}^t\right]_{m\times n} = \begin{array}{c} \\ WA_1 \\ WA_2 \\ \vdots \\ WA_m \end{array} \begin{array}{c} \begin{array}{cccc} WG_1 & WG_2 & \dots & WG_n \end{array} \\ \begin{bmatrix} WR_{11}^t & WR_{12}^t & \dots & WR_{1n}^t \\ WR_{21}^t & WR_{22}^t & \dots & WR_{2n}^t \\ \vdots & \vdots & \vdots & \vdots \\ WR_{m1}^t & WR_{m2}^t & \dots & WR_{mn}^t \end{bmatrix} \end{array} \tag{7}$$

$$WR = \left[WR_{ij}\right]_{m\times n} = \begin{array}{c} \\ WA_1 \\ WA_2 \\ \vdots \\ WA_m \end{array} \begin{array}{c} \begin{array}{cccc} WG_1 & WG_2 & \dots & WG_n \end{array} \\ \begin{bmatrix} WR_{11} & WR_{12} & \dots & WR_{1n} \\ WR_{21} & WR_{22} & \dots & WR_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ WR_{m1} & WR_{m2} & \dots & WR_{mn} \end{bmatrix} \end{array} \tag{8}$$

Based on INNWG, the

$WR = \left[WR_{ij}\right]_{m\times n} = \left(\left[WTL_{ij}, WTR_{ij}\right], \left[WIL_{ij}, WIR_{ij}\right], \left[WFL_{ij}, WFR_{ij}\right]\right)_{m\times n}$ is:

$$\begin{aligned} WR_{ij} &= \left(WR_{ij}^1\right)^{w_1} \otimes \left(WR_{ij}^2\right)^{w_2}, \dots \otimes \left(WR_{ij}^t\right)^{w_t} \\ &= \left(\begin{array}{l} \left[\prod_{k=1}^{t}\left(WTL_{ij}^t\right)^{w_j}, \prod_{k=1}^{t}\left(WTR_{ij}^t\right)^{w_j}\right], \\ \left[1-\prod_{k=1}^{t}\left(WIL_{ij}^t\right)^{w_j}, 1-\prod_{k=1}^{t}\left(WIR_{ij}^t\right)^{w_j}\right], \\ \left[1-\prod_{k=1}^{t}\left(WFL_{ij}^t\right)^{w_j}, 1-\prod_{k=1}^{t}\left(WFR_{ij}^t\right)^{w_j}\right] \end{array}\right) \end{aligned} \tag{9}$$

Step 2. Then we obtained the crisp values.

Step 3. Combined the decision matrix.

Step 4. Normalized the decision matrix.

Step 5. Normalize the decision matrix

$Y_{ij} = \frac{WR_{ij} - \min WR_{ij}}{\max WR_{ij} - \min WR_{ij}}$ for beneficial criteria. (10)

$Y_{ij} = \frac{\max WR_{ij} - WR_{ij}}{\max WR_{ij} - \min WR_{ij}}$ for cost criteria. (11)

Step 6. Compute the criteria weights by using the mean method.

Step 7. Compute the total of the weighted comparability sequence of each alternative.

$X_i = \sum_{j=1}^{n} w_j Y_{ij}$ (12)

Step 8. Compute the whole power weight of compatibility sequence of each alternative.

$Z_i = \sum_{j=1}^{n} \left(Y_{ij}\right)^{w_j}$ (13)

Step 9. Compute the relative weights of the alternatives by using the aggregate strategies.

$K_{ia} = \frac{Z_i + X_i}{\sum_{i=1}^{m}(Z_i + X_i)}$ (14)

$K_{ib} = \frac{X_i}{\min X_i} + \frac{Z_i}{\min Z_i}$ (15)

$$K_{ic} = \frac{\gamma(X_i)+(1-\beta)(Z_i)}{\beta \max X_i+(1-\beta)\max Z_i};\ 0 \le \gamma \le 1 \quad (16)$$

Step 10. Compute the assessment value

$$K_i = (K_{ia}K_{ib}K_{ic})^{\frac{1}{3}} + \frac{1}{3}(K_{ia} + K_{ib} + K_{ic}) \quad (17)$$

Step 11. Rank the alternatives.

4. Numerical example for rural tourism competitiveness evaluation

With the development of the social economy and the improvement of living standards, people's demand for leisure activities and connecting with nature is gradually increasing. Rural tourism, which encompasses rural scenery, rural life, and rural culture, has emerged as a popular trend. The impact of rural tourism on rural society and the economy is multifaceted, promoting increased farmers' income, agricultural efficiency, rural cultural construction, and ecological environment improvement. It significantly drives the adjustment of rural industrial structures and the construction of a modern rural economic development system. Therefore, promoting the high-quality development of rural tourism aligns with contemporary needs. Rural tourism attracts urban tourists primarily because rural areas possess primitive cultures and histories that are rapidly disappearing with urban-rural integration. In developing rural tourism, local governments must comprehensively understand and analyze local cultural resources, systematically extract regional characteristic cultural resources with tourism development value, and focus on the systematic development of intangible cultural heritage. This heritage should be organically integrated with the modern tourism industry to enhance the cultural connotation of rural tourism, form a new development model that achieves mutual benefits, and integrate culture and tourism effectively.

The development of rural tourism relies heavily on the active participation of local residents, especially in a competitive environment. Many regions have rich intangible cultural heritage, and effectively integrating this heritage with the rural tourism industry can further enhance its core competitiveness. To continuously strengthen the core competitiveness of the rural tourism industry, it is crucial for local villagers to actively participate, increase the inheritance and protection of intangible cultural heritage, enhance its influence, and promote the development of the rural tourism industry. Evaluating rural tourism competitiveness in surrounding cities involves MAGDM. This process requires a comprehensive assessment of various attributes and factors that contribute to the attractiveness and sustainability of rural tourism destinations. By employing advanced decision-making approaches such as INN-SHS-CoCoSo technique, local governments and stakeholders can make informed decisions to foster the high-quality development of rural tourism, ensuring it meets the evolving needs and preferences of tourists while preserving and enhancing local cultural heritage.

Five rural tourism demonstration zones assessed with six attributes: Ten rural tourism demonstration zones are assessed with INNs under three experts with equal weight. Table 1 shows the list of criteria and weight.

Table 1. List of criteria.

	Criteria	Values	Weights
C_1	Transport Accessibility	<50%, >100%.	0.174973
C_2	Cultural Heritage Sites	<20, >20	0.159406
C_3	Accommodation Capacity	<50, >50 beds	0.176843
C_4	Visitor Satisfaction	High, Moderate, Low	0.158682
C_5	Environmental Conservation Efforts	<50%, >50%	0.164595
C_6	Tourism Revenue	<$1M, > $100M.	0.1655

We built the decision matrix as shown in Table 2. Then we compute the criteria weights as shown in Table 1.

Table 2. The decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_2	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_3	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
A_4	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_5	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_6	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
A_7	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
A_8	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_9	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_{10}	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_2	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_3	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
A_4	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_5	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_6	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
A_7	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_8	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
A_9	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_{10}	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_2	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_3	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])

A_4	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_5	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_6	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
A_7	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
A_8	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
A_9	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
A_{10}	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])

Based on SuperHyperSoft we can suggest a set of suggestions for sub criteria to rank the alternatives Let F: $(\{< 50\%, > 100\%\}, \{< 20, > 20\}, \{> 50 \text{ beds}\}, \{\text{High}\}, \{> 50\%\}, \{> \$100\text{M}\})$. There are four suggestions such as:

S_1: <50%, <20, >50 beds, high, >50%, <$100m

$S_{2:}$ <50%, >20, >50 beds, high, >50%, <$100m

$S_{3:}$ >50%, <20, >50 beds, high, >50%, <$100m

$S_{4:}$ >50%, >20, >50 beds, high, >50%, <$100m

Then we applied the proposed method for 4 suggestions.

Suggestion 1

We normalized the decision matrix as shown in Table 3.

Then we computed the total of the weighted comparability sequence for each alternative as shown in Table 4.

Then we computed the power weight.

Then we computed the relative weights of the alternatives.

Then we computed assessment values.

Then we ranked the alternatives as shown in Figure 1.

Table 3. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.43	0.31068	0.606796	0.755208	0.730539	0.843318
A_2	0.96	0.349515	0.587379	0.536458	0.742515	1
A_3	0.235	0.694175	0.81068	0	0.107784	0
A_4	1	0.621359	0.742718	0.760417	0.197605	0.672811
A_5	0.885	0.514563	0.723301	0.78125	1	0.539171
A_6	0.625	0.446602	0.121359	0.979167	0.047904	0.75576
A_7	0.675	0.121359	0	0.427083	0.57485	0.820276
A_8	0	0	1	0.505208	0	0.331797
A_9	0.48	1	0.927184	1	0.083832	1
A_{10}	0.91	0.708738	0.650485	0.609375	0.57485	0.179724

Table 4. The weighted comparability sequence.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.075238	0.049524	0.107308	0.119838	0.120243	0.139569
A_2	0.167974	0.055715	0.103874	0.085126	0.122214	0.1655
A_3	0.041119	0.110656	0.143363	0	0.017741	0
A_4	0.174973	0.099049	0.131345	0.120665	0.032525	0.11135
A_5	0.154851	0.082025	0.127911	0.123971	0.164595	0.089233
A_6	0.109358	0.071191	0.021462	0.155376	0.007885	0.125078
A_7	0.118107	0.019345	0	0.067771	0.094618	0.135756
A_8	0	0	0.176843	0.080168	0	0.054913
A_9	0.083987	0.159406	0.163966	0.158682	0.013798	0.1655
A_{10}	0.159225	0.112977	0.115034	0.096697	0.094618	0.029744

Suggestion 2

We normalized the decision matrix as shown in Table 5.

Then we computed the total of the weighted comparability sequence for each alternative as shown in Table 6.

Then we computed the power weight.

Then we computed the relative weights of the alternatives.

Then we computed The assessment values.

Then we ranked the alternatives as shown in Figure 1.

Table 5. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.43	0.626728	0.606796	0.755208	0.730539	0.843318
A_2	0.96	0.557604	0.587379	0.536458	0.742515	1
A_3	0.235	0.331797	0.81068	0	0.107784	0
A_4	1	0.820276	0.742718	0.760417	0.197605	0.672811
A_5	0.885	0.603687	0.723301	0.78125	1	0.539171
A_6	0.625	0.691244	0.121359	0.979167	0.047904	0.75576
A_7	0.675	0.672811	0	0.427083	0.57485	0.820276
A_8	0	0	1	0.505208	0	0.331797
A_9	0.48	1	0.927184	1	0.083832	1
A_{10}	0.91	0.442396	0.650485	0.609375	0.57485	0.179724

Table 6. The weighted comparability sequence.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.075238	0.099904	0.107308	0.119838	0.120243	0.139569
A_2	0.167974	0.088886	0.103874	0.085126	0.122214	0.1655
A_3	0.041119	0.052891	0.143363	0	0.017741	0
A_4	0.174973	0.130757	0.131345	0.120665	0.032525	0.11135
A_5	0.154851	0.096231	0.127911	0.123971	0.164595	0.089233

A_6	0.109358	0.110189	0.021462	0.155376	0.007885	0.125078
A_7	0.118107	0.10725	0	0.067771	0.094618	0.135756
A_8	0	0	0.176843	0.080168	0	0.054913
A_9	0.083987	0.159406	0.163966	0.158682	0.013798	0.1655
A_{10}	0.159225	0.070521	0.115034	0.096697	0.094618	0.029744

Suggestion 3

We normalized the decision matrix as shown in Table 7.

Then we computed the total of the weighted comparability sequence for each alternative as shown in Table 8.

Then we computed the power weight.

Then we computed the relative weights of the alternatives.

Then we computed The assessment values.

Then we ranked the alternatives as shown in Figure 1.

Table 7. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.43	0.31068	0.606796	0.755208	0.730539	0.843318
A_2	0.96	0.349515	0.587379	0.536458	0.742515	1
A_3	0.235	0.694175	0.81068	0	0.107784	0
A_4	1	0.621359	0.742718	0.760417	0.197605	0.672811
A_5	0.885	0.514563	0.723301	0.78125	1	0.539171
A_6	0.625	0.446602	0.121359	0.979167	0.047904	0.75576
A_7	0.675	0.121359	0	0.427083	0.57485	0.820276
A_8	0	0	1	0.505208	0	0.331797
A_9	0.48	1	0.927184	1	0.083832	1
A_{10}	0.91	0.708738	0.650485	0.609375	0.57485	0.179724

Table 8. The weighted comparability sequence.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.075238	0.049524	0.107308	0.119838	0.120243	0.139569
A_2	0.167974	0.055715	0.103874	0.085126	0.122214	0.1655
A_3	0.041119	0.110656	0.143363	0	0.017741	0
A_4	0.174973	0.099049	0.131345	0.120665	0.032525	0.11135
A_5	0.154851	0.082025	0.127911	0.123971	0.164595	0.089233
A_6	0.109358	0.071191	0.021462	0.155376	0.007885	0.125078
A_7	0.118107	0.019345	0	0.067771	0.094618	0.135756
A_8	0	0	0.176843	0.080168	0	0.054913
A_9	0.083987	0.159406	0.163966	0.158682	0.013798	0.1655
A_{10}	0.159225	0.112977	0.115034	0.096697	0.094618	0.029744

Suggestion 4

We normalized the decision matrix as shown in Table 9.

Then we computed the total of the weighted comparability sequence for each alternative as shown in Table 10.

Then we computed the power weight.

Then we computed the relative weights of the alternatives.

Then we computed The assessment values.

Then we ranked the alternatives as shown in Figure 1. We show the alternative 5 is the best.

Table 9. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.43	0.626728	0.606796	0.755208	0.730539	0.843318
A_2	0.96	0.557604	0.587379	0.536458	0.742515	1
A_3	0.235	0.331797	0.81068	0	0.107784	0
A_4	1	0.820276	0.742718	0.760417	0.197605	0.672811
A_5	0.885	0.603687	0.723301	0.78125	1	0.539171
A_6	0.625	0.691244	0.121359	0.979167	0.047904	0.75576
A_7	0.675	0.672811	0	0.427083	0.57485	0.820276
A_8	0	0	1	0.505208	0	0.331797
A_9	0.48	1	0.927184	1	0.083832	1
A_{10}	0.91	0.442396	0.650485	0.609375	0.57485	0.179724

Table 10. The weighted comparability sequence.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.075238	0.099904	0.107308	0.119838	0.120243	0.139569
A_2	0.167974	0.088886	0.103874	0.085126	0.122214	0.1655
A_3	0.041119	0.052891	0.143363	0	0.017741	0
A_4	0.174973	0.130757	0.131345	0.120665	0.032525	0.11135
A_5	0.154851	0.096231	0.127911	0.123971	0.164595	0.089233
A_6	0.109358	0.110189	0.021462	0.155376	0.007885	0.125078
A_7	0.118107	0.10725	0	0.067771	0.094618	0.135756
A_8	0	0	0.176843	0.080168	0	0.054913
A_9	0.083987	0.159406	0.163966	0.158682	0.013798	0.1655
A_{10}	0.159225	0.070521	0.115034	0.096697	0.094618	0.029744

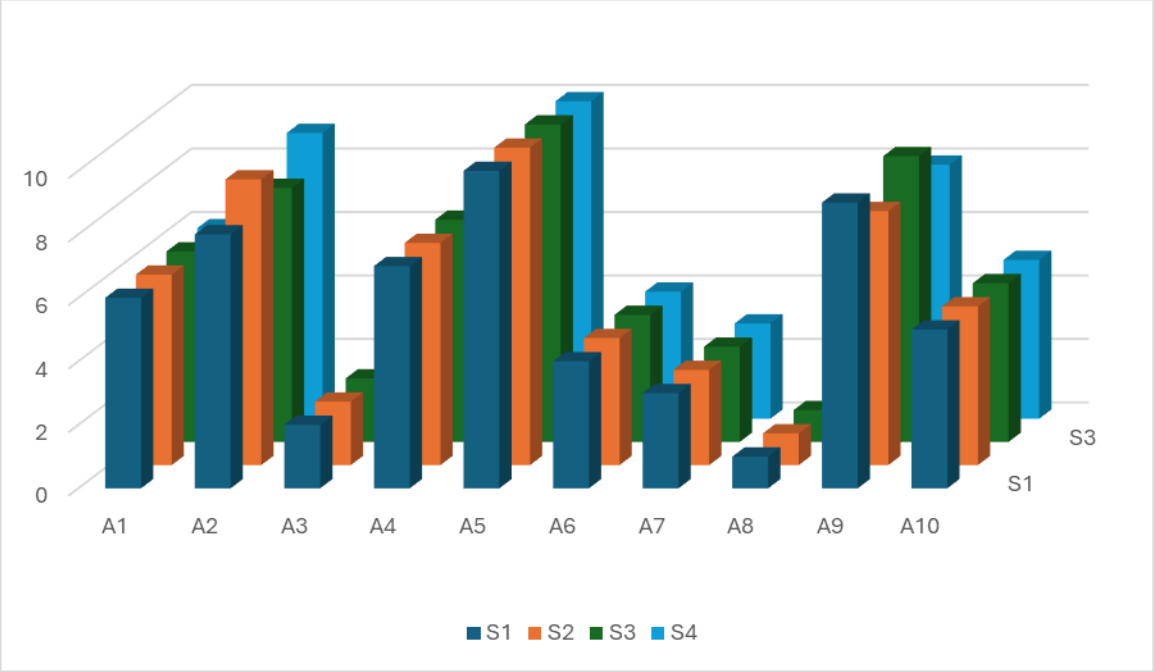


Figure 1. The rank of alternatives.

5. Conclusion and contributions

The evaluation of rural tourism competitiveness around cities aims to analyze and assess the overall performance and attractiveness of rural tourism destinations in the market. With the acceleration of urbanization, more and more urban residents are choosing rural areas as short-term vacation and leisure destinations, making the evaluation of rural tourism competitiveness particularly important. By comprehensively analyzing the resource advantages, visitor experience, market performance, and development potential of rural tourism, stakeholders and operators can better understand their market position and identify potential areas for improvement. At the same time, such evaluations provide a basis for policymakers to optimize rural tourism development strategies, improve the quality of tourism services, and promote the sustainable growth of rural economies. Ultimately, the competitiveness of rural tourism not only affects the number of visitors and their satisfaction but also determines its long-term development prospects in the tourism market. The evaluation of rural tourism competitiveness in surrounding cities involves MAGDM. Recently, the CoCoSo approach has been effectively applied to address MAGDM challenges. In this context, INSs serve as a tool for handling uncertain information during the evaluation process. The proposed INN-SHS-CoCoSo approach utilizes the Hamming distance to address MAGDM under INSs.

This approach integrates the following components:

(1) INSs with SuperHyperSoft for Uncertainty: INSs with SuperHyperSoft effectively manage the uncertainty and vagueness inherent in the evaluation process, allowing for more

nuanced decision-making.

(2) Hamming Distance: The Hamming distance is employed to measure the difference between interval neutrosophic numbers.

A numerical case study is provided to validate the effectiveness of the proposed INN-SHS-CoCoSo approach. This case study demonstrates how the approach can be applied to assess rural tourism competitiveness, thereby offering a robust framework for decision-makers to prioritize and optimize rural tourism development strategies.

6. Research limitations and future research works

Although this paper proposes an effective evaluation method for the competitiveness of rural tourism around cities, there are still some shortcomings. First, the neglect of external factors is a major issue. The paper focuses on internal factors (such as resources and facilities) for evaluation but overlooks the potential impact of external factors like policy support, market demand changes, and technological advancements. These external factors could play a crucial role in real-world operations. Second, the complexity of the model's application limits its practicality. While the use of Interval Neutrosophic Sets (INSs) effectively handles uncertainty, the calculations are complex and highly specialized, making it challenging for ordinary rural tourism managers or decision-makers to apply these methods efficiently. Lastly, the lack of consideration for dynamic changes is another deficiency. The evaluation in this paper is based on static data, without accounting for the dynamic nature of factors such as the rural tourism market over time, which may result in a lack of forward-looking competitiveness assessment.

For future research directions, the following three aspects can be considered:

(1) Incorporating external factors: Future studies should integrate external factors such as policy support, changes in market demand, and technological innovations into the evaluation model of rural tourism competitiveness to improve the comprehensiveness and realism of the assessment. This would not only more accurately reflect competitiveness but also provide stronger references for policymakers.

(2) Simplifying the model and developing practical tools: Further research should consider how to simplify complex evaluation models or develop user-friendly decision-support tools, enabling rural tourism managers to more easily apply these models. This would help improve the operability and applicability of the evaluation methods.

(3) Introducing dynamic evaluation models: Future research could explore how to introduce a time dimension into competitiveness evaluations, developing dynamic models that track changes in rural tourism competitiveness over time. This would help decision-makers adjust strategies in

response to changes in the market and environment, ensuring the sustainable development of rural tourism.

References

[1] Lebe SS, Milfelner B. Innovative organisation approach to sustainable tourism development in rural areas. Kybernetes. 2006;35(7-8):1136-46.

[2] Zasada I, Piorr A. The role of local framework conditions for the adoption of rural development policy: An example of diversification, tourism development and village renewal in brandenburg, germany. Ecological Indicators. 2015;59:82-93.

[3] Pejanovic R, Demirovic D, Glavas-Trbic D, Maksimovic G, Tomas-Simin M. Clusters as a factor of competitiveness of rural tourism destinations in the danube region of the republic of serbia. Tourism Economics. 2017;23(2):475-82.

[4] Feng NP, Wei FF, Zhang KH, Gu DX. Innovating rural tourism targeting poverty alleviation through a multi-industries integration network: The case of zhuanshui village, anhui province, china. Sustainability. 2018;10(7):18.

[5] Long NT, Nguyen TL. Sustainable development of rural tourism in an giang province, vietnam. Sustainability. 2018;10(4):20.

[6] Nabeeh N. Assessment and contrast the sustainable growth of various road transport systems using intelligent neutrosophic multi-criteria decision-making model. Sustainable Machine Intelligence Journal. 2023;2:(2):1-12.

[7] Mohamed R, Ismail MM. Leveraging an uncertainty methodology to appraise risk factors threatening sustainability of food supply chain. Neutrosophic Systems with Applications. 2024;19:30-52.

[8] A. Nabeeh N, M. Sallam K. A combined compromise solution (cocoso) of mcdm problems for selection of medical best bearing ring. Neutrosophic Optimization and Intelligent Systems. 2024;1:1-13.

[9] Lei F, Cai Q, Liao NN, Wei GW, He Y, Wu J, et al. Todim-vikor method based on hybrid weighted distance under probabilistic uncertain linguistic information and its application in medical logistics center site selection. Soft Computing. 2023;27(13):8541-59.

[10] Yazdani M, Zarate P, Zavadskas EK, Turskis Z. A combined compromise solution (cocoso) method for multi-criteria decision-making problems. Management Decision. 2018;57(9):2501-19.

[11] Badi I, Jibril M, Bakır M. A composite approach for site optimization of fire stations. Journal of Intelligent Management Decision. 2022;1(1):28-35.

[12] Badi I, Bouraima MB. Development of mcdm-based frameworks for proactively managing the most critical risk factors for transport accidents: A case study in libya. Spectrum of engineering and management sciences. 2023;1(1):38-47.

[13] Badi I, Stević Ž, Bouraima MB. Evaluating free zone industrial plant proposals using a combined full consistency method-grey-cocoso model. Journal of Industrial Intelligence. 2023;1(2):101-09.

[14] Tešić D, Božanić D, Radovanović M, Petrovski A. Optimising assault boat selection for military operations: An application of the dibr ii-bm-cocoso mcdm model. Journal of Intelligent Management Decision. 2023;2(4):160-71.

[15] Shannon CE. A mathematical theory of communication. Bell System Technical Journal. 1948;27(4):379-423.

[16] Wang H, Smarandache F, Zhang YQ, Sunderraman R. Interval neutrosophic sets and logic: Theory

and applications in computing. Hexis: Phoenix, AZ, USA. 2005.

[17] Wang H, Smarandache F, Zhang YQ, Sunderraman R. Single valued neutrosophic sets. Multispace Multistruct. 2010(4):410-13.

[18] Peng JJ, Wang JQ, Wang J, Zhang HY, Chen XH. Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. International Journal of Systems Science. 2016;47(10):2342-58.

[19] Huang YH, Wei GW, Wei C. Vikor method for interval neutrosophic multiple attribute group decision-making. Information. 2017;8(4):144.

[20] Zhang HY, Wang JQ, Chen XH. Interval neutrosophic sets and their application in multicriteria decision making problems. Scientific World Journal. 2014;2014:645953.

[21] Ye J. Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. Journal of Intelligent & Fuzzy Systems. 2014;26(1):165-72.

Received: Aug 28, 2024. Accepted: Nov 24, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Enhancing Neutrosophic Data Analysis: A Review of Neutrosophic Measures and Applications with Neutrostat

Zahid Khan[1,*] and Katrina Lane Krebs[2]

[1]Department of Quantitative Methods, University of Pannonia, Egyetem str. 10, Veszprem, H-8200, Hungary.; zahidkhan@hu.edu.pk

[2]SNM Social Sciences, Central Queensland University, 160 Ann St, Brisbane, QLD 4000, QLD, Australia; k.lane-krebs@cqu.edu.au

[*]Correspondence: zahidkhan@hu.edu.pk

Abstract. Neutrosophic statistical measures analyze data that is not fully determined, often due to imprecise observations. This type of data presents a major concern in neutrosophic statistics. The existing literature categorizes neutrosophic measures into two types: descriptive and inferential, aiming to broaden the categorization of statistics in these two major areas. Every statistical measure can be contextualized within the neutrosophic framework by acknowledging the inherent imprecision, vagueness, or not fully defined. In this study, the major focus is on reviewing existing neutrosophic measures rather than proposing new ones. The aim is to enhance current neutrosophic structures to make them more beneficial for end users in their analysis. Additionally, one of the contemporary challenges in neutrosophic data analysis is the dimension of data. We develop the R library neutrostat to efficiently describe complex and larger imprecise datasets. Finally, real-world examples are provided to review the effectiveness of the neutrostat package for analyzing neutrosophic data and evaluating existing neutrosophic measures.

Keywords: Descriptive measures; inferential measures; neutrosophic measures; neutrosophic probability; R package

1. Introduction

In data analysis, there are various statistical methods available for analyzing and interpreting data in specific situations [1]. It is important to understand the assumptions and conditions of these methods to choose the most appropriate option for data analysis [2]. The two main types of statistical methods used are descriptive statistics, which summarize data with measures including presentation of data using graphs, measuring central tendency and dispersion, and inferential statistics, which make conclusions based on sample data using tests like the z-test and t-test or constructing confidence intervals [3, 4]. Choosing the appropriate

statistical method is crucial and depends on three key factors: the purpose and goals of the research, the type and distribution of data, and the characteristics of the observations being analyzed [5]. Choosing the right statistical method is crucial when analyzing real-world data [6]. Selecting an inappropriate method can lead to issues in interpreting findings and impact the overall validity and conclusion of the study [7]. It is essential to understand the assumptions and conditions of different statistical methods to select the most correct option for data analysis [8]. Many researchers make avoidable mistakes in statistical analysis by using the wrong approach, such as using a z-test instead of a t-test or chi-square test instead of correlation analysis, using a normal distribution test method for skewed data, applying methods for correlated data but they actually developed under independent identically distribution scenarios—these are but a few common examples [9]. Many statistical software packages, such as SPSS, R, Stata, SAS, S-Plus, MATLAB, etc., are available in the market, which makes analysis easier for high-dimensional data. While all computational tools help with analysis, they do not assist in selecting the appropriate method [10, 11]. Among research communities, R software appears as the most popular compared to other commercial statistical software because it is free and open source. The R software is flexible, easy to use, and allows users to create their own functions or packages for specific tasks [12]. Despite advancements in statistical analysis tools, choosing the right test remains challenging, especially for those without a statistical background. Aside from understanding statistical methods, it is crucial to consider the nature and type of data collected and the study's objective. The statistical methods chosen should align with the study's goal and be suitable for the data at hand.

When all data or some part of it involves imprecise, ambiguous, or incomplete portions, here the methods of neutrosophic statistics comes into play [13, 14, 15]. Neutrosophic statistics provides a new foundation for dealing with real-world scenarios that have indeterminate information. Fundamental concepts of neutrosophy were introduced by Smarandache in 1995 as a generalization for fuzzy logic and intuitionistic fuzzy logic [16]. Expanding traditional statistical measures with neutrosophic logic allows for parameters to have undetermined values, enabling better handling of situations involving imprecise or clear statistical data, particularly when working with vague and imprecise statistical information [17, 18]. According to the concept of neutrosophic logic, crisp values can be represented by a neutrosophic structure with determinate and indeterminate parts. When the indeterminate part of a number becomes zero, it converts to a classical structure [19, 20]. Thus, neutrosophic statistical methods have an advantage over classical methods due to their generalization with the extra benefit of analyzing neutrosophic data. Neutrosophic logic has been utilized to transform various statistical measures to accommodate imprecise datasets[21, 22, 23]. Some descriptive measures have been thoroughly explored in [24], with many methods still pending further examination. For

inferential statistics in the domain of neutrosophic logic, significant contributions can be found in the research conducted by Smarandache [13]. Many statistical distributions that reflect the mathematical data generated models beyond any data source have also been converted into a neutrosophic framework [25, 26, 27].

In this paper, we propose enhancements to the descriptive aspects of neutrosophic structures. We aim to rectify inconsistencies encountered in published literature, where the principles of neutrosophic structures are inaccurately utilized. Our objective is not to critique existing studies but to point out notable discrepancies that require consideration for the successful application of neutrosophic measures in practical situations. Additionally, an orientation of some R functions, which are part of our ongoing project neutrostat package, is also presented. In real-world scenarios, dimensions can become quite large, and this R tool can simplify the calculations. It is not feasible to discuss every neutrosophic measure due to the limited source of this work; however, we suggest that many existing works could benefit from refinement and proper implementation of the neutrosophic framework discussed in this work.

This paper is structured as follows: In Section 2, some neutrosophic data and their algebraic operations are presented. In Section 3, some neutrosophic descriptive measures with major discrepancies are discussed. In Section 4, the neutrostat package with some key functions is described. Finally, the major findings of the work are concluded in Section 5.

2. Neutrosophic Data

Neutrosophic data is a collection of information where some or all of the data is uncertain to a certain extent. This neutrosophic data can be in the form of discrete and continuous. For example, the following data represent the neutrosophic form of data: $2, 3, 5, 6, 3 + I_1, 8, 7 + I_2$ where I_1 and I_2 represent the indeterminate parts of two numbers 3 and 7 with any numerical values say $I_1 = [4, 6]$ and $I_2 = [3, 5]$. Due to these indeterminate parts, we do not know the exact value of some data observations. It could be 7 and 9 if we consider I_1 or it can be 10 and 12 if we consider I_2. So, in a general structure, neutrosophic number can be expressed as:

$$N = D + I \tag{1}$$

where D represents the determinate part of the number and I represents the indeterminate part of the number. It is obvious from (1) that when indeterminate becomes zero, the number turns to a crisp number. In this way, the neutrosophic number has a more generated structure as compared to the crisp numbers. When the data is comprised of neutrosophic numbers, the operations of arithmetic operations have specific operations for interval analysis. To understand this interval analysis let us assume that $Z_1 = [\underline{Z}_1, \overline{Z}_1]$ and $Z_2 = [\underline{Z}_2, \overline{Z}_2]$ be the two

closed bounded neutrosophic numbers with operation of addition, subtraction, multiplication and division denoted by star (*) then:

$$\left[\underline{Z}_1, \overline{Z}_1\right] * \left[\underline{Z}_2, \overline{Z}_2\right] = [\alpha, \beta] \tag{2}$$

where $[\alpha, \beta] = \left\{Z_1 * Z_2 \mid \underline{Z}_1 < Z_1 < \overline{Z}_1, \underline{Z}_2 < Z_2 < \overline{Z}_2\right\}$.

Equation (2) can be further simplified as:

$$\left[\underline{Z}_1, \overline{Z}_1\right] + \left[\underline{Z}_2, \overline{Z}_2\right] = \left[\underline{Z}_1 + \underline{Z}_2, \overline{Z}_1 + \overline{Z}_2\right] \tag{3}$$

$$\left[\underline{Z}_1, \bar{Z}_1\right] - \left[\underline{Z}_2, \bar{Z}_2\right] = \left[\underline{Z}_1 - \bar{Z}_2, \overline{Z}_1 - \underline{Z}_2\right] \tag{4}$$

$$\frac{\left[\underline{Z}_1, \overline{Z}_1\right]}{\left[\underline{Z}_2, \overline{Z}_2\right]} = \left[\underline{Z}_1, \overline{Z}_1\right] * \left[\frac{1}{\underline{Z}_2}, \frac{1}{\overline{Z}_2}\right] \tag{5}$$

Equation (5) assumes that zero does not belong to $\left[\bar{Z}_2, \bar{Z}_2\right]$.

$$[\underline{Z}_1, \overline{Z}_1] \times [\underline{Z}_2, \overline{Z}_2] = [\alpha, \beta] \tag{6}$$

where $\alpha = \min\left\{\underline{Z}_1\underline{Z}_2, \underline{Z}_1\overline{Z}_2, \overline{Z}_1\underline{Z}_2, \overline{Z}_1\overline{Z}_2\right\}$ and $\beta = \max\left\{\underline{Z}_1\underline{Z}_2, \underline{Z}_1\overline{Z}_2, \overline{Z}_1\underline{Z}_2, \overline{Z}_1\overline{Z}_2\right\}$. Multiplication and division expressions can be simplified further if we know that $\underline{Z}_1 > 0, \quad \overline{Z}_2 < 0, \quad \underline{Z}_1 < 0 \quad \text{or} \quad \overline{Z}_2 < 0$.

To understand the neutrosophic numbers and their computation procedure, let us take a real-world example. A metrologist recorded the temperature in Veszprem, Hungary, over 10 days. The data may have some variability due to different measuring instruments and missing data. Some temperatures are reported as ranges, either due to estimates or less precise measurements, while others are precise readings from reliable instruments. Here is the recorded temperature data for the 10 days:

22, [18, 20], 25, [21, 23], 19, 26, [24, 27], 23, [22, 24], [20, 22]

Using the definitions of neutrosophic mean, median, and standard deviation [14], we can find the summary measures of temperature as:

Neutrosophic mean= [22, 23.1]

Neutrosophic median= [22, 23]

Neutrosophic standard deviation= [1.73, 3.36]

Note that we have correctly followed the principles defined for neutrosophic numbers and can compute these basic measures using some defined functions of our neutrostat R package in just a few seconds. This package can handle large and complex neutrosophic data efficiently. To ensure accuracy in statistical measures developed using neutrosophic theory, it is essential to verify the fundamental operations and their extensions defined in neutrosophic theory. Unfortunately, a critical review of various papers in this field has revealed violations of these principles in computation. Consequently, the research community in applied sciences is unable

to benefit from the newly proposed neutrosophic statistical measures. In the following section, we have addressed some discrepancies found in neutrosophic measures to make them more user-friendly and practical for end users.

3. Neutrosophic Measures

In this section, we discuss some previously published studies related to neutrosophic statistical measures, highlighting that their computations do not adhere to the properly defined algebraic operations for neutrosophic data. In the study [28], authors analyzed the neutrosophic coefficient variation for utilizing the temperature data on different cities of the southeastern Anatolia region of Turkey. The neutrosophic coefficient of variation is defined as:

$$\text{CV}_n = \frac{\text{Neutrosophic mean}}{\text{Neutrosophic standard deviation}} \tag{7}$$

The computed neutrosophic coefficient of variation reported in the source [28] is given in Table 1.

TABLE 1. Coefficient of variation reported for temperature data reported for different cities of Turkey

City	**Neutrosophic mean**	**Neutrosophic standard deviation**	CV_n
Adiyaman	[0.66, 7.47]	[2.04, 3.06]	[40.96, 309.06]
Batman	[1.33, 8.23]	[1.54, 2.56]	[31.10, 115.79]
Gaziantep	[0.80, 8.57]	[3.13, 7.08]	[82.61, 391.25]
Killis	[3.80, 10.57]	[2.14, 4.49]	[42.48, 56.31]
Mardin	[3.61, 11.23]	[1.83, 3.77]	[33.57, 50.69]

Now computed values of CV_n using the neutrostat R package for the same data are shown in Table 2.

TABLE 2. Neutrosophic coefficient of variation using neutrostat R package

City	**Adiyaman**	**Batman**	**Gaziantep**	**Killis**	**Mardin**
CV_n	[27.29, 463.65]	[18.71, 192.2]	[36.51, 885]	[20.24, 118.19]	[16.28, 104.42]

The results in Table 2 show significant discrepancies between the accurate results and the presented results from the original source [28]. Note that the neutrosophic coefficient of variation has been calculated using the neutrostat package, assuming that the neutrosophic mean and standard deviation were accurately reported. Incorrect computation of the neutrosophic coefficient of variation can have serious implications for decision-making and statistical analysis. This can mislead researchers or stakeholders into drawing incorrect conclusions about the

consistency or volatility of the data. A question may arise is how can we trust that the neutrostat package provides accurate information? To address this, we implement some relevant functions of the neutrostat package on the example data provided by Smarandache [29]. The data consists of the values: 30, 6, [2, 5], [18, 24].

Now, let us see how we can input this data into the "nmean" and "nstd" functions of the R library "neutrostat":

Input:

$x < -list(30, 6, c(2, 5), c(18, 24))$

We can run the following lines in the R workspace to obtain the output:

"nmean(x)" and "nstd(x)"

Output:

Neutro mean = [14, 16.25] and Neutro standard deviation = [9.18, 12.88]. The output matches the values reported by Smarandache [29] Additionally, the neutrostat package offers the advantage of providing the summary function "nsummary(data)", which performs a more comprehensive analysis of the underlying data. Another study conducted by Shahzadi [30] on the computation of neutrosophic coefficient of determination for temperature data to measure the consistency of temperature across different cities of Pakistan.

TABLE 3. The CV_n values for temperature data for different cities of PaKistan

City	**Gujranwala**	**Lahore**	**Karachi**	**Islamabad**	**Sialkot**
μ_n	[29.68, 37.78]	[30.96, 39.07]	[28.54, 32.00]	[27.43, 35.42]	[27.75, 35.32]

However, we believe that there are major discrepancies in the results reported for neutrosophic standard deviation and reported values are not according to rules designed for interval arithmetic. These discrepancies result in neutrosophic standard deviations, leading to wrong statistics reported as evident in Table 3. The correct reported using the R package can be easily obtained using different built-in functions such as "nmean", "nstd" and "ncv" for reporting neutrosophic mean, standard deviation, and coefficient of variation, respectively, for temperature data. Correct statistics using the neutrsostat package are reported in Tables 4-6.

TABLE 4. Neutrosophic mean of temperature data across different cities in Pakistan

City	**Gujranwala**	**Lahore**	**Karachi**	**Islamabad**	**Sialkot**
μ_n	[29.68, 37.78]	[30.96, 39.07]	[28.54, 32.00]	[27.43, 35.42]	[27.75, 35.32]

Table 5. Neutrosophic variability of temperature data across different cities in Pakistan

City	Gujranwala	Lahore	Karachi	Islamabad	Sialkot
σ_n	[7.89, 9.83]	[7.82, 10.11]	[3.41, 4.38]	[7.91, 9.24]	[7.23, 9.61]

Table 6. Neutrosophic coefficient of variation of temperature data across different cities in Pakistan

City	Gujranwala	Lahore	Karachi	Islamabad	Sialkot
CV_n	[20.89, 33.14]	[20.00, 32.65]	[10.66, 15.37]	[22.33, 33.67]	[20.48, 34.61]

Table 5 displays notable differences between the precise findings and the reported data in the source material [30]. These discrepancies are common in various studies on neutrosophic measures, highlighting the importance of further research to ensure that end users can effectively utilize the innovative approach to neutrosophic statistics. Results obtained using the neutrostat package are more profound and reflective.

4. Overview of neutrostat

In this section, a brief description of neutrostat package with its basic functionalities is discussed. The neutrostat is an R software package with functions developed following the principles of neutrosophic statistics originated by Smarandache [13]. The neutrostat package works as a computational tool to analyze data based on vague and imprecise information. The latest release of version 0.0.1 of the package is available on CRAN at https://CRAN.R-project.org/package=neutrostat and the developmental version is hosted on GitHub at https://github.com/kzst/neutrostat. The initial version of the package is equipped with various descriptive functions for neutrosophic numbers, where indeterminacy, inconsistency, and uncertainty are inherent parts of the data. Along with a basic functions package, it also includes some real-world neutrosophic data such as gold prices across Indian cities (goldprice), dioxin ingestion in Japan (dioxin) and temperature data from different cities of Pakistan (citytemp). Additionally, some interval operators, such as addition, subtraction, multiplication, and division for interval arithmetic commonly used in interval calculations of neutrosophic numbers, are built into the package. These operators not only allow the computational procedures of many functions but also facilitate the addition features, such as the conversion of data to neutrosophic form or sorting of data in ascending or descending order. The summary of some key functions of the neutrostat package is outlined below:

TABLE 7. Key functions of the `neutrostat` package with short descriptions

Function	Description	Usage
`interval_df`	Converting data into neutrosophic numbers	`interval_df(data)`
`interval_add`	Return addition of neutrosophic numbers	`interval_add(data)`
`interval_sub`	Return subtraction of neutrosophic numbers	`interval_sub(data)`
`interval_mul`	Return multiplication of neutrosophic numbers	`interval_mul(data)`
`interval_div`	Return division of neutrosophic numbers	`interval_div(data)`
`sort_interval`	Return sorting of neutrosophic numbers	`sort_interval(data)`
`nmean`	Return mean of neutrosophic numbers	`nmean(data)`
`nvar`	Return variance of neutrosophic numbers	`nvar(data)`
`nstd`	Return standard deviation of neutrosophic numbers	`nstd(data)`
`nmedian`	Return median of neutrosophic numbers	`nmedian(data)`
`nquant`	Return quantiles of neutrosophic numbers	`nquant(data)`
`nsk`	Return Pearson coefficient of skewness of neutrosophic numbers	`nsk(data)`
`nkur`	Return kurtosis coefficient of neutrosophic numbers	`nkur(data)`
`nsummary`	Return basic summary of neutrosophic numbers	`nsummary(data)`

Table 7 indicates that the first version (0.0.1) of the neutrostat package is equipped with the necessary fundamental functions required for an initial glance at data analysis. Each function listed returns an interval value, indicating imprecision in data, making it ideal for real-world scenarios where data uncertainty commonly exist. Additionally, the functions listed in Table 7 are user-friendly. A comprehensive description of each function, and applied examples, can be found in the reference manual of the neutrostat package, which is available on CRAN for public use.

5. Conclusions

This research thoroughly examined various neutrosophic statistical measures and classified them into descriptive and inferential categories. Our findings emphasize the significance of integrating the neutrosophic approach in statistical analyses to accommodate imprecise, vague, or undefined data. By utilizing this framework, we can better represent and analyze such complex data commonly encountered in practical situations. Through this review, we have pinpointed both the advantages and drawbacks of existing neutrosophic metrics and proposed methods to improve their functionality for users. Additionally, we have underscored the importance of utilizing R software for managing intricate datasets with imprecise values. We have developed the neutrostat R package to make the neutrosophic data analysis easier for public use with its key functions outlined in this work. With neutrostat library, individuals

can effectively conduct neutrosophic data analysis, even in situations where uncertainties and indeterminacies are present in high-dimension neutrosophic information.

As neutrosophic statistical methods continue to evolve, extending the neutrostat library to incorporate these advancements will be a key focus of future work.

Acknowledgments: This work has been implemented by the TKP2021-NVA-10 project with support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, financed under the 2021 Thematic Excellence Programme funding scheme.

Conflicts of Interest: The authors declare no conflict of interest.

Data availbility

The data used in this work is available within the manuscript.

References

1. Burr, I.W. Applied Statistical Methods; Elsevier, 2014.
2. Bastardoz, N.; Matthews, M.J.; Sajons, G.B.; Ransom, T.; Kelemen, T.K.; Matthews, S.H. Instrumental variables estimation: Assumptions, pitfalls, and guidelines. *Leadersh. Q.* **2023**, *34*, 101673.
3. Shatz, I. Assumption-checking rather than (just) testing: The importance of visualization and effect size in statistical diagnostics. *Behav. Res. Methods* **2024**, *56*, 826–845.
4. Casella, G.; Berger, R. Statistical Inference; CRC Press, 2024.
5. Choi, D.-A.; Tagore, P.; Siddiq, F.; Park, K.; Ewing, R. Descriptive statistics and visualizing data. In *Basic Quantitative Research Methods for Urban Planners*; Routledge, 2020; pp. 107–132.
6. Ravid, R. Practical Statistics for Educators; Rowman & Littlefield, 2024.
7. Di Leo, G.; Sardanelli, F. Statistical significance: p-value, 0.05 threshold, and applications to radiomics—reasons for a conservative approach. *Eur. Radiol. Exp.* **2020**, *4*, 1–8.
8. McShane, B.B.; Bradlow, E.T.; Lynch Jr, J.G.; Meyer, R.J. "Statistical significance" and statistical reporting: Moving beyond binary. *J. Mark.* **2024**, *88*, 1–19.
9. Greenland, S.; Senn, S.J.; Rothman, K.J.; Carlin, J.B.; Poole, C.; Goodman, S.N.; Altman, D.G. Statistical tests, p-values, confidence intervals, and power: A guide to misinterpretations. *Eur. J. Epidemiol.* **2016**, *31*, 337–350.
10. Kopyt, M. Basic operations in the R software. In *Applied Spatial Statistics and Econometrics*; Routledge, 2020; pp. 1–35.
11. Siebert, J. Applications of statistical causal inference in software engineering. *Inf. Softw. Technol.* **2023**, *159*, 107198.
12. James, G.; Witten, D.; Hastie, T.; Tibshirani, R.; Taylor, J. An Introduction to Statistical Learning: With Applications in Python; Springer, 2023.
13. Smarandache, F. Neutrosophic logic—a generalization of the intuitionistic fuzzy logic. *Multispace Multistruct. Neutrosophic Transdisciplinarity (100 Collected Papers Sci.)* **2010**, *4*, 396.
14. Broumi, S.; Bakali, A.; Bahnasse, A. Neutrosophic sets: An overview; Infinite Study, 2018.
15. Garg, H. A new exponential-logarithm-based single-valued neutrosophic set and their applications. *Expert Syst. Appl.* **2024**, *238*, 121854.
16. Smarandache, F. A unifying field in logics: Neutrosophic logic. In *Philosophy*; American Research Press, 1999; pp. 1–141.

17. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability; Infinite Study, 2005.
18. Alhabib, R.; Ranna, M.M.; Farah, H.; Salama, A. Some neutrosophic probability distributions. *Neutrosophic Sets Syst.* **2018**, *22*, 30–38.
19. Khan, Z.; Al-Bossly, A.; Almazah, M.M.; Alduais, F.S. On statistical development of neutrosophic gamma distribution with applications to complex data analysis. *Complexity* **2021**, *2021*, 3701236.
20. Duan, W.-Q.; Khan, Z.; Gulistan, M.; Khurshid, A. Neutrosophic exponential distribution: Modeling and applications for complex data analysis. *Complexity* **2021**, *2021*, 5970613.
21. Sherwani, R.A.K.; Shakeel, H.; Saleem, M.; Awan, W.B.; Aslam, M.; Farooq, M. A new neutrosophic sign test: An application to COVID-19 data. *PLoS ONE* **2021**, *16*, 0255671.
22. Khan, Z.; Almazah, M.M.; Odhah, H.H.; Alshanbari, H.M. Generalized Pareto model: Properties and applications in neutrosophic data modeling. *Math. Probl. Eng.* **2022**, *2022*, 3686968.
23. Ibrahim, A.M.M.; Khan, Z. Neutrosophic Laplace distribution with properties and applications in decision making. *Int. J. Neutrosophic Sci.* **2024**, *1*, 73–3.
24. Smarandache, F.; Abdel-Basset, M. Neutrosophic Operational Research: Methods and Applications; Springer, 2021.
25. Khan, Z.; Gulistan, M.; Kausar, N.; Park, C. Neutrosophic Rayleigh model with some basic characteristics and engineering applications. *IEEE Access* **2021**, *9*, 71277–71283.
26. Ibrahim, A.M.M.; Alduais, F.S.; Khan, Z.; Amin, A.; LaneKrebs, K. Neutrosophic structure of the geometric model with applications. *Neutrosophic Sets Syst.* **2023**, *61*, 382–394.
27. Al-Duais, F.S. Neutrosophic log-gamma distribution and its applications to industrial growth. *Neutrosophic Sets Syst.* **2024**, *72*, 22.
28. Kandemir, H.S.; Aral, N.D.; Karakas, M.; Et, M. Neutrosophic statistical analysis of temperatures of cities in the southeastern Anatolia region of Turkey. *Neutrosophic Syst. Appl.* **2024**, *14*, 50–59.
29. Smarandache, F. Introduction to Neutrosophic Statistics; Infinite Study, 2013.
30. Shahzadi, I. Neutrosophic statistical analysis of temperature of different cities of Pakistan; Infinite Study, 2023.

Received: Aug 29, 2024. Accepted: Nov 20, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Robust Neutrosophic Model for Risk Management in Livestock Supply Chains: A Case Study Towards Sustainable Practices

Abdullah Ali Salamai
Department of Management, Applied College, Jazan University, Jazan, KSA, Saudi Arabia
Email: absalamai@jazanu.edu.sa

Abstract

This study proposed a decision machining methodology to identify and evaluate the risks based on the livestock supply chain (LSC). The risks related to LSC are identified and collected by the opinion of the experts and decision-makers. We collected 11 LSC criteria and nine alternatives. We proposed an uncertainty framework to deal with uncertainty and vague information in the evaluation process. We used the type-2 neutrosophic set (T2NSs) to deal with vague data. The T2NSs were proposed under multi-objective optimization based on simple ratio analysis (MOOSRA) to rank the alternatives. We identified nine strategies to reduce risks in LSC. The results show that operational risks have the highest weights. We applied sensitivity analysis to show the different ranks of alternatives under different criteria weights. The results show the proposed method is stable under different weights. Finally, we conducted a comparative analysis between other methods to show the robustness of the proposed method.

Keywords: Neutrosophic Set; Livestock; Supply Chain (SC); Sustainability; Risk Management.

1. Introduction

Disruptions to the supply chain (SC) have affected business performance in recent years. In this field, the Ericsson example is well known. In 2000, a fire at a Phillips semiconductor plant interrupted production, ultimately resulting in a $400 million loss for Ericsson. Toyota's output fell by 40,000 automobiles because of the 2011 earthquake, tsunami, and nuclear disaster that struck Japan, losing the company $72 million every day[1], [2]. The October 2011 devastating floods in Thailand had an impact on the SC of hard drive-dependent computer firms. Much effort is made in the field of supply chain risk management (SCRM) in both academic and practitioner circles to regulate and lessen the adverse impacts brought on by such hazards[3], [4], [5]. The livestock supply chain (LSC) is susceptible to various risks at different stages, from production to consumption. Conducting a risk assessment in this domain requires evaluating specific criteria that reflect these vulnerabilities and identifying potential mitigation strategies[6], [7]. A multi-criteria decision-making (MCDM) problem that considers several criteria and options is thus how risk management is defined in LSC. The MCDM approaches using neutrosophic fuzzy sets have been used to better address uncertainty for a variety of applications[8], [9], [10].

Multi-objective optimization techniques include multi-objective optimization based on simple ratio analysis (MOOSRA) and multi-objective optimization by ratio analysis with the full multiplicative form (MULTIMOORA). While the MOOSRA technique computes the simple ratio of advantageous and non-beneficial criteria during the decision-making process, the MULTIMOORA approach summarizes the MOORA method, including the ratio system, reference point, and the full multiplicative form. Each criterion is given a weight based on the decision maker's assessment of its significance[11], [12].

Among the multi-objective optimization techniques is the MOOSRA approach. When the MOOSRA and MOORA methods are compared, the MOOSRA technique is less sensitive to significant fluctuations in the criterion values and does not exhibit the negative performance scores of the MOORA approach[13], [14], [15]. It was utilized in the creation of a multi-criteria decision-making framework to determine the best cutting parameters for surface roughness, choose the best-cutting fluid from three different types for a gear operation, choose materials, and use non-traditional machinery[16], [17].
The main contribution of this study is organized as follows:

- It is the first time to propose the MOOSRA method under type-2 neutrosophic sets to rank the alternatives in this study.
- We proposed a decision-making model for risk management in LSC to compute the criteria weights and rank the alternatives.
- The type-2 neutrosophic sets are used to deal with vague data.
- This study proposed a set of strategies to overcome the risks in LSC.

The rest of this paper is organized as follows: Section 2 shows the proposed methodology. Section 3 shows the results and discussion. Section 4 shows the sensitivity analysis. Section 5 shows the comparative analysis. Section 6 shows a comparative analysis. Section 7 shows the conclusions.

2. Type-2 Neutrosophic Methodology

This section shows some definitions of T2NSs and the steps of the proposed methodology.

Definition 1.

We can define the T2NSs [18]as:

$P = \{x, A(x), B(x), C(x) | x \in X\}$

Where $A(x) = \left(A_A(X), A_B(X), A_C(X)\right), B(x) = \left(B_A(X), B_B(X), B_C(X)\right), and\ C(x) = \left(C_A(X), C_B(X), C_C(X)\right)$

Definition 2 [19], [20], [21]**.**

Let $A_1 = \left(\left(A_{A_1}(X), A_{B_1}(X), A_{C_1}(X)\right), \left(B_{A_1}(X), B_{B_1}(X), B_{C_1}(X)\right), \left(C_{A_1}(X), C_{B_1}(X), C_{C_1}(X)\right)\right)$ and

$A_2 = \left(\left(A_{A_2}(X), A_{B_2}(X), A_{C_2}(X)\right), \left(B_{A_2}(X), B_{B_2}(X), B_{C_2}(X)\right), \left(C_{A_2}(X), C_{B_2}(X), C_{C_2}(X)\right)\right)$ are two T2NNs and operations can be defined as:

$$A_1 \oplus A_2 = \begin{pmatrix} \begin{pmatrix} A_{A_1}(X) + A_{A_2}(X) - A_{A_1}(X)A_{A_2}(X), \\ A_{B_1}(X) + A_{B_2}(X) - A_{B_1}(X)A_{B_1}(X), \\ A_{C_1}(X) + A_{C_2}(X) - A_{C_1}(X)A_{C_2}(X) \end{pmatrix}, \\ \left(B_{A_1}(X)B_{A_2}(X), B_{B_1}(X)B_{B_2}(X), B_{C_1}(X)B_{C_1}(X)\right), \\ \left(C_{A_1}(X)C_{A_2}(X), C_{B_1}(X)C_{B_2}(X), C_{C_1}(X)C_{C_2}(X)\right) \end{pmatrix} \quad (1)$$

$$A_1 \otimes A_2 = \begin{pmatrix} \left(A_{A_1}(X)A_{A_2}(X), A_{B_1}(X)A_{B_2}(X), A_{C_1}(X)A_{C_2}(X)\right), \\ \begin{pmatrix} B_{A_1}(X) + B_{A_2}(X) - B_{A_1}(X)B_{A_2}(X), \\ B_{B_1}(X) + B_{B_2}(X) - B_{B_1}(X)B_{B_2}(X), \\ B_{C_1}(X) + B_{C_2}(X) - B_{C_1}(X)B_{C_2}(X) \end{pmatrix} \\ \begin{pmatrix} C_{A_1}(X) + C_{A_2}(X) - C_{A_1}(X)C_{A_2}(X), \\ C_{B_1}(X) + C_{B_2}(X) - C_{B_1}(X)C_{B_2}(X), \\ C_{C_1}(X) + C_{C_2}(X) - C_{C_1}(X)C_{C_2}(X) \end{pmatrix} \end{pmatrix} \quad (2)$$

$$\rho A_1 = \begin{pmatrix} \left(\left(1 - \left(1 - A_{A_1}(X)\right)\right)^{\rho}, 1 - \left(1 - A_{B_1}(X)\right)^{\rho}, 1 - \left(1 - A_{B_1}(X)\right)^{\rho}\right), \\ \left(\left(B_{A_1}(X)^{\rho}\right), \left(B_{B_1}(X)^{\rho}\right), \left(B_{B_1}(X)^{\rho}\right)\right), \\ \left(\left(C_{A_1}(X)^{\rho}\right), \left(C_{B_1}(X)^{\rho}\right), \left(C_{B_1}(X)^{\rho}\right)\right) \end{pmatrix} \quad (3)$$

$$A_1^{\rho} = \begin{pmatrix} \left(\left(A_{A_1}(X)\right)^{\rho}, \left(A_{B_1}(X)\right)^{\rho}, \left(A_{C_1}(X)\right)^{\rho}\right), \\ \left(1 - \left(1 - B_{A_1}(X)\right)^{\rho}, 1 - \left(1 - B_{B_1}(X)\right)^{\rho}, 1 - \left(1 - B_{C_1}(X)\right)^{\rho}\right), \\ \left(1 - \left(1 - C_{A_1}(X)\right)^{\rho}, 1 - \left(1 - C_{B_1}(X)\right)^{\rho}, 1 - \left(1 - C_{C_1}(X)\right)^{\rho}\right) \end{pmatrix} \quad (4)$$

Definition 3.

The score function can be computed as follows[22]:

$$S(A_1) = \frac{1}{12}\begin{pmatrix} 8 + \left(A_{A_1}(X) + 2A_{B_1}(X) + A_{C_1}(X)\right) \\ -\left(B_{A_1}(X) + 2B_{B_1}(X) + B_{C_1}(X)\right) \\ -\left(C_{A_1}(X) + 2C_{B_1}(X) + C_{C_1}(X)\right) \end{pmatrix} \quad (5)$$

Definition 4.

The accuracy function can be computed as follows:

$$S(A_1) = \frac{1}{4}\begin{pmatrix} \left(\left(A_{A_1}(X)\right) + 2\left(A_{B_1}(X)\right) + \left(A_{C_1}(X)\right)\right) - \\ \left(C_{A_1}(X)\right) + 2\left(C_{B_1}(X)\right) + \left(C_{C_1}(X)\right) \end{pmatrix} \quad (6)$$

3.1 The T2N-MOOSRA Method

We used the T2N- MOOSRA methodology for ranking the strategies to reduce risks in LSC. We used a set of criteria and alternatives in this study. The criteria are presented as $RRY_j(i = 1,2,\ldots.,m)$ and the alternatives can be represented as $RRX_i(i = 1,2,\ldots.,n)$. The steps of the proposed methodology are presented as:

Step 1. Construct the decision matrix.

The decision matrix is constructed by the opinions of experts.

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix} \quad (7)$$

Step 2. Determine the score function

We obtained the score values in the decision matrix.

Step 3. Combine the decision matrix.

Step 4. The normalization of decision matrix is computed as:

$$h_{ij} = \frac{r_{oj}}{\sqrt{\sum_{i=1}^{m} r_{ij}^2}} \quad (8)$$

Step 5. Compute the criteria weights.

The criteria weights are computed using the average method.

Step 6. Compute the weighted normalized decision matrix.

$$T_{ij} = w_j * h_{ij} \quad (9)$$

Step 7. Compute the score values.

$$S_i = \frac{\sum_{j=1}^{m} U_j(w_j)T_{ij}}{\sum_{j=m+1}^{n} T_j(w_j)T_{ij}} \quad (10)$$

Where U refers to the benefit and cost criteria.

Step 8. Rank the alternatives.

3. Results and Discussion

In this section, we present the outcomes derived from applying the proposed methodology for risk management in the LSC. The analysis is based on 11 key criteria and nine potential alternatives, which were carefully selected to reflect the most significant factors impacting the SC's sustainability and efficiency.

3.1 Criteria for Risk Management

The study identified and classified 11 critical risk categories, each representing a distinct aspect of the LSC that could potentially affect its overall stability. These criteria were chosen based on their relevance and impact on the SC's functionality. Below is a brief explanation of each criterion:

- **Technological Risks**: These risks arise from the potential failure of technology integration or the obsolescence of current systems. The rapid pace of technological advancements means that SC must be continuously updated to avoid vulnerabilities.
- **Policy Risks**: Policy risks refer to the uncertainties related to changes in government regulations or policies. These can significantly affect the operation of SC, especially when new laws or trade agreements are introduced.
- **Regulatory Risks**: Closely related to policy risks, regulatory risks involve compliance with industry standards, environmental laws, and health regulations. Non-compliance could lead to legal penalties or disruptions in operations.

- **Transportation Risks**: This criterion encompasses risks associated with the movement of goods, including disruptions caused by natural disasters, logistical failures, or delays in transportation networks.
- **Infrastructure Risks**: These risks are related to the physical and organizational infrastructure required to operate the SC. Any failure in infrastructure, such as transportation networks, storage facilities, or equipment, could cause severe delays or operational bottlenecks.
- **Market and Demand Risks**: This involves uncertainty around market conditions, consumer demand fluctuations, and market access. Unexpected changes in demand can lead to either surplus or shortage situations, both of which can be costly.
- **Operational Risks**: Operational risks arise from inefficiencies or disruptions within the internal operations of the SC. This includes issues like workforce challenges, equipment malfunctions, or production delays.
- **Environmental Risks**: These risks are linked to natural disasters, climate change, and other environmental factors that could harm livestock or disrupt SC activities. Environmental risks are becoming increasingly important as sustainability and climate adaptation become global priorities.
- **Supply Risks**: Supply risks refer to disruptions in the availability of resources, such as raw materials, feed, or livestock. These risks can be caused by both external factors (e.g., weather conditions) and internal factors (e.g., poor supplier performance).
- **Financial Risks**: Financial risks deal with the economic aspects of the SC, including exchange rate fluctuations, price volatility, and the availability of financial resources to support operations.
- **Social Risks**: Social risks include labor disputes, changes in workforce availability, and community relations. These risks affect the workforce and the community that supports the SC's operations.

3.2 Alternatives for Risk Mitigation

The study also identified nine potential alternatives for managing and mitigating the risks in the LSC. These alternatives were selected based on their ability to address the various criteria effectively:

- **Technology Integration**: This alternative focuses on the adoption of modern technologies, such as automation, data analytics, and blockchain, to streamline operations and reduce technological risks. Integrating technology improves efficiency and enhances transparency in the SC.
- **Market Strategies**: Adapting market strategies involves aligning the SC to meet shifting demand and market conditions. This strategy aims to address market and demand risks by diversifying customer bases and improving market forecasting techniques.
- **Environmental Adaptation**: This alternative emphasizes the need to adjust operations in response to environmental challenges. Implementing sustainable practices and climate-resilient strategies can help manage environmental risks, ensuring that the SC is prepared for changes in weather patterns and environmental regulations.
- **Workforce and Community Engagement**: Fostering strong relationships with the workforce and the surrounding community can reduce social risks. By ensuring fair working conditions and promoting community engagement, organizations can mitigate labor-related disruptions and strengthen public support for the SC.
- **Infrastructure and Logistics**: This alternative focuses on strengthening the physical infrastructure and logistical systems. Investing in more robust storage, transportation, and processing facilities helps minimize infrastructure risks and ensures a smoother flow of goods throughout the SC.

- **Policy Advocacy and Compliance**: Engaging in policy advocacy ensures that the SC is not caught off guard by regulatory changes. By staying involved in policy discussions and complying with relevant regulations, businesses can reduce both policy and regulatory risks.
- **Diversification and Resilience**: Diversification involves spreading risk across different markets, suppliers, or products. This strategy helps mitigate supply risks and financial risks by reducing dependency on a single source or market and increasing resilience to market shifts.
- **Financial Instruments**: Utilizing financial instruments, such as insurance, hedging, or credit lines, can help address financial risks. These tools provide a buffer against unforeseen economic challenges, including fluctuations in currency exchange rates or commodity prices.
- **Health and Disease Management**: This alternative focuses on minimizing health and disease risks, particularly those that affect livestock. Implementing proactive health measures, including vaccinations and biosecurity protocols, helps to ensure the well-being of animals and the stability of the SC.

The proposed methodology was applied to assess the effectiveness of these 11 criteria and 9 alternatives in mitigating risks within the LSC. The results of the evaluation showed that certain criteria, such as Environmental Risks and Supply Risks, were consistently identified as the most critical in maintaining the stability of the SC. These findings suggest that addressing environmental challenges, such as climate change, and ensuring reliable supply sources are central to reducing vulnerability.

Among the alternatives, Technology Integration Diversification, and Resilience were identified as the most impactful strategies for managing risks. Technology integration allows for greater efficiency and real-time monitoring of the SC, while diversification provides a safety net against market disruptions or supply shortages. Furthermore, Market Strategies Policy Advocacy, and Compliance were also found to be crucial for long-term sustainability. Businesses that actively engage with market trends and anticipate regulatory changes are better equipped to handle uncertainties and adapt to new challenges.

3.3 Discussion and Evaluation

The proposed methodology involves several structured steps to evaluate and rank alternatives for risk management in the LSC. Each step is crucial in transforming the decision matrix into actionable results. Below is a detailed explanation of the process, based on the specific formulas and techniques used.

- *Step 1: Building the Decision Matrix*

The first step involves creating the decision matrix with input from three experts who assess the alternatives based on a set of criteria. In this case, the decision matrix is built using Type-2 Neutrosophic Numbers (T2NNs), which help capture uncertainty and imprecision in the evaluation process. These numbers allow for a more accurate representation of expert opinions, especially when dealing with subjective or uncertain data. The decision matrix built using the T2NNs is shown in Table 1. This matrix forms the foundation for all subsequent calculations and evaluations in the methodology.

- *Step 2: Obtaining Crisp Values Using the Score Function*

Once the decision matrix is created, Equation (5) is applied to convert the T2NNs into crisp values. The score function is used for this conversion, which helps simplify the data for further processing. This is an essential step, as it transforms the initial decision matrix into clear, quantifiable values that can be analyzed effectively.

- *Step 3: Combining the Decision Matrix Using the Average Method*

In the next step, the decision matrix is combined by applying the average method. This method aggregates the input from the three experts to generate a consolidated decision matrix. By averaging the individual ratings, this step ensures that collective judgment is considered, thus reducing individual biases and uncertainties.

- *Step 4: Normalizing the Decision Matrix*

To ensure comparability across different criteria and alternatives, the decision matrix is normalized using Equation (8). Normalization standardizes the values so that all criteria are on a comparable scale, typically between 0 and 1. This step, as shown in Table 2, ensures that no single criterion disproportionately influences the results due to differences in measurement units or scale.

- *Step 5: Calculating Criteria Weights Using the Average Method*

The next step involves calculating the criteria weights using the average method. By averaging the normalized values, we can determine the relative importance of each criterion in the decision-making process. This step is crucial as it helps in identifying which risks (criteria) are most critical to the LSC's sustainability.

According to the results, Operational Risks were found to have the highest weights, meaning they are considered the most important criteria in this study. On the other hand, Infrastructure Risks were found to have the lowest weights, indicating that they are less significant in terms of risk management. This information is visually represented in Figure 1, which provides a clear depiction of the relative importance of each criterion.

- Step 6: Computing the Weighted Normalized Decision Matrix

The weighted normalized decision matrix is computed using Equation (9). This matrix reflects the importance of each criterion by multiplying the normalized values by the corresponding criteria weights. The result is a matrix that accounts for both the normalized performance of each alternative and the importance of the criteria, as shown in Table 3.

- *Step 7: Computing Score Values*

Once the weighted normalized decision matrix is obtained, Equation (10) is applied to compute the score values. These score values represent the overall performance of each alternative based on the weighted criteria. The higher the score, the better the alternative is in terms of risk management effectiveness.

- *Step 8: Ranking the Alternatives*

Finally, the alternatives are ranked based on the computed score values. As shown in Figure 2, the alternatives are ordered from the best to the worst, with the Health and Disease Management alternative emerging as the top-ranked option. This suggests that focusing on health and disease management is the most effective strategy for mitigating risks in the LSC.

The methodology applied in this study provides a clear and structured process for evaluating risk management strategies in the LSC. By using T2NNs and the average method, the study effectively integrates expert opinions, normalizes the data, and calculates the importance of different criteria. The results show that Health and Disease Management is the most effective alternative for managing risks, while Infrastructure Risks are considered the least significant in the context of this SC. These findings offer

valuable insights into how risk management strategies can be optimized for better sustainability and resilience.

Table 1. The opinions of experts.

	RRX_1	RRX_2	RRX_3	RRX_4	RRX_5	RRX_6	RRX_7	RRX_8	RRX_9	RRX_{10}	RRX_{11}
RRY_1	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
RRY_2	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
RRY_3	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
RRY_4	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
RRY_5	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
RRY_6	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
RRY_7	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
RRY_8	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
RRY_9	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.50,0.75,0.80), (0.50,0.75,0.65))
	RRX_1	RRX_2	RRX_3	RRX_4	RRX_5	RRX_6	RRX_7	RRX_8	RRX_9	RRX_{10}	RRX_{11}
RRY_1	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
RRY_2	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
RRY_3	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
RRY_4	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
RRY_5	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
RRY_6	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.25,0.15))
RRY_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
RRY_8	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
	RRX_1	RRX_2	RRX_3	RRX_4	RRX_5	RRX_6	RRX_7	RRX_8	RRX_9	RRX_{10}	RRX_{11}
RRY_1	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
RRY_2	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))
RRY_3	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))
RRY_4	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))
RRY_5	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))
RRY_6	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))
RRY_7	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))
RRY_8	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))
RRY_9	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.20,0.20,0.10), (0.65,0.80,0.85), (0.45,0.80,0.70))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.95,0.90,0.95), (0.10,0.10,0.05), (0.05,0.05,0.05))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))

Table 2. The normalized decision matrix.

	RRX_1	RRX_2	RRX_3	RRX_4	RRX_5	RRX_6	RRX_7	RRX_8	RRX_9	RRX_{10}	RRX_{11}
RRY_1	0.15289	0.298644	0.094548	0.116521	0.221013	0.179085	0.16074	0.188222	0.23609	0.188684	0.163429
RRY_2	0.208523	0.167402	0.226127	0.125969	0.160205	0.115527	0.165225	0.172721	0.20327	0.189344	0.099415
RRY_3	0.142084	0.175883	0.111882	0.144864	0.278897	0.123378	0.13694	0.215902	0.23609	0.093682	0.158095
RRY_4	0.156893	0.208024	0.180429	0.32332	0.166052	0.138333	0.135215	0.139137	0.060346	0.112155	0.185252
RRY_5	0.208123	0.199989	0.20446	0.16061	0.160205	0.166748	0.164535	0.136553	0.149277	0.12469	0.226958
RRY_6	0.113667	0.212042	0.151277	0.22202	0.191778	0.125995	0.165225	0.164602	0.109752	0.126669	0.279333
RRY_7	0.195716	0.099102	0.148913	0.194202	0.207565	0.250121	0.191095	0.172721	0.131279	0.168233	0.324434
RRY_8	0.216528	0.076335	0.263553	0.268734	0.311055	0.143567	0.153152	0.125481	0.132337	0.190004	0.082927
RRY_9	0.187311	0.171419	0.148913	0.194727	0.19704	0.250121	0.209376	0.207782	0.169039	0.220682	0.10766

Table 3. The weighted normalized decision matrix.

	RRX_1	RRX_2	RRX_3	RRX_4	RRX_5	RRX_6	RRX_7	RRX_8	RRX_9	RRX_{10}	RRX_{11}
RRY_1	0.014346	0.025555	0.008719	0.009229	0.016997	0.016978	0.016392	0.018443	0.022674	0.019205	0.013022
RRY_2	0.019566	0.014324	0.020853	0.009978	0.01232	0.010953	0.016849	0.016924	0.019522	0.019272	0.007922
RRY_3	0.013332	0.01505	0.010317	0.011474	0.021448	0.011697	0.013964	0.021155	0.022674	0.009536	0.012597
RRY_4	0.014721	0.0178	0.016639	0.025609	0.01277	0.013115	0.013789	0.013634	0.005796	0.011416	0.014761
RRY_5	0.019529	0.017113	0.018855	0.012721	0.01232	0.015809	0.016778	0.01338	0.014336	0.012692	0.018084
RRY_6	0.010666	0.018144	0.01395	0.017585	0.014748	0.011945	0.016849	0.016129	0.010541	0.012893	0.022257
RRY_7	0.018364	0.00848	0.013732	0.015382	0.015962	0.023713	0.019487	0.016924	0.012608	0.017124	0.025851
RRY_8	0.020317	0.006532	0.024304	0.021285	0.023921	0.013611	0.015618	0.012295	0.01271	0.01934	0.006608
RRY_9	0.017576	0.014668	0.013732	0.015424	0.015153	0.023713	0.021351	0.02036	0.016234	0.022462	0.008578

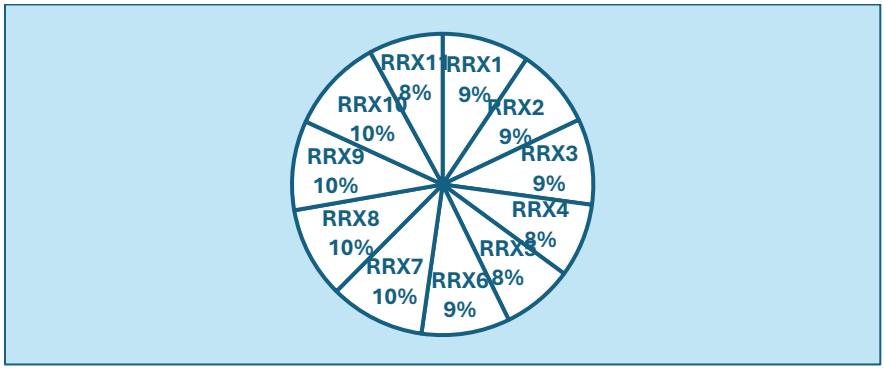


Figure 1. The criteria weights.

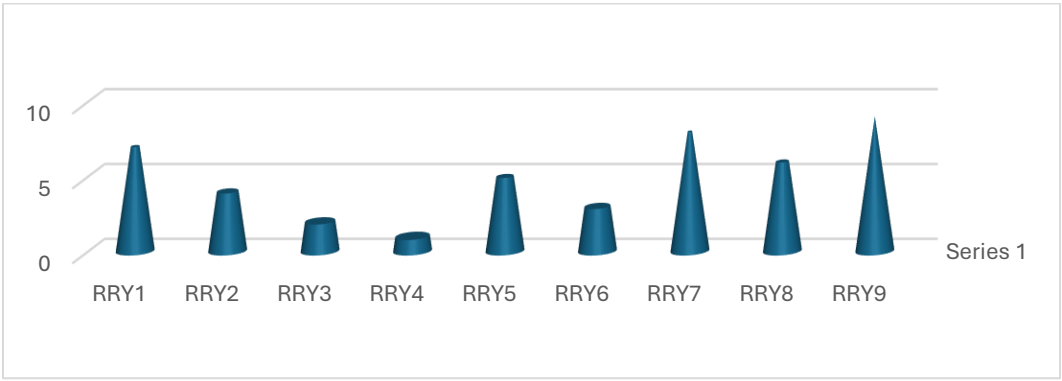


Figure 2. The rank of alternatives.

4. Sensitivity Analysis

In this section, we conduct a sensitivity analysis to evaluate how changes in the criteria weights affect the ranking of the alternatives. Sensitivity analysis is crucial in decision-making processes, as it helps to understand the robustness of the results under varying conditions. By altering the weights assigned to the criteria, we can assess whether the proposed method provides consistent and reliable rankings or if it is highly sensitive to changes in these weights.

4.1 Altering Criteria Weights Across Different Cases

To perform the sensitivity analysis, we systematically change the criteria weights across 12 different cases, as shown in Figure 3. Each case represents a different set of criteria weights, reflecting various possible

scenarios or priorities that might be assigned to the factors influencing risk management in the LSC. The idea behind this approach is to explore how the rankings of the alternatives change when the importance of different criteria is adjusted. These changes might occur due to shifting market conditions, changes in policy or regulations, or evolving priorities within the SC.

4.2 Applying the Proposed Method Under Different Weights

After altering the criteria weights in the 12 different cases, we apply the proposed method to compute the rankings of the alternatives under each set of weights. The results of these calculations are shown in Figure 4. For each case, the ranking of the alternatives is updated based on the new weightings, and the resulting rankings are compared.

4.3 Analysis of Stability

The key observation from the sensitivity analysis is that the proposed method is stable under different criteria weights. This means that, despite the changes in the importance given to various risk factors, the overall ranking of the alternatives remains consistent. Specifically, Health and Disease Management continues to rank as the top alternative in most cases, indicating its robustness as a critical risk mitigation strategy in the LSC. This stability highlights the reliability and generalizability of the method in real-world applications, where the relative importance of different criteria may fluctuate over time. The consistency of the rankings under different weight scenarios suggests that the proposed method provides a solid basis for decision-making, even when external factors influence the criteria weights.

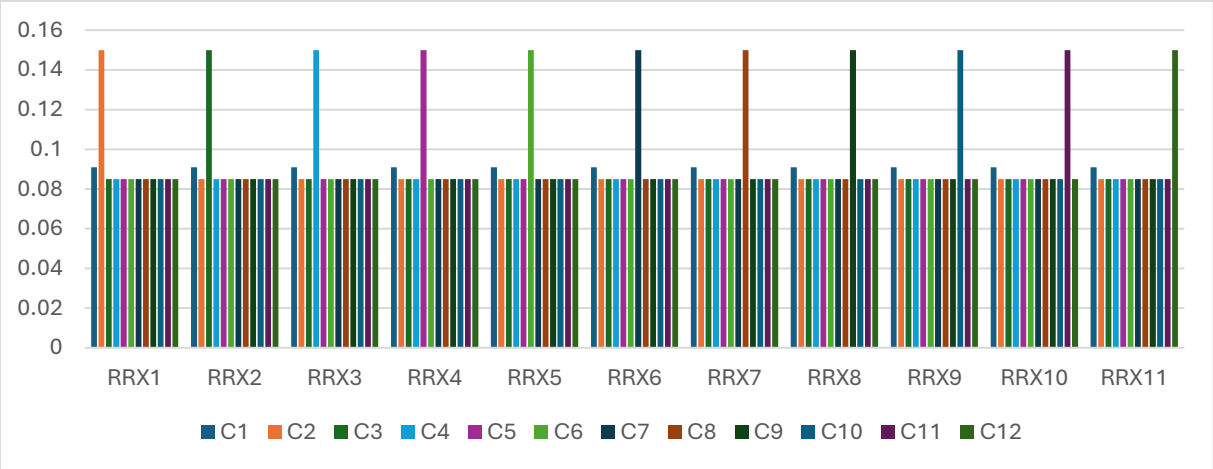


Figure 3. The different criteria weights.

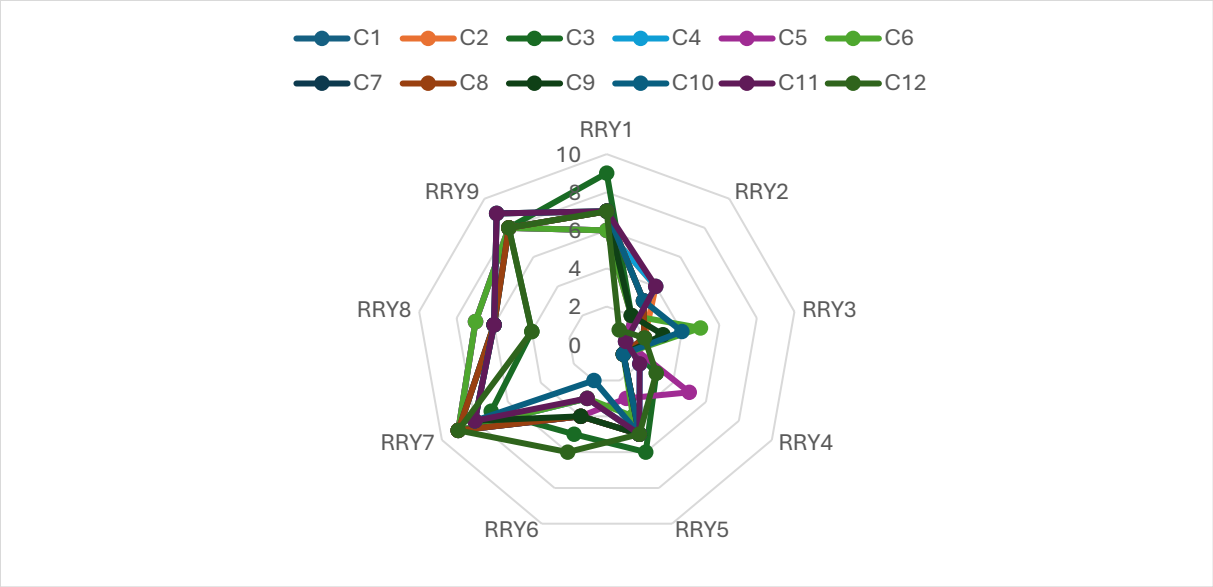


Figure 4. The different ranks of alternatives.

5. Comparative Analysis

In this section, we conduct a comparative analysis between the proposed method and other established Multi-Criteria Decision-Making (MCDM) methods. The goal of this analysis is to evaluate the robustness and effectiveness of the proposed method in comparison to widely used MCDM techniques such as TOPSIS, MABAC, and VIKOR. By applying the same set of alternatives and criteria across these methods, we can determine how well the proposed method performs relative to the others in terms of ranking alternatives and handling decision complexities.

To compare the methods, we used the same set of criteria and alternatives that were evaluated in the proposed method. The decision matrix was constructed using the same T2NNs for the proposed method and the standard input for the other MCDM methods. After processing the data through each method, the rankings of the alternatives were obtained. We then compared the rankings produced by each method to assess how each method handled the decision problem. Key points of comparison included the consistency of rankings, the ability to handle uncertainty, and the overall stability of the results across different weighting schemes.

The results of the comparative analysis are shown in Figure 5. From the analysis, the proposed method demonstrated several advantages over the traditional MCDM methods:

- The proposed method showed a high degree of stability and consistency across varying criteria weights and decision scenarios. While the other methods (TOPSIS, MABAC, and VIKOR) sometimes produced different rankings when criteria weights were altered, the proposed method consistently ranked Health and Disease Management as the best alternative. This indicates that the proposed method is less sensitive to changes in criteria weights, making it more robust in dynamic decision-making environments.
- The proposed method, based on T2NNs, was particularly effective at capturing and handling uncertainty and imprecision in the decision-making process. While the other methods are effective for

crisp data, they struggle to handle the uncertainty present in real-world decision problems as well as the proposed method does.

- The proposed method accounted for a wider range of risk factors and provided a more comprehensive evaluation of alternatives, whereas methods like TOPSIS, MABAC, and VIKOR tend to focus on fewer risk categories or handle them with less granularity.

Figure 5 clearly illustrates the comparative performance of the methods, showing how the alternatives were ranked by each method. The ranking results confirm that, while all methods were able to identify Health and Disease Management as a high-ranking alternative, the proposed method was the most consistent and stable across different scenarios and changes in criteria weights.

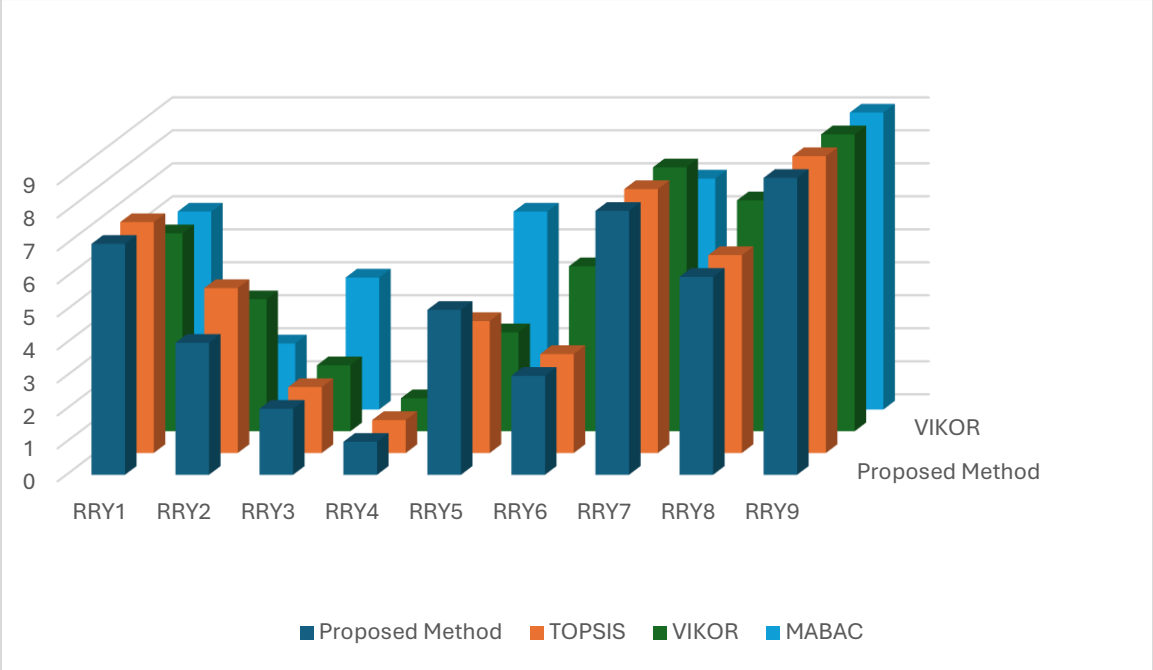


Figure 5. Comparative analysis.

6. Managerial Implications

Evaluating risks within the LSC is crucial for managers, as it provides them with valuable insights into potential vulnerabilities and allows them to design strategies that mitigate these risks effectively. By utilizing the findings from risk assessments, managers can enhance decision-making, streamline operations, and ensure sustainability across the SC. Below are the key managerial implications grouped into focus areas that will help guide practical applications of the results from this study.

Risk assessment provides managers with data-driven insights that allow them to make informed decisions on where to focus their risk mitigation efforts. For example, the analysis may reveal significant vulnerabilities in areas such as transportation risks or disease control, enabling managers to prioritize actions in these areas. For instance, if transportation risks are found to be high, managers could consider investing in more reliable vehicles, optimizing routes, or creating contingency plans for transportation disruptions. Similarly, for disease-related risks, managers may decide to invest in enhanced biosecurity protocols, vaccination programs, or partnerships with veterinary experts to mitigate potential outbreaks.

By understanding which risks are most critical, managers can implement targeted interventions that provide the highest return in terms of risk reduction.

By identifying and assessing operational risks, such as feed shortages, logistical challenges, or labor-related issues, managers can take proactive steps to minimize disruptions in the SC. For example, if feed shortages are identified as a potential risk, managers might explore diversifying feed suppliers, establishing contingency plans, or even adjusting inventory management practices to mitigate such shortages. If infrastructure risks related to storage or processing facilities are highlighted, investing in infrastructure upgrades or creating backup facilities could ensure operational continuity. Proactively addressing these risks helps maintain a smoother flow of operations, reduces downtime, and ensures that critical SC activities are not interrupted.

Risk assessment aids managers in making better decisions regarding resource allocation by highlighting the severity and likelihood of various risks. Resources such as vaccines, feed, labor, and financial support can be allocated more effectively based on the risk priorities. For instance, if health and disease management risks are found to be a major concern, resources may be directed toward improving animal health monitoring systems, vaccine procurement, and biosecurity measures. Similarly, in the case of transportation risks, managers may choose to allocate resources toward improving fleet management, upgrading equipment, or negotiating better contracts with logistics providers. By aligning resource distribution with identified risks, managers can ensure that the right resources are available at the right time, which ultimately contributes to more efficient and resilient SC operations.

In the context of sustainability, risk assessment empowers managers to make decisions that not only mitigate immediate risks but also promote long-term resilience. For example, focusing on environmental risks and adopting sustainable practices in sourcing, production, and waste management can help mitigate the impact of climate change and environmental degradation. Similarly, financial risks may lead managers to adopt more robust financial strategies, such as hedging, diversifying revenue streams, or creating financial reserves to weather economic fluctuations. By taking a proactive approach to managing these risks, managers can build a more sustainable and resilient LSC that is better prepared for future challenges.

Lastly, implementing risk management strategies requires continuous monitoring and adjustment. Managers should establish regular reviews of the risk landscape, incorporating new data, technologies, and evolving market conditions into the risk assessment process. This ongoing evaluation will allow them to stay ahead of emerging risks, refine existing strategies, and adapt to changing circumstances, ensuring that the SC remains resilient in the long run.

7. Conclusions

This study used the MCDM methodology for risk management in LSC risks. We used the T2NSs to deal with the uncertainty problem in the decision-making problem. We used the MOOSRA method to rank alternatives. 11 criteria and nine alternatives were collected from the previous study. Three experts are invited to evaluate the criteria and alternatives. Then we normalized the decision matrix between criteria and alternatives. The results show that operational risks have the highest weights of criteria. The alternatives Health and Disease Management has the highest alternatives. We conducted a sensitivity analysis to show the stability of the rank of alternatives under different weights. We proposed 12 cases in the criteria weights. We show the rank of alternatives is stable. We conducted a comparative analysis

between the proposed method and other MCDM methods. We proposed the method is robust compared with other MCDM methods.

References

[1] N. M. Hashem, A. González-Bulnes, and A. J. Rodriguez-Morales, "Animal welfare and livestock supply chain sustainability under the COVID-19 outbreak: An overview," *Front. Vet. Sci.*, vol. 7, p. 582528, 2020.

[2] S. J. Staal, "Livestock marketing and supply chain management of livestock products," *Indian J. Agric. Econ.*, vol. 70, no. 1, pp. 42–63, 2015.

[3] L. Gwaka and J. Dubihlela, "The resilience of smallholder livestock farmers in sub-saharan Africa and the risks imbedded in rural livestock systems," *Agriculture*, vol. 10, no. 7, p. 270, 2020.

[4] P. Alarcon *et al.*, "Urban livestock keeping in the city of Nairobi: diversity of production systems, supply chains, and their disease management and risks," *Front. Vet. Sci.*, vol. 4, p. 171, 2017.

[5] S. Jaffee, P. Siegel, and C. Andrews, "Rapid agricultural supply chain risk assessment: A conceptual framework," *Agric. Rural Dev. Discuss. Pap.*, vol. 47, no. 1, pp. 1–64, 2010.

[6] W. Khan, S. Khan, A. Dhamija, M. Haseeb, and S. A. Ansari, "Risk assessment in livestock supply chain using the MCDM method: a case of emerging economy," *Environ. Sci. Pollut. Res.*, vol. 30, no. 8, pp. 20688–20703, 2023.

[7] C. M. Godde, D. Mason-D'Croz, D. E. Mayberry, P. K. Thornton, and M. Herrero, "Impacts of climate change on the livestock food supply chain; a review of the evidence," *Glob. Food Sec.*, vol. 28, p. 100488, 2021.

[8] G. Demir, P. Chatterjee, and D. Pamucar, "Sensitivity analysis in multi-criteria decision making: A state-of-the-art research perspective using bibliometric analysis," *Expert Syst. Appl.*, vol. 237, p. 121660, 2024.

[9] S. S. Yildiz, "Spatial multi-criteria decision making approach for wind farm site selection: A case study in Balıkesir, Turkey," *Renew. Sustain. Energy Rev.*, vol. 192, p. 114158, 2024.

[10] M. Bani-Doumi, J. Serrano-Guerrero, F. Chiclana, F. P. Romero, and J. A. Olivas, "A picture fuzzy set multi criteria decision-making approach to customize hospital recommendations based on patient feedback," *Appl. Soft Comput.*, vol. 153, p. 111331, 2024.

[11] A. Sarkar, S. C. Panja, D. Das, and B. Sarkar, "Developing an efficient decision support system for non-traditional machine selection: an application of MOORA and MOOSRA," *Prod. Manuf. Res.*, vol. 3, no. 1, pp. 324–342, 2015.

[12] Z. Sitorus, A. Karim, A. H. Nasyuha, and M. H. Aly, "Implementation of MOORA and MOORSA Methods in Supporting Computer Lecturer Selection Decisions," *J. INFOTEL*, vol. 16, no. 3, pp. 554–566, 2024.

[13] N. Vasić, M. Kilibarda, and M. Andrejić, "Selecting the e-commerce distribution channel by applying the integrated FAHP and MOOSRA methods," *Int. J. Shipp. Transp. Logist.*, 2024.

[14] V. Pajic, M. Andrejic, and M. Poledica, "A novel approach based on CRITIC-MOOSRA methods for evaluation and selection of cold chain monitoring devices," *J. Intell Manag. Decis*, vol. 3, no. 2, pp.

68–76, 2024.

[15] F. Hu, "An integrated group decision making framework for business performance evaluation of performing arts enterprises with probabilistic hesitant fuzzy IDOCRIW-MOOSRA method," *J. Intell. Fuzzy Syst.*, no. Preprint, pp. 1–13, 2024.

[16] S. Narayanamoorthy *et al.*, "A novel assessment of bio-medical waste disposal methods using integrating weighting approach and hesitant fuzzy MOOSRA," *J. Clean. Prod.*, vol. 275, p. 122587, 2020.

[17] M. Mesran and F. T. Waruwu, "Comparative Analysis of MOORA and MOOSRA Methods in Determining Prospective Students Recipient of the Indonesian Smart Card (KIP)," *J. Inf. Syst. Res.*, vol. 3, no. 4, pp. 499–506, 2022.

[18] M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number," *Appl. Soft Comput.*, vol. 77, pp. 438–452, 2019.

[19] V. Simic, I. Gokasar, M. Deveci, and A. Karakurt, "An integrated CRITIC and MABAC based type-2 neutrosophic model for public transportation pricing system selection," *Socioecon. Plann. Sci.*, vol. 80, p. 101157, 2022.

[20] V. Simić, B. Milovanović, S. Pantelić, D. Pamučar, and E. B. Tirkolaee, "Sustainable route selection of petroleum transportation using a type-2 neutrosophic number based ITARA-EDAS model," *Inf. Sci. (Ny).*, vol. 622, pp. 732–754, 2023.

[21] Z. Wang, Q. Cai, and G. Wei, "Modified TODIM method based on cumulative prospect theory with Type-2 neutrosophic number for green supplier selection," *Eng. Appl. Artif. Intell.*, vol. 126, p. 106843, 2023.

[22] M. Deveci, N. Erdogan, U. Cali, J. Stekli, and S. Zhong, "Type-2 neutrosophic number based multi-attributive border approximation area comparison (MABAC) approach for offshore wind farm site selection in USA," *Eng. Appl. Artif. Intell.*, vol. 103, p. 104311, 2021.

Received: Jul 13, 2024. Accepted: Nov 29, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Novel Approach to Assessing Art Education Teaching Quality in Vocational Colleges Based on Double-Valued Neutrosophic Numbers and Multi-Attribute Decision-Making with Tree Soft Sets

Heng He*

Chongqing Industry & Trade Polytechnic, Fuling, 408000, Chongqing, China

*Corresponding author, E-mail: 18983313993@163.com

*https://orcid.org/0009-0000-4293-4635

Abstract: To promote stable and sustainable growth in vocational education, it is crucial to innovate and adapt the teaching methods of art education to align with modern educational trends. A comprehensive analysis of the art education curriculum is necessary to drive reform and modernize teaching methodologies, ensuring the continued relevance and effectiveness of art instruction. By updating art education, students can enhance their aesthetic appreciation, develop a refined understanding of high-quality artworks, and experience consistent growth in their artistic development. Moving beyond traditional teaching models is essential for the ongoing advancement of art education in vocational colleges. The evaluation of teaching quality in these institutions is approached using a well-established Multi-Attribute Decision Making (MADM) framework. Recently, the CoCoSo method, combined with average techniques, has been applied to address the complexities of MADM. To manage uncertain or ambiguous information during the evaluation process, Double-Valued Neutrosophic Sets (DVNSs) have also been utilized. This study introduces the integration of the CoCoSo method with DVNSs, leading to the development of the Double-Valued Neutrosophic Number CoCoSo (DVNN-CoCoSo) method for MADM. We used the Tree Soft Set (TSS) with the CoCoSo method to deal with the main and sub criteria to obtain the relation between them (DVNN-TSS-CoCoSo). To illustrate the practical application of this approach, a numerical example is provided to assess the quality of art education in vocational colleges. The DVNN-TSS-CoCoSo method offers a significant improvement in evaluating teaching quality, marking an important step toward refining evaluation processes and enhancing educational strategies in vocational settings.

Keywords: Multiple-attribute decision-making (MADM); DVNSs; CoCoSo approach; teaching quality evaluation; Tree Soft Set.

1. Introduction

Art is a special course. In order for students to learn art well, art teachers need to improve their art appreciation, cognitive ability, and artistic creation ability, in order to achieve comprehensive development of students[1]. Therefore, art teachers need to have a longer-term perspective when carrying out teaching activities. In the past teaching models, some teachers placed too much emphasis on conducting classroom teaching and imparting theoretical knowledge to students. Although this form can improve students' painting skills, it is difficult to enhance their practical abilities and develop their artistic literacy[2]. Therefore, teachers need to actively guide students to participate in social practice activities, so that students can improve their artistic literacy and develop their abilities in practical activities[3]. For example, college teachers could encourage students to

participate in local or international painting competitions. By participating in art related competition activities, students can enhance their competitive awareness, actively learn art knowledge, and improve their own literacy and abilities. In addition, teachers can also lead or encourage students to visit art galleries and museums, allowing them to accumulate humanistic knowledge during the visit process, broaden their horizons by appreciating famous works, and improve their painting skills and artistic literacy[4]. Teachers should practice new educational concepts in art teaching in vocational colleges, in order to enrich students' learning experience and enable them to master art knowledge in a joyful environment. The new curriculum reform advocates more flexible forms of art teaching activities, which can create a good atmosphere for students[5]. In the process of promoting the new curriculum reform, teachers should clarify the purpose of art teaching, understand the key and difficult points of art subjects in vocational colleges, and demonstrate the new value of art courses. There are obvious differences between art courses in vocational colleges and ordinary high schools. Art teaching in vocational colleges has strong operability, and students can participate in creative practice, relying on visual and tactile senses to feel and create art works[6]. Therefore, teachers should comprehensively study art teaching in order to enhance the art teaching effectiveness. Currently, with the rapid development for information technology, electronic products have played an important role for people's lives and become an indispensable part of their daily lives. Numerous sectors have embraced modern information technology, with its impact being particularly noticeable in the education industry. The integration of modern information technology in educational settings has notably enhanced teaching effectiveness [7]. Art teachers in vocational colleges need to use multimedia technology in the classroom, which not only attracts students' attention, but also allows them to better experience the charm of art works, thereby improving their artistic literacy. In the process of teaching, art teachers should pay attention to combining theory with practice, allowing students to master theoretical knowledge while improving their practical abilities[8]. In addition to using multimedia technology, teachers can also introduce micro lessons in art teaching. Micro lessons mainly present targeted content to students in the form of short videos. This teaching method is not only concise, but also helps students consolidate their knowledge. Art teachers in vocational colleges can make reasonable use of micro courses to help students master relevant art theory knowledge. This not only saves more classroom time and allows students to engage in more practical activities, but also enables students to apply the theoretical knowledge they have learned to practice in a timely manner, thereby helping students improve their practical abilities and ultimately improve classroom teaching effectiveness[9]. Designing micro courses requires a certain level of computer operation ability from art teachers. Therefore, teachers should improve their computer application level and design more valuable and targeted micro courses[10]. In the process of teaching, art teachers for vocational colleges not only need to strengthen students' artistic foundation, but also need to integrate folk art into teaching and set more distinctive teaching content. Art is an art discipline, and in order to deepen students' learning of art, teachers should understand their learning interests and integrate content closely related to their daily lives into art teaching, in

order to better enhance students' artistic literacy[11]. Teachers should avoid teaching fixed content according to textbooks, as this can limit students' thinking and vision. Teachers can explore the integration of folk art into art teaching, allowing students to be exposed to diverse art works and styles. In order to better demonstrate the importance of folk art, teachers can play movies related to folk art for students during classroom teaching, so that students can understand the development background of folk art, feel the characteristics of folk art, and better learn relevant art knowledge and skills.

MADM is a decision analysis technique primarily used to select the best option from multiple alternatives, especially when these alternatives vary across multiple evaluation dimensions[12, 13]. This method is applicable to various complex decision-making situations such as project selection, resource allocation, strategy formulation, and spans multiple fields including industry, economics, management, healthcare, environmental protection, and policy analysis[14, 15]. The core of multi-attribute decision-making lies in how to handle and synthesize multiple attributes or criteria to achieve a comprehensive evaluation and rational choice. In traditional decision-making processes, decision-makers might subjectively evaluate each option, whereas multi-attribute decision-making offers a systematic and quantitative framework that makes the decision-making process more objective, transparent, and verifiable[16]. The basic steps of MADM include establishing a decision matrix that lists all the alternatives and their performances on different attributes; determining the weight of each attribute to reflect its importance to the overall decision; and using appropriate decision rules (such as weighted sum, ideal solution distance, etc.) to assess and rank the options. Since the mid-20th century, when multi-attribute decision-making began to develop, various methods and techniques have emerged. Initially, the focus was on how to appropriately assign weights to attributes and how to handle quantitative data[17]. As theory and application evolved, researchers began exploring how to handle qualitative data, how to consider changes in decision-makers' preferences, and how to make decisions when facing uncertainty and fuzzy information. For example, the Analytic Hierarchy Process (AHP) introduced by Saaty in the 1970s, uses a hierarchical structure model to analyze and address decision issues, allowing complex decisions to be broken down into smaller, more manageable parts. Additionally, techniques such as TOPSIS and fuzzy comprehensive evaluation have been developed to handle more types of data and more complex decision environments. With the improvement of computational capabilities and the development of information technology, multi-attribute decision-making methods continue to evolve. Modern multi-attribute decision-making tools and software can handle large data sets, support more complex models and algorithms. Moreover, the development of artificial intelligence and machine learning provides new analytical tools for multi-attribute decision-making, such as automatically identifying attribute weights through data mining techniques or finding optimal decisions through optimization algorithms [18, 19]. Today, multi-attribute decision-making has become an indispensable tool for decision-makers facing complex issues. It not only helps decision-makers evaluate the pros and cons of different options but also promotes the scientification and

standardization of the decision-making process, enhancing the rationality and effectiveness of decisions. With further globalization and technological advancement, MADM is expected to demonstrate its unique value in more fields and a wider range of application scenarios.

The art education teaching quality evaluation in vocational colleges is a well-established field of research in the context of MADM. Recently, the CoCoSo approach [20] and entropy [21] have been widely adopted to address MADM challenges. To characterize fuzzy information inherent in art education teaching quality evaluation, the utilization of DVNSs [22] has gained attention. However, most existing approaches have independently employed CoCoSo [20] and entropy [21] without considering their integration under DVNSs. Therefore, in this study, we propose the establishment of the DVNN-TSS-CoCoSo approach to effectively manage MADM. This model fills the gap by combining the CoCoSo [20] and entropy [21] under the framework of DVNSs.

The major motivations of this research are illustrated: (1) Introducing a novel MADM approach based on the integration of CoCoSo and average approaches under DVNSs with the TSS to deal with the criteria and sub criteria, thereby enhancing the accuracy and robustness of the evaluation process. (2) Incorporating objective weights through the utilization of the average approach, enabling a more comprehensive and unbiased assessment of the art education teaching quality in vocational colleges. (3) Proposing a new MADM approach, the DVNN-TSS-CoCoSo approach, specifically tailored for the evaluation of art education teaching quality in vocational colleges, considering the unique characteristics of DVNSs. (4) Demonstrating the effectiveness of the DVNN-TSS-CoCoSo model through a practical numerical example, thereby providing empirical evidence and validation for its applicability in the field of art education teaching quality evaluation in vocational colleges.

The framework of this study has been established and consists of the following sections: In Section 2, the concept of DVNSs is introduced. Section 3 focuses on the construction of the DVNN-TSS-CoCoSo approach. Section 4, numerical example, is presented to prove the practical application of DVNN-TSS. Finally, the study concludes in Section 5.

2. Preliminaries

Wang et al. [23] coped with the SVNSs.

Definition 1 [23]. The SVNSs is illustrated:

$$CA=\left\{\left(y,CT_A\left(y\right),CI_A\left(y\right),CF_A\left(y\right)\right)\middle|,y\in Y\right\} \tag{1}$$

where $CT_A\left(y\right),CI_A\left(y\right),CF_A\left(y\right)$ is truth-membership (TM), indeterminacy membership (IM) and falsity-membership (FM), $CT_A\left(y\right),CI_A\left(y\right),CF_A\left(y\right)\in\left[0,1\right]$, $0\le CT_A\left(y\right)+CI_A\left(y\right)+CF_A\left(y\right)\le 3$.

Kandasamy [22] illustrated the DVNSs.

Definition 2 [22]. The DVNSs is put forward:

$$CA=\left\{\left(\theta,CT_A(\theta),CIT_A(\theta),CIF_A(\theta),CF_A(\theta)\right)|\theta\in\Theta\right\}. \quad (2)$$

where $CT_A(\theta),CIT_A(\theta),CIF_A(\theta),CF_A(\theta)$ is TM, IM leaning towards TM, IM leaning towards FM, FM, $CT_A(\theta),CIT_A(\theta),CIF_A(\theta),CF_A(\theta)\in[0,1]$, $0\le CT_A(\theta)+CIT_A(\theta)+CIF_A(\theta)+CF_A(\theta)\le 4$.

The DVNN is illustrated as $CA=(CT_A,CIT_A,CIF_A,CF_A)$, where $(CT_A,CIT_A,CIF_A,CF_A)\in[0,1]$, $0\le CT_A+CIT_A+CIF_A+CF_A\le 4$.

Definition 3[22]. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ be DVNN, score value is illustrated:

$$SV(CA)=\frac{(2+CT_A+CIT_A-CIF_A-CF_A)}{4},\ SV(CA)\in[0,1]. \quad (3)$$

Definition 4[22]. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ be DVNN, accuracy value is illustrated:

$$AV(CA)=\frac{(CT_A+CIT_A+CIF_A+CF_A)}{4},\ AV(CA)\in[0,1]\ . \quad (4)$$

The order for different DVNNs is illustrated.

Definition 5[22]. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$, $SV(CA)=\frac{(2+CT_A+CIT_A-CIF_A-CF_A)}{4}$, $SV(CB)=\frac{(2+CT_B+CIT_B-CIF_B-CF_B)}{4}$, $AV(CA)=\frac{(CT_A+CIT_A+CIF_A+CF_A)}{4}$, $AV(CB)=\frac{(CT_B+CIT_B+CIF_B+CF_B)}{4}$, $SV(CA)<SV(CB)$, $CA<CB$; $SV(CA)=SV(CB)$, (1) $AV(CA)=AV(CB)$, $CA=CB$; (2) $AV(CA)<AV(CB)$, $CA<CB$.

Definition 6[22]. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$ be DVNNs, the operations are illustrated:

(1) $CA\oplus CB=(CT_A+CT_B-CT_ACT_B,CIT_A+CIT_B-CIT_ACIT_B,CIF_ACIF_B,CF_ACF_B)$;

(2) $CA\otimes CB=(CT_ACT_B,CIT_ACIT_B,CIF_A+CIF_B-CIF_ACIF_B,CF_A+CF_B-CF_ACF_B)$;

(3) $\lambda CA=\left(1-(1-CT_A)^{\lambda},1-(1-CIT_A)^{\lambda},(CIF_A)^{\lambda},(CF_A)^{\lambda}\right),\lambda>0$;

(4) $(CA)^{\lambda}=\left((CT_A)^{\lambda},(CIT_A)^{\lambda},1-(1-CIF_A)^{\lambda},1-(1-CF_A)^{\lambda}\right),\lambda>0.$

Definition 7[22]. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$, the Hamming distance for $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$ is illustrated:

$$HAMMD(CA,CB)=\frac{1}{4}\begin{pmatrix}|CT_A-CT_B|+|CIT_A-CIT_B|\\+|CIF_A-CIF_B|+|CF_A-CF_B|\end{pmatrix} \tag{5-a}$$

Definition 8 [22]. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$, the Euclidean distance for $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$ is illustrated:

$$EUCLD(CA,CB)=\sqrt{\frac{1}{4}\begin{pmatrix}|CT_A-CT_B|^2+|CIT_A-CIT_B|^2\\+|CIF_A-CIF_B|^2+|CF_A-CF_B|^2\end{pmatrix}} \tag{5-b}$$

Definition 9. Let $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$, the Hellinger distance for $CA=(CT_A,CIT_A,CIF_A,CF_A)$ and $CB=(CT_B,CIT_B,CIF_B,CF_B)$ is illustrated:

$$HELLD(CA,CB)=\sqrt{\frac{1}{4}\begin{pmatrix}\left|\sqrt{CT_A}-\sqrt{CT_B}\right|^2+\left|\sqrt{CIT_A}-\sqrt{CIT_B}\right|^2\\+\left|\sqrt{CIF_A}-\sqrt{CIF_B}\right|^2+\left|\sqrt{CF_A}-\sqrt{CF_B}\right|^2\end{pmatrix}} \tag{5-c}$$

Definition 10

The TSS methodology is used to show the relationship between the criteria and sub criteria. The basic idea of this methodology can be expressed as:

Let P be universe of discourse, and Y non-empty subset of P with the power set of Y $P(Y)$. Let X be a set of criteria for main nodes as $X=\{x_1,x_2,\dots,x_n\}\ where\ n\geq 1$ this is called the first level.

The sub criteria are located in the second level of TSS as $\{x_{1-1},x_{2-1},\dots,x_{n-1}\}$.

3. DVNN-TSS-CoCoSo approach for MADM with average

The DVNN-TSS-CoCoSo approach is illustrated for MADM. Let $CA=\{CA_1,CA_2,\cdots,CA_m\}$ be different alternatives, $CG=\{CG_1,CG_2,\cdots,CG_n\}$ be different attributes with weight uw, $cw_j\in[0,1]$, $\sum_{j=1}^{n}cw_j=1$. The DVNNs are illustrated: $CR=(CR_{ij})_{m\times n}=(CT_{ij},CIT_{ij},CIF_{ij},CF_{ij})_{m\times n}$.

Step 1. Illustrate the DVNN-matrix $CR=(CR_{ij})_{m\times n}=(CT_{ij},CIT_{ij},CIF_{ij},CF_{ij})_{m\times n}$.

$$CR = \left[CR_{ij} \right]_{m \times n} = \begin{bmatrix} CR_{11} & CR_{12} & \dots & CR_{1n} \\ CR_{21} & CR_{22} & \dots & CR_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ CR_{m1} & CR_{m2} & \dots & CR_{mn} \end{bmatrix} \tag{6}$$

Step 2. Normalize the decision matrix

$N_{ij} = \frac{CR_{ij} - \min CR_{ij}}{\max CR_{ij} - \min CR_{ij}}$ for beneficial criteria. (7)

$N_{ij} = \frac{\max CR_{ij} - CR_{ij}}{\max CR_{ij} - \min CR_{ij}}$ for cost criteria. (8)

Step 3. Compute the criteria weights by using the average method.

Step 4. Compute the total of the weighted comparability sequence (weighted normalized decision matrix) and the whole of the power weight of comparability of each alternative.

$S_i = \sum_{j=1}^{n} w_j N_{ij}$ (9)

$P_i = \sum_{j=1}^{n} (N_{ij})^{w_j}$ (10)

Step 5. Compute the relative weights of the alternatives as:

$U_{ia} = \frac{P_i + S_i}{\sum_{i=1}^{m} (P_i + S_i)}$ (11)

$U_{ib} = \frac{S_i}{\min S_i} + \frac{P_i}{\min P_i}$ (12)

$U_{ic} = \frac{\gamma(S_i) + (1-\gamma)(P_i)}{\gamma \max S_i + (1-\gamma) \max P_i}; \quad 0 \le \gamma \le 1$ (13)

$U_i = (U_{ia} U_{ib} U_{ic})^{\frac{1}{3}} + \frac{1}{3}(U_{ia} + U_{ib} + U_{ic})$ (14)

Step 7. Rank the alternatives.

4. Example study for art education teaching quality evaluation of vocational colleges

Art teachers in vocational colleges are mostly graduates from normal universities. Some teachers directly enter the workforce after graduation. Although they have mastered certain professional abilities and qualities during their university years and achieved certain results, they lack teaching experience and are difficult to effectively impart the knowledge they have mastered to students, making it difficult to provide effective guidance to students. Some vocational colleges have overlooked the improvement and further training of teacher abilities, and some teachers have not yet formulated clear career plans, resulting in limited opportunities for teachers to participate in training. At the same time, some art teachers lack learning ability, making it difficult to recognize the importance of improving teaching ability, which limits the improvement of art teaching quality. Currently, many vocational colleges have offered art courses, but some vocational colleges focus on cultivating and enhancing students' painting abilities in art courses, without combining the current hot topics of art education and the specific learning situation of students to carry out teaching. The teaching form and content are relatively single. This not only causes waste of art teaching resources, but also to some extent affects the learning effectiveness of students. There is a

high demand for diversified art talents in the current society. However, in teaching, some art teachers only rely on textbooks to explain knowledge, without combining it with current art hotspots, which is detached from the actual teaching needs of vocational colleges and not conducive to cultivating students' social adaptability. This also affects the improvement of the quality of art education to a certain extent. The educational function of art is reflected in improving students' personality and thinking, enhancing their aesthetic abilities, and helping to develop their comprehensive qualities. In the current social context, the demand for art professionals in society is constantly increasing, and the previous concept of talent cultivation is no longer able to adapt to the development needs of the new era. In this situation, vocational college teachers need to explore education models that adapt to social development, improve professional course teaching, and enhance students' professional abilities and literacy through art course education. Although the current art courses have achieved good results and trained art professionals with good qualities and abilities, some vocational colleges find it difficult to integrate art courses with the content of other courses, and the actual teaching effect is not satisfactory, which to some extent limits the development of students and leads to a relative lack of innovative and practical abilities in the trained talents. The art education teaching quality evaluation of vocational colleges is MADM. Six possible art education colleges are depicted with 16 attributes as shown in Table 1.

Table 1. The relationship between criteria and sub criteria.

	Criteria	Sub criteria	weights
N_1	Faculty Professional Experience	N_{11}<50%, N_{12}>50%	0.062484
N_2	Facility Quality	N_{21}<50%, N_{22}>50%	0.061977
N_3	Creative Skill Development	N_{31}<50%, N_{32}>50%	0.062611
N_4	Employment in Art-Related Fields	N_{41}<50%, N_{42}>50%	0.062484
N_5	Student-Centered Teaching Approaches	N_{51}<50%, N_{52}>50%	0.062636
N_6	Curriculum Relevance	N_{61}<60%, N_{62}>60%	0.063042
N_7	Innovation in Teaching Materials	N_{71}<50%, N_{72}>50%	0.06289
N_8	Practical Components	N_{81}<50%, N_{82}>50%	0.062814
N_9	Community Projects	N_{91} 2, N_{92} 5, N_{93} 10 projects	0.062332
N_{10}	Ongoing Training	$N_{10,1}$<50%, $N_{10,2}$>50%	0.061825
N_{11}	Student Participation in Exhibitions	$N_{11,1}$<50%, $N_{11,2}$>50%	0.061977
N_{12}	Student Satisfaction	$N_{12,1}$ 3, $N_{12,2}$ 4, and $N_{12,3}$ 5 rate	0.062484
N_{13}	Graduation Rate	$N_{13,1}$<50%, $N_{13,2}$>50%	0.062611
N_{14}	Use of Technology in Teaching	$N_{14,1}$<50%, $N_{14,2}$>50%	0.062687
N_{15}	Teacher-Student Ratio	$N_{15,1}$ <10, $N_{15,2}$ >10	0.062256
N_{16}	Interdisciplinary Integration	$N_{16,1}$ 3, $N_{16,2}$ 7, and $N_{16,3}$ 10 courses	0.06289

Table 2. The opinions of three experts.

	A_1	A_2	A_3	A_4	A_5	A_6
N_{12}	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.73, 0.18, 0.06, 0.03)	(0.65, 0.22, 0.08, 0.05)	(0.68, 0.15, 0.12, 0.05)
N_{21}	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.59, 0.26, 0.10, 0.05)	(0.68, 0.15, 0.12, 0.05)	(0.59, 0.26, 0.10, 0.05)	(0.52, 0.33, 0.08, 0.07)
N_{31}	(0.73, 0.18, 0.06, 0.03)	(0.76, 0.11, 0.07, 0.06)	(0.76, 0.11, 0.07, 0.06)	(0.65, 0.22, 0.08, 0.05)	(0.76, 0.11, 0.07, 0.06)	(0.59, 0.26, 0.10, 0.05)
N_{42}	(0.71, 0.20, 0.07, 0.02)	(0.59, 0.26, 0.10, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.73, 0.18, 0.06, 0.03)	(0.47, 0.38, 0.09, 0.06)	(0.76, 0.11, 0.07, 0.06)
N_{52}	(0.80, 0.10, 0.06, 0.04)	(0.52, 0.33, 0.08, 0.07)	(0.76, 0.11, 0.07, 0.06)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)	(0.47, 0.38, 0.09, 0.06)
N_{61}	(0.47, 0.38, 0.09, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)
N_{71}	(0.76, 0.11, 0.07, 0.06)	(0.65, 0.22, 0.08, 0.05)	(0.71, 0.20, 0.07, 0.02)	(0.47, 0.38, 0.09, 0.06)	(0.73, 0.18, 0.06, 0.03)	(0.71, 0.20, 0.07, 0.02)
N_{82}	(0.59, 0.26, 0.10, 0.05)	(0.73, 0.18, 0.06, 0.03)	(0.73, 0.18, 0.06, 0.03)	(0.76, 0.11, 0.07, 0.06)	(0.65, 0.22, 0.08, 0.05)	(0.73, 0.18, 0.06, 0.03)
N_{93}	(0.52, 0.33, 0.08, 0.07)	(0.71, 0.20, 0.07, 0.02)	(0.65, 0.22, 0.08, 0.05)	(0.59, 0.26, 0.10, 0.05)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
$N_{10,2}$	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.68, 0.15, 0.12, 0.05)	(0.52, 0.33, 0.08, 0.07)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)
$N_{11,1}$	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.59, 0.26, 0.10, 0.05)	(0.68, 0.15, 0.12, 0.05)	(0.59, 0.26, 0.10, 0.05)	(0.52, 0.33, 0.08, 0.07)
$N_{12,3}$	(0.73, 0.18, 0.06, 0.03)	(0.76, 0.11, 0.07, 0.06)	(0.76, 0.11, 0.07, 0.06)	(0.65, 0.22, 0.08, 0.05)	(0.76, 0.11, 0.07, 0.06)	(0.59, 0.26, 0.10, 0.05)
$N_{13,1}$	(0.71, 0.20, 0.07, 0.02)	(0.59, 0.26, 0.10, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.73, 0.18, 0.06, 0.03)	(0.47, 0.38, 0.09, 0.06)	(0.76, 0.11, 0.07, 0.06)
$N_{14,2}$	(0.80, 0.10, 0.06, 0.04)	(0.52, 0.33, 0.08, 0.07)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)	(0.47, 0.38, 0.09, 0.06)
$N_{15,1}$	(0.47, 0.38, 0.09, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.73, 0.18, 0.06, 0.03)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)
$N_{16,3}$	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.73, 0.18, 0.06, 0.03)	(0.71, 0.20, 0.07, 0.02)
	A_1	A_2	A_3	A_4	A_5	A_6
N_{12}	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
N_{21}	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
N_{31}	(0.73, 0.18, 0.06, 0.03)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.59, 0.26, 0.10, 0.05)
N_{42}	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.76, 0.11, 0.07, 0.06)
N_{52}	(0.80, 0.10, 0.06, 0.04)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)
N_{61}	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)

N_{71}	(0.76, 0.11, 0.07, 0.06)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.71, 0.20, 0.07, 0.02)
N_{82}	(0.59, 0.26, 0.10, 0.05)	(0.73, 0.18, 0.06, 0.03)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
N_{93}	(0.52, 0.33, 0.08, 0.07)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.65, 0.22, 0.08, 0.05)
$N_{10,2}$	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)
$N_{11,1}$	(0.65, 0.22, 0.08, 0.05)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.52, 0.33, 0.08, 0.07)
$N_{12,3}$	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.76, 0.11, 0.07, 0.06)	(0.59, 0.26, 0.10, 0.05)
$N_{13,1}$	(0.71, 0.20, 0.07, 0.02)	(0.59, 0.26, 0.10, 0.05)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
$N_{14,2}$	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
$N_{15,1}$	(0.47, 0.38, 0.09, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)
$N_{16,3}$	(0.76, 0.11, 0.07, 0.06)	(0.52, 0.33, 0.08, 0.07)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.73, 0.18, 0.06, 0.03)	(0.71, 0.20, 0.07, 0.02)
	A_1	A_2	A_3	A_4	A_5	A_6
N_{12}	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.73, 0.18, 0.06, 0.03)	(0.65, 0.22, 0.08, 0.05)	(0.68, 0.15, 0.12, 0.05)
N_{21}	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.52, 0.33, 0.08, 0.07)
N_{31}	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)
N_{42}	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.47, 0.38, 0.09, 0.06)	(0.76, 0.11, 0.07, 0.06)
N_{52}	(0.80, 0.10, 0.06, 0.04)	(0.52, 0.33, 0.08, 0.07)	(0.76, 0.11, 0.07, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)
N_{61}	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.80, 0.10, 0.06, 0.04)
N_{71}	(0.76, 0.11, 0.07, 0.06)	(0.65, 0.22, 0.08, 0.05)	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)
N_{82}	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.73, 0.18, 0.06, 0.03)
N_{93}	(0.52, 0.33, 0.08, 0.07)	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.65, 0.22, 0.08, 0.05)
$N_{10,2}$	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)
$N_{11,1}$	(0.65, 0.22, 0.08, 0.05)	(0.47, 0.38, 0.09, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.52, 0.33, 0.08, 0.07)
$N_{12,3}$	(0.73, 0.18, 0.06, 0.03)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.76, 0.11, 0.07, 0.06)	(0.59, 0.26, 0.10, 0.05)
$N_{13,1}$	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)
$N_{14,2}$	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)
$N_{15,1}$	(0.47, 0.38, 0.09, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.65, 0.22, 0.08, 0.05)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)
$N_{16,3}$	(0.76, 0.11, 0.07, 0.06)	(0.68, 0.15, 0.12, 0.05)	(0.80, 0.10, 0.06, 0.04)	(0.71, 0.20, 0.07, 0.02)	(0.73, 0.18, 0.06, 0.03)	(0.71, 0.20, 0.07, 0.02)

This study has 16 criteria and six alternatives, and three experts have evaluated the criteria and alternatives as shown in Table 2. Then we selected the highest related sub criteria as $N_{12} \times N_{21} \times N_{31} \times N_{42} \times N_{52} \times N_{61} \times N_{71} \times N_{82} \times N_{93} \times N_{10,2} \times N_{11,1} \times N_{12,3} \times N_{13,1} \times N_{14,2} \times N_{15,1} \times N_{16,3}$

Then we obtained the normalized decision matrix as shown in Table 3. Then we obtained the weighted comparability sequence as shown in Table 4. Then we obtained the relative weights of the alternatives. Then we obtained the final rank as shown in Figure 1.

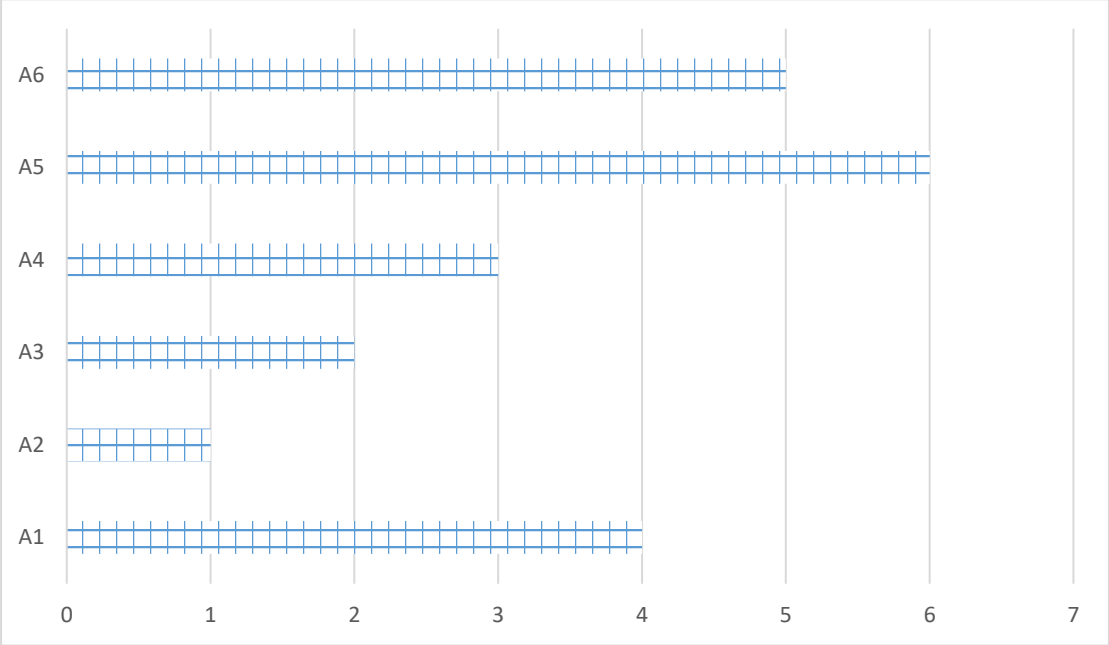


Figure 1. The rank of the alternatives.

Table 3. The normalized decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6
N_{12}	0	1	0.952381	0.857143	0.380952	0.190476
N_{21}	1	0	0	0.5	0.666667	0.333333
N_{31}	1	0.916667	0.833333	0	0.916667	0.666667
N_{42}	1	0	0.3125	1	0.125	0.5
N_{52}	1	0	0.153846	0	0.769231	0.307692
N_{61}	0	0.166667	0.5	0.666667	1	0.833333

N_{71}	0.454545	0.454545	0.727273	0	0.772727	1
N_{82}	0	1	0.421053	0.473684	0.421053	0.842105
N_{93}	0	1	0	0.214286	0.428571	0.428571
$N_{10,2}$	0.25	1	0	0.8125	0.75	0
$N_{11,1}$	0.8	0.4	0	0.3	1	0.2
$N_{12,3}$	1	0	0.357143	0.714286	0.428571	0
$N_{13,1}$	1	0.285714	0.714286	0.142857	0	1
$N_{14,2}$	0.111111	0.111111	1	0	0.444444	0.444444
$N_{15,1}$	0.315789	0	0.631579	0.526316	0.789474	1
$N_{16,3}$	0.4	0	0.15	0.6	1	1

Table 4. The weighted comparability sequence.

	A_1	A_2	A_3	A_4	A_5	A_6
N_{12}	0	0.062484	0.059509	0.053558	0.023803	0.011902
N_{21}	0.061977	0	0	0.030989	0.041318	0.020659
N_{31}	0.062611	0.057393	0.052176	0	0.057393	0.041741
N_{42}	0.062484	0	0.019526	0.062484	0.007811	0.031242
N_{52}	0.062636	0	0.009636	0	0.048182	0.019273
N_{61}	0	0.010507	0.031521	0.042028	0.063042	0.052535
N_{71}	0.028586	0.028586	0.045738	0	0.048597	0.06289
N_{82}	0	0.062814	0.026448	0.029754	0.026448	0.052896
N_{93}	0	0.062332	0	0.013357	0.026714	0.026714
$N_{10,2}$	0.015456	0.061825	0	0.050233	0.046369	0
$N_{11,1}$	0.049582	0.024791	0	0.018593	0.061977	0.012395
$N_{12,3}$	0.062484	0	0.022316	0.044632	0.026779	0
$N_{13,1}$	0.062611	0.017889	0.044722	0.008944	0	0.062611
$N_{14,2}$	0.006965	0.006965	0.062687	0	0.027861	0.027861
$N_{15,1}$	0.01966	0	0.03932	0.032766	0.049149	0.062256
$N_{16,3}$	0.025156	0	0.009433	0.037734	0.06289	0.06289

Sensitivity Analysis

This part shows the sensitivity analysis by changing in the lambda values between 0 to 1. Then we obtained different ranks of alternatives as shown in Figure 2. The different values of lambdas show the optimal alternatives is stable under different values. The results show alternative 5 is the best under different values.

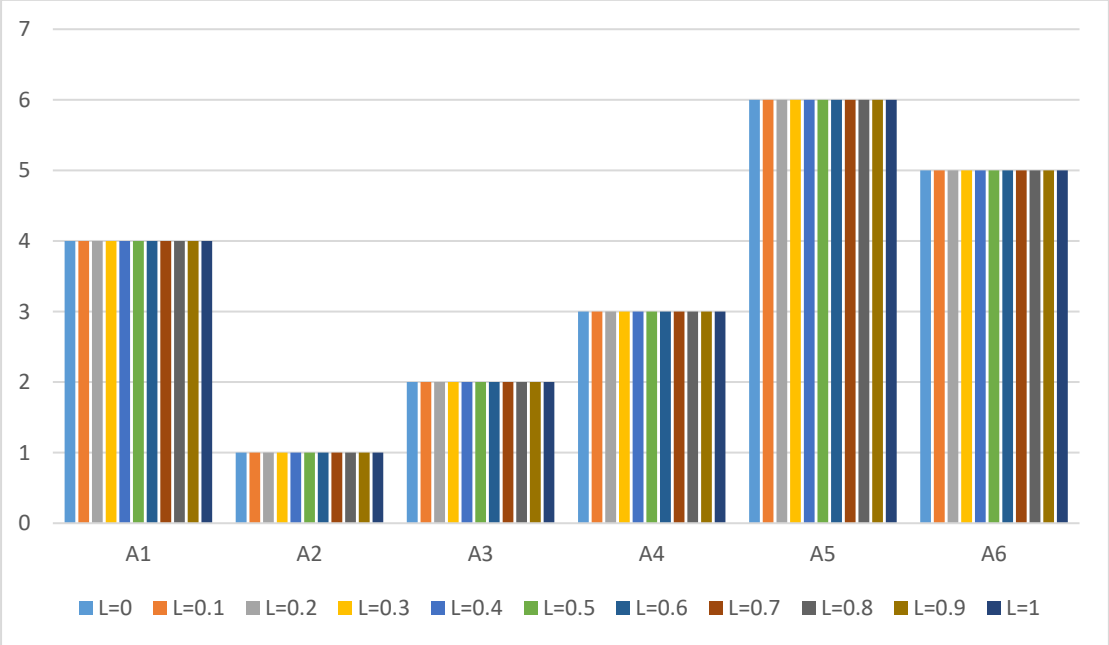


Figure 2. The rank of alternatives under different values of lambda.

5. Conclusion

In art education, teachers tailor their instruction to match the varying levels and circumstances of their students, a practice that demands robust teaching skills. Universities, reflecting their unique visions for student futures and differing art faculty competencies, set distinct benchmarks for training art educators. Art teachers are expected not only to possess advanced practical artistic skills but also to have a comprehensive understanding of art theory, effectively merging theory with practice during instruction. Given that art teachers interpret and convey artistic knowledge in diverse ways, students are encouraged to selectively engage with teachings that resonate with their personal learning styles. To enhance the sustainability of art education, it is essential to deeply explore both the content and methodologies of art instruction, identify the most appropriate courses for vocational college students, and subsequently refine the overall art curriculum. The evaluation of art education teaching quality in vocational colleges is approached through MADM. In this study, we introduce the application of the CoCoSo method for MADM within the framework of DVNSs. We further develop the DVNN-TSS-CoCoSo model for MADM. To illustrate the practical application of this model, we provide the numerical example that assesses the quality of art education in vocational colleges.

The primary research contributions of this paper are summarized: (1)The creation of a novel MADM approach that integrates the CoCoSo approach and average strategies using DVNSs with the Tree Soft Set to deal with the relationship between criteria and sub criteria; (2) The incorporation of objective weights with the average approach; (3) The introduction of the DVNN-TSS-CoCoSo approach specifically tailored for evaluating the quality of art education in vocational colleges; (4) The validation of the DVNN-TSS-CoCoSo model through a practical numerical example, underscoring its efficacy in assessing the quality of art education.

In assessing the quality of art education in vocational colleges, it's crucial to recognize potential shortcomings that could be explored in future studies. Several key areas offer valuable opportunities for deeper investigation: (1) Consensus Management: Future research could aim to develop strategies for achieving consensus in evaluating the quality of art education in vocational colleges, within the scope of DVNSs. This would involve integrating the diverse opinions and perspectives of various stakeholders to reach a unified conclusion. Developing and refining methodologies that support consensus building could significantly improve the reliability and validity of the evaluation processes. (2) Regret Theory: Another promising area for research is the application of regret theory to the evaluation of teaching quality in art education at vocational colleges, also employing DVNSs. Regret theory examines the decision-making process by accounting for the potential regret that might follow from choosing a specific option. By integrating regret theory into the evaluative framework, researchers can gain a more nuanced understanding of the compromises and dynamics involved in making quality assessments. Addressing these areas in subsequent research would not only augment the efficacy and thoroughness of the evaluation methods used for art education

teaching quality in vocational colleges but also push forward the field, offering critical insights to educational institutions and policymakers.

References

[1] A. Clarke, K. Budge, Listening for creative voices amid the cacophony of fiscal complaint about art and design education, International Journal of Art & Design Education, 29 (2010) 153-162.

[2] F. McArdle, New maps of learning for quality art education: What pre-service teachers should learn and be able to do, Australian Educational Researcher, 39 (2012) 91-106.

[3] C. Christophersen, Changes and challenges in music education: Reflections on a norwegian arts-in-education programme, Music Education Research, 17 (2015) 365-380.

[4] A. Bautista, A. Moreno-Nunez, R. Bull, F. Amsah, S.F. Koh, Arts-related pedagogies in preschool education: An asian perspective, Early Childhood Research Quarterly, 45 (2018) 277-288.

[5] K. Burke, The challenges of facilitating arts learning in home education, British Educational Research Journal, 45 (2019) 961-978.

[6] X.X. Ling, Model for evaluating the art education teaching quality with uncertain information, Journal of Intelligent & Fuzzy Systems, 37 (2019) 1967-1972.

[7] G.F. Qu, W.B. Hu, W.X. Jiao, J.B. Jin, Application of deep learning-based integrated trial-error plus science, technology, reading/writing, engineer, arts, mathematics teaching mode in college entrepreneurship education, Frontiers in Psychology, 12 (2021) 12.

[8] C.R. Tan, The nascent case for adopting jupyter notebooks as a pedagogical tool for interdisciplinary humanities, social science, and arts education, Computing in Science & Engineering, 23 (2021) 107-113.

[9] Q.Y. Li, Z.H. Li, J. Han, H.M. Ma, Quality assurance for performing arts education: A multi-dimensional analysis approach, Applied Sciences-Basel, 12 (2022) 23.

[10] V. Pavlou, A. Castro-Varela, E-learning canvases: Navigating the confluence of online arts education and sustainable pedagogies in teacher education, Sustainability, 16 (2024) 19.

[11] X.X. Fan, J.Y. Li, Artificial intelligence-driven interactive learning methods for enhancing art and design education in higher institutions, Applied Artificial Intelligence, 37 (2023) 20.

[12] R. Verma, E. Alvarez-Miranda, Multiple-attribute group decision-making approach using power aggregation operators with critic-waspas method under 2-dimensional linguistic intuitionistic fuzzy framework, Applied Soft Computing, 157 (2024) 33.

[13] M. Abouhawwash, M. Jameel, Evaluation factors of solar power plants to reduce cost under neutrosophic multi-criteria decision making model, Sustainable Machine Intelligence Journal, 2 (2023) 1-11.

[14] C. Jana, H. Garg, M. Pal, B. Sarkar, G.W. Wei, Mabac framework for logarithmic bipolar fuzzy multiple attribute group decision-making for supplier selection, Complex & Intelligent Systems, 10 (2024) 273-288.

[15] O. Dawlet, Y.L. Bao, Normalized hesitant fuzzy aggregation operators for multiple attribute decision-making, International Journal of Fuzzy Systems, (2024) 16.

[16] H.L. Wang, T. Mahmood, K. Ullah, Improved cocoso method based on frank softmax aggregation operators for t-spherical fuzzy multiple attribute group decision-making, International Journal of

Fuzzy Systems, 25 (2023) 1275-1310.

[17] G.-F. Yu, D.-F. Li, D.-C. Liang, G.-X. Li, An intuitionistic fuzzy multi-objective goal programming approach to portfolio selection, International Journal of Information Technology & Decision Making, 20 (2021) 1477-1497.

[18] M.R. Feylizadeh, A. Mahmoudi, M. Bagherpour, D.F. Li, Project crashing using a fuzzy multi-objective model considering time, cost, quality and risk under fast tracking technique: A case study, Journal of Intelligent & Fuzzy Systems, 35 (2018) 3615-3633.

[19] G.F. Yu, D.F. Li, J.M. Qiu, Y.F. Ye, Application of satisfactory degree to interval-valued intuitionistic fuzzy multi-attribute decision making, Journal of Intelligent & Fuzzy Systems, 32 (2017) 1019-1028.

[20] M. Yazdani, P. Zarate, E.K. Zavadskas, Z. Turskis, A combined compromise solution (cocoso) method for multi-criteria decision-making problems., Management Decision, 57 (2018) 2501-2519.

[21] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[22] I. Kandasamy, Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm, Journal of Intelligent Systems, 27 (2018) 163-182.

[23] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

Received: Aug 27, 2024. Accepted: Nov 25, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Novel Analysis on the Perception of Physical Environmental Quality: An In-Depth Study Comparing the Neutrosophic Likert Scale with Score Function and the Classic Likert Scale - Statistical Analysis of Data from a Restaurant Survey

Seher BODUR[1], Serpil KAYA [2] and Selçuk TOPAL[3]

[1] Department of Statistics, Bitlis Eren University, Bitlis 13000, Türkiye; skorkmaz@beu.edu.tr
[2] Gastronomy and Culinary Arts Department, İstanbul Rumeli University, İstanbul 34421, Türkiye; serpil.kaya@rumeli.edu.tr
[3*] Department of Mathematics, Gebze Technical University, Kocaeli 41000, Türkiye; stopal@gtu.edu.tr
* Correspondence: stopal@gtu.edu.tr

Abstract: There are studies examining the quality of the physical environment in the restaurant industry. However, these studies have only been conducted using Likert scale. Since the neutrosophic scale has not been used in the restaurant industry before, it is considered to be an important gap. For this reason, it is aimed to investigate the perceptions of physical environmental quality in restaurants by using Likert and neutrosophic scales for the first time. By applying Exploratory Factor Analysis, evidence is presented that a single factor adequately incorporates both scales, thus optimising the assessment procedure. Their consistency and reliability in assessing environmental perceptions is confirmed by the robust positive correlation between the two methods. Further investigations with one-way and two-way ANOVA reveal notable demographic differences, in particular the increased sensitivity of individuals aged 45 years and older towards the quality of the physical environment in restaurants. In contrast, gender has no effect on how environmental quality is perceived, suggesting that improvements in this area may resonate universally with consumers of all genders. The results of this study provide practical recommendations for restaurant managers seeking to improve business operations and customer satisfaction by emphasising environmental hygiene, especially for older customers. This research not only contributes to the body of knowledge regarding methodological approaches to environmental perception, but also sheds light on specific demographic preferences that can shed light on focused improvements in the restaurant industry.

Keywords: ANOVA, Likert scale, neutrosophic Likert scale, exploratory factor analysis, physical environment quality.

1. Introduction

As the increase in the number of restaurant establishments brings with it competition, it has become important for restaurants to be able to offer a physical service experience. The physical environment is an important element not only to maintain the loyalty of existing customers but also

to attract new customers [1]. The physical environment, which refers to the environment where product or service delivery takes place [2], is defined as an environment created by service providers, including general layout, design, decoration, and aesthetics [3]. As a key factor in attracting and satisfying customers in restaurant businesses, the physical environment has attracted attention among business managers [4]. It consists of all tangible and intangible elements inside or outside a restaurant [5]. It is stated that it is an important part of customers' evaluation of their satisfaction with the services they receive [6]. In this context, it is possible to say that the physical environment plays a critical role in differentiating the image of restaurant businesses, providing competitive advantage and influencing customer behavior [7-8-9-10-11-12]. Moreover, the physical environment can positively or negatively affect the customer's mood [13]. Restaurant businesses attach importance to the quality of the physical environment in order to differentiate from their competitors and increase their preferability. The physical environment has become a strategic element that restaurants pay attention to both in differentiating the atmosphere, changing the perception of the product or service, and ensuring that the first impression is positive. For this reason, it has been the subject of study by many researchers. This is because the quality of the physical environment is effective in the success of the restaurant business [14]. In this direction, studies on the use of the physical environment variable, which is the subject of the research, in Likert scales were examined. In the study conducted to examine the effects of physical environment, perceived value and image on customers' behavioural intentions in resort hotels in Malaysia, 280 questionnaires were distributed to hotels in two different tourist destinations. In this study, a five-point Likert scale was used. As a result of the study, it was found that guests with high perceptions of the physical environment were more likely to have positive perceived images, values and behavioural intentions [15]. A total of 340 questionnaires were distributed and analysed to test the relationships between physical environment, price perception, consumption emotions and customer satisfaction in resort hotels in China. The results of the analyses reveal that physical environment is an important determinant of consumption emotions and price perceptions, which in turn affect customer satisfaction. Moreover, consumption emotions and price perceptions significantly mediate the relationship between physical environment and customer satisfaction [16]. To examine the impact of physical environment on emotions, customer satisfaction and behavioural intentions in the Chinese resort hotel industry, 170 questionnaires were analysed. Respondents were asked to respond to all items using a 5-point Likert scale.

To examine the effect of hotel-restaurant image and physical environment, service, and food quality on satisfaction and intention, 308 questionnaires were distributed and analyzed. The study showed that the variables are generally significantly related; quality dimensions and satisfaction have a mediating role; satisfaction and overall image have a greater impact on decision-making than other variables [17]. To investigate the importance of the physical environment in terms of customer satisfaction in Bali, 105 participants were surveyed in 22 different restaurants. According to the research results, the most determinant factor of customer satisfaction is the atmosphere factor, followed by the aesthetic factor of the facility. The variable that determines customer satisfaction the most is comfortable room temperature, followed by clean furniture [18]. To reveal the relationship between the physical environment of a restaurant and price perception and to determine the effect of price perception on satisfaction and loyalty, a questionnaire was collected from 475 people who experienced first-class restaurants in Istanbul. As a result of the data obtained from this study using a seven-point Likert scale, it was concluded that there was no significant relationship between ambiance and decoration and price perception. In contrast, a significant relationship was found between layout and price perception [19].

To examine the effect of the physical environment and food quality on satisfaction and behavioral intentions, a questionnaire was applied to 130 participants who visited McDonald's point of sale. According to the results of the research, it was determined that food quality has a positive and significant effect on consumer satisfaction and physical environment has a positive and

significant effect on consumer satisfaction. The results also show that food quality has a positive and insignificant effect on behavioral intentions through consumer satisfaction, while the physical environment has a positive and insignificant effect on behavioral intentions through consumer satisfaction [20]. As a result of the literature review, it was determined that the neutrosophic scale has not been applied in the restaurant sector before. For this reason, it is aimed to evaluate how restaurant businesses that try to differentiate with their physical environment respond to two different questionnaires (Likert + neutrosophic) consisting of the same questions.

2. Likert Scale

The Likert rating scale, which is widely used in social sciences and attitude scores, can be a useful and reliable tool for measuring self-efficacy [21-22]. This scale developed by Rensis Likert was defined for the evaluation of attitudes [23]. It requires individuals to respond to a series of statements as strongly disagree, disagree, undecided, agree, or strongly agree. A point value is assigned to each response and the score of the individuals is determined by summing the point values of all statements [22-24]. Here, all statements reveal a particular dimension of attitude toward the topic, so they are necessarily interrelated [25]. The Likert scale is recognized as an easy and reliable scaling technique [26], where it is easy to measure and understand participant perception [27]. It is stated to be more manageable and fast compared to other scales [28]. However, there is a debate among researchers about the optimum number of options in a Likert-type scale. Some researchers prefer scales with 7-item responses [29].

However, although the Likert scale seems to be useful, it also has some disadvantages. There are uncertainties about whether the responses collected on the Likert scale should be ordinal or interval level [23-30] and how the data obtained should be analyzed [27]. This confusion has led many to use statistical methods such as mean and standard deviation, which are inappropriate for Likert scales [30]. In addition, participants' responses may be influenced by previous questions and their tendency to avoid selecting extreme possibilities on the scale is also problematic. Due to these uncertainties, it is thought that the Likert scale may not be the most appropriate scale to use in research. Therefore, this situation leads researchers to propose different scale types. To obtain more acceptable measurement findings, the Neutrosophic approach based on fuzzy sets theory was proposed as an alternative to the Likert scale in this study.

3. Neutrosophy and Neutrosophic Set

The philosophical and mathematical framework known as Neutrosophy [31] was created by Florentin Smarandache during the latter part of the 20th century. This particular field of study pertains to issues and principles that encompass indeterminacy, ambiguity, and contradictions. Neutrosophic logic offers an extension to classical, fuzzy, and intuitionistic fuzzy logic, enabling the representation of information that is uncertain, contradictory, and ambiguous. Classical logic entails assertions that can either be true or false, whereas neutrosophic logic allows claims to be true, false, and indeterminate at the same time. This enables a more sophisticated and adaptable approach to logical thinking. Neutrosophic set theory is an extension of classical, fuzzy, and intuitionistic fuzzy set theory that addresses the challenges posed by sets containing indeterminate or uncertain members. In classical set theory, an element is classified as either a member of a set or not. In neutrosophic set theory, an element can possess different levels of membership in a set, have no degree of membership in the set, or have no degree of membership in the set. Neutrosophic probability is an extension of the conventional theory of probability that addresses the challenges posed by unpredictable and uncertain events. In classical probability theory, events are characterized by well-defined probabilities ranging from 0 to 1. The correlation between events in neutrosophic probability and degrees of truth, falsehood, and indeterminacy allows for a more comprehensive depiction of uncertainty.

Neutrosophy has been utilized in many domains such as artificial intelligence [32], decision-making [33-34-35], information fusion [36], and risk analysis [37], where the effective management of uncertainty and ambiguity is of utmost importance. The paradigm presented offers a structured approach to addressing scenarios in which classical logic and probability theory may necessitate revision as a result of conflicting or uncertain data. In classical set theory, an element is classified as either a member of a set or not. The interpretation of the membership of elements in a set is based on binary terms, as per the binary case.

The single valued neutrosophic set [31-38] offers a comprehensive framework that encompasses intuitionistic fuzzy sets, classical sets, fuzzy sets, dialetheist sets, paradoxist sets, tautological sets, and intuitionistic fuzzy sets, all of which are rooted in the principles of Neutrosophy. An member x(T, I, F) is considered true in the set when its degree is T ∈ [0,1], ambiguous when its degree is I ∈ [0,1], and false when its degree is F ∈ [0,1].

In this section, we will provide fundamental definitions and concepts pertaining to single-valued neutrosophic sets, fuzzy sets, and intuitionistic fuzzy sets.

Definition 1. [39] A fuzzy set X in $\boldsymbol{U}$ is a set of ordered pairs, defined as $\boldsymbol{X} = \{(\boldsymbol{x}, \mu_{\boldsymbol{X}}(\boldsymbol{x}))|\boldsymbol{x} \in \boldsymbol{U}\}$, where $\mu_{\boldsymbol{X}} : \boldsymbol{U} \to [\boldsymbol{0}, \boldsymbol{1}]$ is termed the membership function of X, and $\mu_{\boldsymbol{X}}(\boldsymbol{x})$ is the degree of membership of the element x in X given a universal set U and a generic element, represented by x.

Definition 2. [40] An intuitionistic fuzzy set X exists over a discourse-level world. The representation of U is given by $\boldsymbol{X} = \{(\boldsymbol{x}, \mu_{\boldsymbol{X}}(\boldsymbol{x}), \boldsymbol{\nu}_{\boldsymbol{X}}(\boldsymbol{x}))|\boldsymbol{x} \in \boldsymbol{U}\}$, where the terms "membership function of X" and "non-membership function of X" for x in X are, respectively, $\mu_{\boldsymbol{X}} : \boldsymbol{U} \to [\boldsymbol{0}, \boldsymbol{1}]$ and $\boldsymbol{\nu}_{\boldsymbol{X}} : \boldsymbol{U} \to [\boldsymbol{0}, \boldsymbol{1}]$. The formula for determining the degree of non-membership of an element, x, in X is $\mu_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{\nu}_{\boldsymbol{X}}(\boldsymbol{x}) \leq \boldsymbol{1}$. The hesitation degree of an element x defined by $\boldsymbol{\pi}_{\boldsymbol{X}}(\boldsymbol{x}) = \boldsymbol{1} - (\mu_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{\nu}_{\boldsymbol{X}}(\boldsymbol{x}))$.

Definition 3. [31-38] Let U be a discourse universe. $\boldsymbol{N} = \{(\boldsymbol{x}, \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{I}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})) : \boldsymbol{x} \in \boldsymbol{U}\}$ is a neutrosophic set, denoted by a truth-membership function, $\boldsymbol{T}_{\boldsymbol{N}} : \boldsymbol{U} \to]^{-}\boldsymbol{0}, \boldsymbol{1}^{+}[$; an indeterminacy-membership function, $\boldsymbol{I}_{\boldsymbol{N}} : \boldsymbol{U} \to]^{-}\boldsymbol{0}, \boldsymbol{1}^{+}[$; and a falsity-membership function, $\boldsymbol{F}_{\boldsymbol{N}} : \boldsymbol{U} \to]^{-}\boldsymbol{0}, \boldsymbol{1}^{+}[$.

Definition 4. [31-38] Let U be a discourse universe. A single-valued neutrosophic set is defined as $\boldsymbol{N} = \{(\boldsymbol{x}, \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{I}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})) : \boldsymbol{x} \in \boldsymbol{U}\}$, which is identified by a truth-membership function, $\boldsymbol{T}_{\boldsymbol{N}} : \boldsymbol{U} \to [\boldsymbol{0}, \boldsymbol{1}]$; indeterminacy-membership function, $\boldsymbol{I}_{\boldsymbol{N}} : \boldsymbol{U} \to [\boldsymbol{0}, \boldsymbol{1}]$; and falsity-membership function, $\boldsymbol{F}_{\boldsymbol{N}} : \boldsymbol{U} \to [\boldsymbol{0}, \boldsymbol{1}]$, with $\boldsymbol{0} \leq \boldsymbol{T}_{\boldsymbol{N}}(\boldsymbol{x}) + \boldsymbol{I}_{\boldsymbol{N}}(\boldsymbol{x}) + \boldsymbol{F}_{\boldsymbol{N}}(\boldsymbol{x}) \leq \boldsymbol{3}$.

3.1. Neutrosophy in Social Sciences

Neutrosophic Sociology, also known as Neutrosociology, refers to the application of neutrosophic scientific procedures in the study of sociology, as defined by Smarandache [41]. The questionnaire is widely recognized as a crucial tool in surveys for assessing the state of opinions within social groups. While it is widely acknowledged that fuzzy responses are preferable to crisp responses in surveys, it is important to acknowledge that fuzzy processing may not accurately capture the intended meaning of the responder due to uncertainties, confusion, and unclear thinking. The utilization of neutrosophic sets in modeling such a scenario provides responders with an expanded range of potential responses, hence enhancing its relevance. [42]

In this paper, we propose a methodology for generating single-valued neutrosophic sets based on questionnaires administered to social groups. The study [43] provided a definition, demonstration, and introduction of neutrosophic statistical methods in the field of Social Sciences. Frequently, the data presented in the field of Social Sciences exhibit inconsistencies due to errors, conflicts in information and knowledge sources, lack of objectivity in some viewpoints, and various other factors. Consequently, it was asserted that under some conditions, the inclusion of interval data may be necessary.

3.2. Neutrosophic Score Function

Martinez et al. [42] employed the scoring function $s: [0,1] \rightarrow [0,3]$, denoted as $s(a) = 2+ T - I - F$, to quantify neutrosophic qualities and conduct a comparative analysis within the framework of social science. Nevertheless, our primary emphasis lies in the interpretation of the measurement pertaining to the impact of group decision-making on social choices. It is worth noting that this particular score function was not employed in a Likert-type study. The score function was initially employed in a Likert essay, and [42] demonstrated its safe applicability in the field of social sciences. In this study, we employ the $s(a) = (1 + T - 2I - F)/2$ function from the [44] publication. We believe it is appropriate for a starting workspace to consider the negative, neutral, and positive effects by distributing the score values across the [−1, 1] range, which aligns with the approach utilized in neutrosophic research.

3.3. Neutrosophic Likert Scale

The neutrosophic Likert scale was applied for the first time in the study [45]. Classical satisfaction with life scale Likert questions were transformed through numerical value between 0 and 100 was expected by participants to be given the options "I agree with this statement (…)", "I am neutral (or indeterminacy) about this statement (…)", "I disagree with this statement (…)". The results in the paper showed that the neutrosophic scale is also reliable, which supports the reliability of the classical scale because Cronbach's Alpha constant is an acceptable level for the three dimensions.

In our current study, we transform within the framework of direct percentage understanding to get closer to natural language. There is a connection between fuzzy sets and Neutrosophic Likert scales in the way they handle imprecision and uncertainty. Neutrosophic sets, and thus, Neutrosophic Likert scales, enable an even wider representation that incorporates indeterminacy as a core component, whereas fuzzy sets enable the representation and manipulation of data that is not exactly specified. For survey responses where participants' opinions are not only varied across a spectrum (as accommodated by fuzzy sets) but may also include a degree of indecision or neutrality that is difficult to capture by traditional fuzzy logic or crisp Likert scales, Neutrosophic Likert scales are therefore particularly well-suited.

4. Data and Likert Questionnaires

The survey questionnaire was administered to customers in the restaurant with the understanding that the identity of the respondents would be kept confidential, that participation could be withdrawn at any time, and that the data collected would be used only for the purposes of this study. G*Power 3.1.9.4 software determined that 135 participants were necessary for the survey. In addition, a 22-question survey was administered to 135 restaurant customers and four of them were selected to help assess the quality of the physical environment using a 5-point Likert scale (ranging from 1 for "strongly disagree" to 5 for "strongly agree") as shown in Table 1. Participants were directed to choose one of the five agreement levels while responding to the survey.

Table 1. Likert Scale Questions (responses should be marked with a check for one choice)

Questions	Strongly Disagree	Disagree	Neither Agree Nor Disagree	Agree	Strongly Agree
The restaurant has an attractive interior design and decor.	☐	☐	☑	☐	☐
The background music is pleasant.	☐	☑	☐	☐	☐
The dining areas are very clean.	☑	☐	☐	☐	☐
Staff are clean and well dressed.	☐	☐	☐	☑	☐

The neutrosophic scale (a: degree of agreement; b: degree of indeterminacy; c: degree of disagreement) was applied toward measuring physical environmental quality. These selected questions are shown in Table 2. Survey participants (the same participants previously surveyed, as shown in Table 1) selected a, b, c, and one percentile value each. According to the single-valued neutrosophic set approach, the sum of the percentiles (% values) of a + b + c was between 0 and 300; then, we took one percent of these percentage values and mapped them to the closed interval [0, 1].

Table 2. Neutrosophic Scale Questions (responses are required to be completed in percentages)

Questions	Agreement Degree	Indeterminacy Degree (Neither Agree Nor Disagree)	Disagreement Degree
The restaurant has an attractive interior design and decor.	40	60	50
The background music is pleasant.	20	40.5	10
The dining areas are very clean.	0	100	20
Staff are clean and well dressed.	0.5	25	75

Table 3 displays the demographic details of the survey respondents. The responses gathered from participants utilizing both the Likert and neutrosophic scales are illustrated in the tables below.

Table 3. Profile Information of the Survey Respondents (N = 135)

Variables	Categories	Number of Cases	Percentage (%)
Gender	Female	69	51.11
	Male	66	48.89
Age	18–25	44	32.59
	26-34	31	22.96
	35-44	24	17.78
	45-54	17	12.59
	55^{+}	19	14.07

Table 4. Participants' Responses on a Likert-Type Scale

	Q1	Q2	Q3	Q4
Strongly disagree	16	22	3	6
Disagree	7	25	4	1
Neither agree nor disagree	35	27	18	13
Agree	41	25	51	37
Strongly agree	36	36	59	78

Table 5. Participants' Responses on a Neutrosophic Scale

	Q1	Q2	Q3	Q4
Disagree	3	3	1	1
Neither agree nor disagree	50	59	30	19
Agree	82	73	104	115

Table 6. Responses Based on Likert Scale for all Questions

	All Questions
Strongly disagree	7
Disagree	10
Neither agree nor disagree	14
Agree	52
Strongly agree	52

Table 7. Responses Based on Neutrosophic Scale for all Questions

	All Questions
Disagree	15
Neither agree nor disagree	32
Agree	88

Agreeing, represented by the responses "Agree" and "Strongly Agree" on the Likert scale, is selected at a higher frequency compared to neutrosophic agreement. Likewise, disagreement, represented by "Disagree" and "Strongly Disagree" responses on the Likert scale, is selected at a higher frequency compared to the neutrosophic scale. In contrast, it is observed that the number of undecided responses (32) obtained with the Neutrosophic scale exceeds the number of undecided responses (14) obtained with the Likert scale.

5. Statistical Analysis

Quantitative data were analyzed using SPSS (IBM Corporation) and MATLAB software. Exploratory factor analysis (EFA) is a statistical method used to help classify comparable variables into sub-dimensions and to make observations less dimensional in order to make the data easier to understand. For this purpose, a factor analysis was conducted to determine the factor structure of Likert and neutrosophic scales to assess the quality of the physical environment. Kaiser-Meyer-Olkin (KMO) and Bartlett's test were conducted to assess the suitability of the scales for exploratory factor analysis. Kaiser-Meyer-Olkin (KMO) values were calculated as 0.686 and 0.683, respectively, and exceeded the acceptable value of 0.60 in exploratory factor analysis for both scales. In addition, Bartlett's test results for Likert and neutrosophic scales were $x^2 = 218.486$ ($p < 0.05$) and $x^2 = 211.023$ ($p < 0.05$), respectively, indicating that there were significant correlations between the items, in other words, the variances were not homogeneous, supporting the suitability of the data for exploratory factor analysis. Following the exploratory factor analysis, the Likert and neutrosophic scales, each consisting of 4 items, explained 64.082% and 63.176% of the total variance. These results,

as indicated in Figure 1 and Figure 2, suggest that a single component represents these scales well enough.

Figure 1. Scree Plot of the Likert Scale

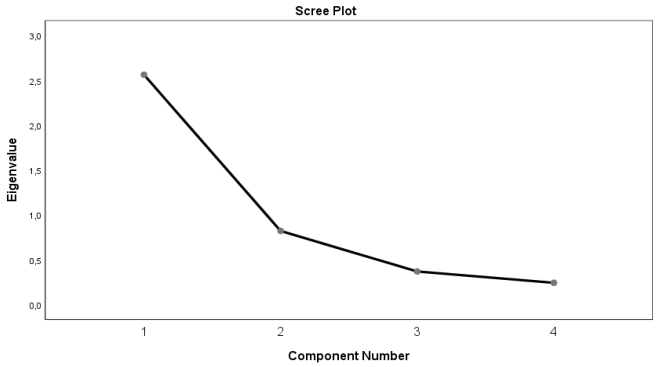
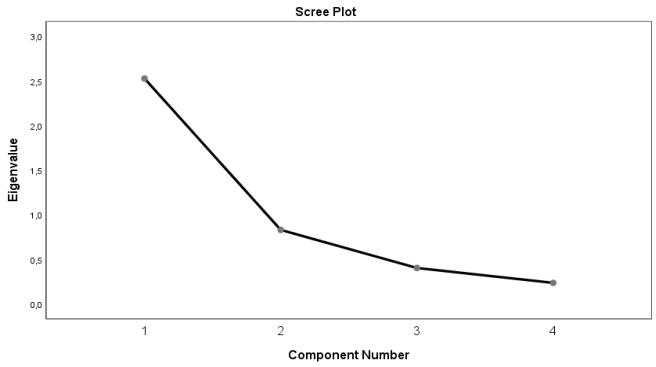


Figure 2. Scree Plot of the Neutrosophic Scale



Descriptive statistics for 135 participants, calculated for both scales used to assess physical environmental quality, are provided in Table 8.

Table 8. Descriptive Statistics

	Likert Scale	Neutrosophic Scale
Mean	15.267	2.321
Median	16.000	2.450
Mode	16.000	4.000
Std. deviation	3.702	1.247
Variance	13.704	1.555

Skewness	−0.935	-0.417
Std. error of skewness	0.209	0.209
Kurtosis	0.596	−0.896
Std. error of kurtosis	0.414	0.414
Range	16.000	4.400
Minimum	4.000	−0.400
Maximum	20.000	4.000

Regarding descriptive statistics, the Likert scale with values ranging from 1 to 5 had a mean of 15.267, while the neutrosophic scale ranging from -1 to 1 resulted in a mean of 2.321. Furthermore, the variance for the Likert scale is 13.704, while it is 1.555 for the neutrosophic scale, which shows that the neutrosophic scale has a lower arithmetic mean and standard deviation. Since the same sample size was used in both scales, the standard errors of skewness and kurtosis are the same. Skewness and kurtosis are used to describe the shape of a distribution.

The findings in Table 9 show that the Likert scale and the neutrosophic scale differ from the normal distribution at the 0.05 statistical significance level. However, especially in social sciences, skewness and kurtosis values can be an important tool for making statistical inferences about data distributions. For most psychometric purposes, skewness and kurtosis values of ±1.0 are considered perfect [36]. Therefore, given that the skewness and kurtosis values of both scales were ±1.0, parametric analyses were performed assuming that they conformed to a normal distribution.

Table 9. Normal Distribution Testing Using the Kolmogorov–Smirnov Test

	Kolmogorov–Smirnov		
	Statistic	**Df**	**Sig.**
Likert scale	0.136	135	0.000
Neutrosophic scale	0.093	135	0.006

Cronbach's alpha coefficient was calculated to assess the reliability of the scales [36]. According to the results in Table 10, it is seen that the scores obtained from the neutrosophic scale are similarly reliable to the Likert scale.

Table 10. Cronbach's Alpha Coefficient

Cronbach's Alpha Constant	**Variables**
0.798	SCORE1, SCORE2, SCORE3, SCORE4
0.785	VAR1a, VAR2a, VAR3a, VAR4a
0.700	VAR1b, VAR2b, VAR3b, VAR4b
0.804	VAR1c, VAR2c, VAR3c, VAR4c
0.795	VAR1, VAR2, VAR3, VAR4

Table 11. Independent Variable Importance

	Importance	**Normalized Importance**
Score1	0.227	65.8%
Score2	0.345	100.0%
Score3	0.215	62.4%
Score4	0.213	61.7%

According to the artificial neural network analysis findings of the two scales, Score4 had the least effect on the Likert scale with 61.7%, while Score2 had the greatest effect with 100.0%.

Table 12. Correlation among Classical Items, Neutrosophic Items, and Scores

		VAR1a	**VAR1b**	**VAR1c**	**Score1**
VAR1	Correlation Coefficient	0.779 **	-0.216 *	−0.660 **	0.753 **
	P	0.000	0.012	0.000	0.000
	N	135	135	135	135
		VAR2a	**VAR2b**	**VAR2c**	**Score2**
VAR2	Correlation Coefficient	0.803 **	-0.064	−0.742 **	0.749 **
	P	0.000	0.458	0.000	0.000
	N	135	135	135	135
		VAR3a	**VAR3b**	**VAR3c**	**Score3**
VAR3	Correlation Coefficient	0.652 **	-0.371 **	−0.543 **	0.625 **
	P	0.000	0.000	0.000	0.000
	N	135	135	135	135
		VAR4a	**VAR4b**	**VAR4c**	**Score4**
VAR4	Correlation Coefficient	0.670 **	-0.423 **	−0.436 **	0.609 **
	P	0.000	0.000	0.000	0.000
	N	135	135	135	135

** Correlation is significant at the 0.01 level. * Correlation is significant at the 0.05 level.

When Spearman's rho correlation coefficient is analyzed in Table 12, a significant positive correlation was found between the Likert scale and the neutrosophic scale for the "agree" option and scores. On the other hand, a significant negative correlation was found between Likert scale and neutrosophic scale for "undecided" and "disagree" options. In general, it is observed that there is a significant relationship between the items of the two scales.

According to the results in Table 13, there is a statistically significant and strong positive relationship between Likert and neutrosophic scales.

Table 13. Correlation between Neutrosophic Scale and Likert Scale

		Likert Scale
Neutrosophic scale	Pearson Correlation (r)	0.668 **
	P	0.000
	N	135

** Correlation is significant at the 0.01 level.

Table 14. Paired Samples T-Test Results for the Likert and Neutrosophic Scales

		Mean	**SD**	**t**	**Df**	***p***	**d**
Pair 1	Likert scale-Neutrosophic scale	12.946	3.016	49.879	134	0.000	4.292

d = Effect size.

According to the results of Table 14, paired samples t-test were significant, t(134) = 49.879 showing that there was a significant difference in the Likert scale, Mean = 15.267, SD = 3.702, n = 135, compared to the neutrosophic scale, Mean = 2.321, SD = 1.247, n = 135. As per Cohen's guidelines, the effect size was identified as considerably large [39]. The mean difference was 12.946, with the 95% confidence interval for the difference between the means ranging from 12.433 to 13.459. Consequently, the null hypothesis that there is no significant difference between the two scales was rejected.

In Table 15, the mean and standard deviation are calculated to describe the central tendency and spread of the two scales, respectively (Table 7, [39]). These values provide valuable information about data distribution and variability, allowing researchers to make informed decisions and draw meaningful conclusions.

Table 15. Mean and Standard Deviation of the Population's Attitudes toward Physical Environmental Quality through both the Likert Scale and Neutrosophic Scale

Item	Likert Scale		Neutrosophic Scale	
	μ	σ	μ	σ
The restaurant has an attractive interior design and decor.	3.548	1.268	0.488	0.418
The background music is pleasant.	3.207	1.436	0.423	0.444
The dining areas are very clean.	4.178	0.929	0.681	0.364
Staff are clean and well dressed.	4.333	1.000	0.729	0.347

μ = Arithmetic mean, σ = Standard deviation.

Table 15 shows that the arithmetic mean and standard deviation of the Neutrosophic scale are lower than the Likert scale. A low standard deviation indicates that the data points tend to be closer to the mean, while a high standard deviation indicates that the data points are spread over a wider range. Therefore, it can be said that the neutrosophic scale has a distribution closer to the mean, while the Likert scale has a wider distribution.

Table 16. One-Way ANOVA Results for the Likert and Neutrosophic Scales

Scale	Effect	Mean Square	F	Significance Level
Likert scale	Age	54.319	4.361	0.002
Neutrosophic scale	Age	4.765	3.272	0.014

A one-way ANOVA with five age groups was conducted to explore the singular effect. The effect of age groups on the Likert scale and neutrosophic scale was statistically significant (respectively, $F = 4.361$, $p < 0.05$; $F=3.272$, $p < 0.05$). In two scales, the difference between the averages in the 45-54 age group and the averages in the over 55 age group was found to be statistically significant ($p < 0.05$). The Likert scale score of clients in the 45-54 age group (Mean = 17.471) is higher than that of clients in the over 55 (Mean = 12.632). Similarly, clients aged between 45 and 54 have a higher neutrosophic scale score (Mean = 2.943) compared to clients aged over 55 (Mean = 1.689).

Table 17. Two-Way ANOVA Results for the Likert and Neutrosophic Scales

Scale	Single and Interaction Effect	Mean Square	F	Significance Level
Likert scale	Gender	10.180	0.810	0.370
	Age	54.870	4.366	0.002
	Gender X Age	6.472	0.515	0.725
Neutrosophic scale	Gender	2.159	1.471	0.227
	Age	4.810	3.277	0.014
	Gender X Age	0.743	0.506	0.731

Two-way ANOVA, two (gender) X five (age), were conducted to explore the individual effects as well as the interaction effects between gender and age. The individual effect of gender ($F = 0.810$, $p > 0.05$) and the interaction effect of gender and age ($F = 0.515$, $p > 0.05$) on the Likert scale were statistically insignificant. However, the single effect of age on the Likert scale was significant ($F =$

4.366, $p < 0.05$). A statistically significant difference was found between the averages of clients in the 45-54 age group and those in the over 55 age group ($p < 0.05$). While clients in the 45-54 age group exhibited the highest Likert scale scores, clients in the over 55 age group demonstrated the lowest Likert scale scores.

Similarly, the single effects of gender ($F = 1.471$, $p > 0.05$) and the interaction between gender and age ($F = 0.506$, $p > 0.05$) on the Neutrosophic scale were not statistically significant. However, the individual effect of the age group on the Neutrosophic scale was statistically significant ($F = 3.277$, $p < 0.05$). Furthermore, there was a significantly difference in the average scores between clients aged 45-54 and those over 55 ($p < 0.05$). The neutrosophic scale score of the clients in 45-54 age group (Mean = 2.943) was higher than the client in 55 over age group (Mean = 1.689). According to the results of the research, studies examining the possible role of demographic differences in physical environmental quality are very rare. Restaurant responses to the quality of the physical environment and employee reactions may differ according to demographic characteristics.

According to the results of both one-way and two-way ANOVA analyses conducted in this study, it was found that 45-54 and 55+ age groups pay attention to the quality of the physical environment (interior design-decor of the restaurant, music, cleanliness of dining areas and employees) when choosing a restaurant. It has also been observed in the literature that mature customers care more about the quality of the physical environment than younger customers [11]. In another study, it is possible to say that the perception of physical environmental elements increases with the increase in the age of consumers. In addition, it was determined that there was no significant difference between the gender status of the participants and their perception of physical environmental elements [46].

6. Conclusions

This research has conducted astute analyses by employing Exploratory Factor Analysis, correlation analysis, and ANOVA to assess the efficacy of Likert and neutrosophic scales in gauging restaurant patrons' perceptions of the quality of the physical environment. The results of the exploratory factor analysis suggest that a solitary component is sufficient to encompass the responses on both measures, thereby streamlining the process of assessing perceived environmental quality. Moreover, a robust positive correlation is observed between the Likert and neutrosophic scales, which substantiates the reliability and consistency of the data collected per these methodologies. The demographic differences in perceptions were emphasized by the ANOVA results, which revealed that individuals aged 45-54 and 55 above exhibit a notably greater level of concern regarding the physical environment's quality in restaurants. This discovery implies the existence of a demographic niche that restaurant proprietors and managers may be able to target in order to increase client loyalty and satisfaction. On the contrary, the absence of gender disparities in the responses suggests that the significance of environmental quality is consistent for both male and female consumers, thereby bolstering the case for a comprehensive strategy towards environmental enhancements.

The restaurant industry can benefit from the practical implications of these findings. A strategic priority could be the enhancement of the physical environment's quality in order to attract and retain a more mature clientele, who are known to place a high value on this aspect. Potentially, implementing modifications that appeal to this age group will result in increased customer satisfaction and enhanced business performance. In its entirety, this study emphasizes the significance of the physical setting in the context of dining and establishes a rigorous methodological structure for evaluating it through the utilization of Likert and neutrosophic scales.

In a prospective future investigation, our intention is to employ machine learning models for the purpose of forecasting demographic variables, including age, education, marital status, and sex, based on the responses obtained from the classical Likert and Neutrosophic Likert scales. Another potential avenue for future research involves utilizing the recently created RANCOM approach, which specifically targets the assessment of Neutrosophic Likert scale data by experts in various fields [47]. A potential future investigation may involve conducting a comprehensive analysis of the data using various scoring functions and comparing the results with the score function employed in this article.

References

[1] Canny, I. U. (2013). The Role of Food Quality, Service Quality, and Physical Environment on Customer Satisfaction and Future Behavıoral Intentions in Casual Dining Restaurant, The 7th National Research Management Conference, Palembang.

[2] Nguyen, N. and Leblanc, G. (2002). Contact Personnel, Physical Environment and the Perceived Corporate Image of Intangible Services by New Clients, *International Journal of Service Industry Management*, 13(3), 242-262.

[3] Lee, S. and Jeong, M. (2012). Effects of E-Servicescape on Consumers' Flow Experiences. *Journal of Hospitality and Tourism Technology*, 3(1), 47-59.

[4] Han, H. and Ryu, K. (2009). The Roles of the Physical Environment, Price Perception, and Customer Satisfaction in Determining Customer Loyalty in the Family Restaurant Industry. *Journal of Hospitality and Tourism Research*, 33(4), 487-510.

[5] Kaya, S. (2023). The Effect of Gastronomic Experience on Restaurant Image, Customer Perceived Value, Customer Satisfaction and Customer Loyalty, Eskisehir Osmangazi University Institute of Social Sciences, PhD Thesis, Eskisehir.

[6] Ali, F. and Amin, M. (2014). The Influence of Physical Environment on Emotions, Customer Satisfaction and Behavioural Intentions in Chinese Resort Hotel Industry. *Journal Global Business Advancement*, 7(3), 249-266.

[7] Harris, L. and Ezeh, C. (2008). Servicescape and Loyalty Intentions: An Empirical Investigation. *European Journal of Marketing*, 42(3), 390-422.

[8] Hu, H., Kandampully, J. and Juwaheer, T. D. (2009). Relationships and Impacts of Service Quality, Perceived Value, Customer Satisfaction, and Image: An Empirical Study. *The Service Industries Journal*, 29(2), 111-125.

[9] Chen, P. T. and Hu, H. H. (2010). How Determinant Attributes of Service Quality Influence Customer-Perceived Value: An Empirical Investigation of the Australian Coffee Outlet Industry. *International Journal of Contemporary Hospitality Management*, 22(4), 535-551.

[10] Clemes, M., Gan, C. and Ren, M. (2011). Synthesizing the Effects of Service Quality, Value, and Customer Satisfaction on Behavioral Intentions in the Motel Industry: An Empirical Analysis. *Journal of Hospitality and Tourism Research*, 35(4), 530-568.

[11] Ryu, K., Lee, H. and Kim, W. (2012). The Influence of the Quality of the Physical Environment, Food, and Service on Restaurant Image, Customer Perceived Value, Customer Satisfaction, and Behavioral Intentions. *International Journal of Contemporary Hospitality Management,* 24(2), 200-223.

[12] Amin, M., Yahya, Z., Ismayatim, W. F. A., Nasharuddin, S. Z. and Kassim, E. (2013). Service Quality Dimension and Customer Satisfaction: An Empirical Study in the Malaysian Hotel Industry. *Services Marketing Quarterly,* 34(2), 115-125.

[13] Namasivayam, K. and Mattila, A.S. (2007) Creating Visible Customer Value. *Cornell Hotel Restaurant Administration Quarterly,* 31, 3-18.

[14] Kotler, P. (1973). Atmospherics as a Marketing Tool, *Journal of Retailing,* 49(4), 48-64.

[15] Ali, F., Omar, R. and Amin, M. (2013). An Examination of the Relationships between Physical Environment, Perceived Value, Image and Behavioural Intentions: A SEM Approach towards Malaysian Resort Hotels, *Journal of Hotel and Tourism Management*, 27(2), 9-26.

[16] Ali, F., Amin, M. and Ryu, K. (2015). The Role of Physical Environment, Price Perceptions, and Consumption Emotions in Developing Customer Satisfaction in Chinese Resort Hotels, *Journal of Quality Assurance in Hospitality & Tourism,* 1-28.

[17] Han, H. and Hyun, S. S. (2017). Impact of Hotel-Restaurant Image and Quality of Physical-Environment, Service, and Food on Satisfaction and Intention, *International Journal of Hospitality Management,* 63, 82-92.

[18] Hendriyani, I. G. A. D. (2018). The Importance of Physical Environment for Guest Satisfaction on Restaurant in Bali, *Advances in Economics, Business and Management Research,* 52, 148-158.

[19] Özdemir-Güzel, S. and Baş, Y. N. (2020). Understanding the Relationship Between Physical Environment, Price Perception, Customer Satisfaction and Loyalty in Restaurants, *Journal of Tourism and Gastronomy Studies,* 8(2), 762-776.

[20] Siswanto, B., Rahmawati, and Kuleh, J. (2022). The Influence of Food Quality and Physical Environment on Behavior Intention Through Customer Satisfaction at Visitors To Mcdonald's Store In Samarinda, *International Journal of Business and Management Invention (IJBMI),* 11(12), 79-99.

[21] Maurer, T. J. and Pierce, H. R. (1998). A Comparison of Likert Scale and Traditional Measures of Self efficacy. *Journal of Applied Psychology,* 83(2), 324-329.

[22] Gay, L. R., Mills, G. E. and Airasian, P. (2009). Educational Research: Competencies for Analysis and Applications. Columbus, OH: Merrill.

[23] Likert, R. (1932). A Technique for the Measurement of Attitudes. *Archives of Psychology,* 22(140), 1-55.

[24] Jamieson, S. (2004). Likert Scales: How to (ab) Use Them. *Medical Education,* 38(12), 1217-1218.

[25] Singh Y. K. (2006). Fundamental of Research Methodology and Statistics. New Delhi: Newage International (P) Ltd. Publisher.

[26] Royeen, C. B. (1985). Adaptation of Likert Scaling for Use with Children. *The Occupational Therapy Journal of Research,* 5(1), 59-69.

[27] Subedi, B. S. (2016). Using Likert Type Data in Social Science Research: Confusion, Issues and Challenges. *International Journal of Contemporary Applied Sciences,* 3(2), 36-49.

[28] Van Laerhoven, H., Van der Zaag-Loonen, H. and Derks, B. H. F. (2004). A Comparison of Likert Scale and Visual Analogue Scales as Response Options in Children's Questionnaires. *Acta Paediatrica,* 93(6), 830-835.

[29] Cohen, L., Manion, L. and Morrison, K. (2000). Research Methods in Education (5th ed.). London: Routledge Falmer.

[30] Edmondson, D. R. (2005). Likert Scales: A History, Charm, 127-133.

[31] Smarandache, F. A. (1999). Unifying Field in Logics: Neutrosophic Logic. In: Philosophy, *American Research Press*, 1-141.

[32] Elhassouny, A., Idbrahim, S. and Smarandache, F. (2019). Machine Learning in Neutrosophic Environment: A survey. *Infinite Study*.

[33] Jha, S., Kumar, R., Son, L. H., Chatterjee, J. M., Khari, M. and Yadav, N. and Smarandache, F. (2019). Neutrosophic Soft Set Decision Making for Stock Trending Analysis. *Evolving Systems*, 10, 621–627.

[34] Liu, P. and Liu, X. (2018). The Neutrosophic Number Generalized Weighted Power Averaging Operator and its Application in Multiple Attribute Group Decision Making. *International Journal of Machine Learning and Cybernetics*, 9, 347-358.

[35] Abdel-Basset, M., Saleh, M., Gamal, A. and Smarandache, F. (2019). An Approach of Topsis Technique for Developing Supplier Selection with Group Decision Making under Type-2 Neutrosophic Number. *Applied Soft Computing*, 77, 438–452.

[36] Bhattacharya, S. (2005). Neutrosophic Information Fusion Applied to the Options Market. *Investment Management and Financial Innovations*, (2, Iss. 1), 139–145.

[37] Chang, K. H. A (2022). Novel Risk Ranking Method Based on the Single Valued Neutrosophic Set. *Infinite Study*.

[38] Wang, H., Smarandache, F., Zhang, Y. and Sunderraman, R. (2012). Single Valued Neutrosophic Sets. *Review of the Air Force Academy*.

[39] Zadeh, L. A. (1965). Fuzzy Sets. *Information and Control*, 8(3), 338-353.

[40] Atanassov, K. (1983). Intuitionistic fuzzy sets. In Proceedings of VII ITKR's Session, Sofia,

[41] Smarandache, F. (2019). Introduction to Neutrosophic Sociology (Neutrosociology). *Infinite Study (2019), ISBN: 978-1-59973-605-1*.

[42] Martínez, C. R., Hidalgo, G. A., Matos, M. A. and Smarandache, F. (2020). Neutrosophy for Survey Analysis in Social Sciences. *Neutrosophic Sets and Systems*, 37, 1.

[43] Jarr´ın, A.A.A.; Tamayo, D.S.P.; Giler, S.A.M.; Zambrano, J.C.A.; Fernandez, D.M.M. (2021). Neutrosophic statistics applied in social science. *Neutrosophic Sets and Systems*, 44, 01–09.

[44] Şahin, R. (2011). Multi-Criteria Neutrosophic Decision Making Method Based on Score and Accuracy Functions under Neutrosophic Environment. *Applied Mathematics Information Sciences*, 5.

[45] Duran, V., Topal, S., and Smarandache, F. (2021). An Application of Neutrosophic Logic in the Confirmatory Data Analysis of the Satisfaction with Life Scale. *Journal of Fuzzy Extension and Applications*, 239(3), 262–282.

[46] Ayazlar, R. A. and Gün, G. (2017). Perception of Physical Environment Elements in the Restaurant, *Journal of Tourism and Gastronomy Studies,* 5/Special Issue 2, 138-150.

[47] Więckowski, J., Kizielewicz, B., Shekhovtsov, A. and Sałabun, W. (2023). RANCOM: A novel approach to identifying criteria relevance based on inaccuracy expert judgments. Engineering Applications of Artificial Intelligence, ScienceDirect. Vol 122, 106114.

Received: Aug 25, 2024. Accepted: Nov 26, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

A Refined Dual-Valued Neutrosophic COPRAS-Based Approach for Holistic Evaluation of English Teaching Standards

Peng Tong[1]*, Bin Xie[2]

[1]Nanchang Institute of Science and Technology, Nanchang, 330108, Jiangxi, China

[2]Jiujiang University, Jiujiang, 332005, Jiangxi, China

*Corresponding author, E-mail: 13767061850@163.com; guohechu@ncpu.edu.cn

Abstract

The evaluation of college English teaching (CET) quality helps to improve teaching standards, ensure the achievement of teaching objectives, and enhance students' language skills and overall competence. Through feedback from evaluations, teachers can refine their teaching methods, optimize course design, and meet students' needs. Additionally, teaching quality evaluation promotes the professional development of teachers, facilitates the rational allocation of educational resources, and ensures effective teaching outcomes. The evaluation of CET quality in vocational colleges in the new era involves the multiple-attribute decision-making (MADM) process. Currently, COPRAS methods are employed to address Multi-Attribute Decision-Making (MADM) challenges. To handle uncertain data in this evaluation, double-valued neutrosophic sets (DVNSs) have been introduced. This study proposes the double-valued neutrosophic number combined COPRAS (DVNN-COPRAS) technique, utilizing the DVNN, to address the MADM problem under DVNSs. A numerical study focused on CET quality evaluation is conducted to validate the proposed method.

Keywords: Multiple-attribute decision-making (MADM); DVNSs; COPRAS technique; CET quality evaluation

1. Introduction

The evaluation of university English teaching quality holds significant importance, directly influencing teaching effectiveness and the development of students' language skills. First, teaching quality evaluation helps educational administrators and teachers understand the actual outcomes of

classroom instruction, identifying strengths and weaknesses in teaching, thus enabling targeted improvements. Through a well-structured evaluation system, teachers can better assess students' proficiency in various areas such as language knowledge, listening, speaking, reading, and writing, allowing them to adjust teaching methods and content to enhance classroom efficiency. Second, teaching quality evaluation ensures that educational objectives are achieved. The goal of university English courses is not only to improve students' language abilities but also to cultivate their cross-cultural communication skills, critical thinking, and independent learning capabilities. Through quality evaluation, the degree to which these goals are met can be measured, ensuring that course design and teaching practices align with the expected outcomes. Finally, teaching quality evaluation promotes professional development and innovation among teachers. Continuous feedback and reflection enable teachers to optimize teaching strategies, adopt new educational concepts and technologies, and improve their teaching skills. Additionally, evaluation results provide data support for schools to formulate teaching reform policies, contributing to systematic improvements in English teaching. Therefore, the evaluation of university English teaching quality is essential not only for student development but also for the enhancement of the entire educational system.

In 1965, Zadeh [1] introduced the pioneering theory of fuzzy sets (FSs) to handle various types of uncertainty. The concept of "neutrosophy" refers to the study of neutral thought, which distinguishes it from fuzzy and intuitionistic fuzzy logic and sets. Neutrosophic logic, introduced by Florentin [2], evaluates each proposition using three components: a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F). In 2005, Wang, Smarandache, Zhang, and Sunderraman [3] introduced the concept of Single-Valued Neutrosophic Sets (SVNSs). In this framework, the truth, indeterminacy, and falsity degrees of each element are confined to the standard unit interval [0, 1]. SVNSs generalize several key concepts, including classical sets, fuzzy sets, intuitionistic fuzzy sets, and paraconsistent sets, expanding the range of tools available for addressing complex real-world problems[4-9]. The purpose of MADM is to assist decision-makers in selecting the optimal choice when faced with multiple conflicting attributes or criteria [9-12]. In real-world problems, decisions often involve various factors such as quality, cost, and efficiency, which may have conflicting or difficult-to-quantify relationships. MADM provides a systematic approach to comprehensively consider these factors, helping decision-makers balance pros and cons

and choose the option that best aligns with overall objectives [2, 13, 14]. The significance of MADM lies in its broad applicability. It can be applied in various fields, such as engineering management, economics, environmental protection, healthcare, and supply chain management. MADM methods not only provide a systematic and transparent decision-making process but also enhance the scientific and rational nature of decisions [15-17]. By assigning weights to different attributes and ranking alternatives, MADM effectively avoids personal bias, ensuring fairness in the decision-making process. Moreover, MADM can handle complex and uncertain information environments[18-20]. Particularly when dealing with vague, incomplete, or uncertain data, MADM methods introduce tools such as fuzzy sets, grey system theory, or neutrosophic sets to offer more flexible solutions[14, 21]. Therefore, MADM plays a crucial role in improving decision quality, reducing decision risk, and optimizing resource allocation. The evaluation of CET quality falls within the realm of MADM. Recently, researchers have leveraged the TODIM and GRA [22] techniques to tackle MADM challenges in this area. Additionally, DVNSs [23] have been introduced to handle uncertain data during the evaluation process.

In this study, we utilize the DVNN-COPRAS technique to address MADM problems involving DVNSs. To validate this approach, we conduct a numerical study focused on CET quality evaluation. The primary objectives and motivations of this study are: (1) Using mean to determine weight values under DVNSs; (2) Implementing the DVNN-COPRAS technique to effectively manage MADM; (3) Conducting a numerical example to demonstrate the effectiveness of the DVNN-COPRAS technique in evaluating CET quality.

The structure of this study is organized as follows: Section 2 introduces the DVNSs. Section 3 details the application of the DVNN-COPRAS technique under DVNSs. Section 4 presents a numerical example related to CET quality evaluation, along with a comparative analysis. Finally, Section 5 offers several remarks and conclusions.

2. Literature review

The evaluation of university English teaching quality is a comprehensive analysis of teaching effectiveness and student learning outcomes. Its core lies in assessing the teaching methods of instructors, the design of course content, and the improvement of students' language abilities. The evaluation process typically employs various methods such as student surveys, classroom observations, and analysis of teaching outcomes. Student feedback is a crucial reference factor, as it directly reflects the actual effectiveness of the teaching. Additionally, the evaluation of teaching

quality also focuses on the professional development of teachers and the utilization of teaching resources. Through a scientific evaluation system, universities can identify strengths and weaknesses in teaching, thereby optimizing course design, enhancing teaching quality, and ultimately promoting the overall development of students and the cultivation of an international perspective. Since 2014, research on the quality of college English teaching has deepened and diversified. He and Wu [24] were among the first to emphasize the necessity of reforming college English teaching, highlighting that reforms could enhance teaching efficiency and effectiveness while cultivating high-quality talents. Following this, drawing from the input-output hypothesis, Song and Xia [25] explored how optimizing input and strengthening output could improve vocabulary teaching, addressing students' challenges in memorizing vocabulary. Lin [26] proposed improving teaching quality and efficiency through reforms in educational philosophy, teaching methods, and quality management. By 2016, Zhao [27] developed a comprehensive evaluation system for college English teachers' classroom performance, integrating student, peer, and self-assessments to systematically enhance teaching quality. Gao [28] in the same year, studied participatory teaching models and constructed a quality evaluation system using the analytic hierarchy process to refine the assessment criteria for college English teaching. In 2017, the focus on reforms in college English teaching intensified. Li [29] emphasized the importance of building effective teaching models and cultivating students' self-learning abilities. Meanwhile, Ma [30] advocated for innovative teaching approaches to meet societal demands for English talents. Xu [31] comprehensively analyzed the effectiveness of ongoing teaching reforms and pointed out the pivotal role of these measures in improving teaching quality. By 2018, researchers turned their attention to the specific factors influencing the quality of college English teaching. Peng [32] analyzed multiple factors affecting teaching quality and proposed strategies for improvement. Focusing on student-centered learning, Guo [33] suggested enhancing teaching quality through feedback mechanisms. In 2019, drawing on the Outcome-Based Education (OBE) framework, Geng [34] proposed a comprehensive evaluation mechanism integrating formative and summative assessments. That same year, Zheng [35] conducted empirical research demonstrating that a chunk-based situational teaching model significantly improved spoken English proficiency. With the rise of online teaching, Gao, Yan and Kang [36] introduced a quality evaluation system tailored for SPOC-based online English teaching, advancing online teaching quality frameworks. In 2021, Lv [37] analyzed the issues in teaching English to international students at Huaqiao University and proposed strategies to improve teaching quality in this context. In 2022, Xu [38] constructed an internal quality assurance system for college English courses, emphasizing the pivotal role of teaching management. In recent years, formative assessment and blended learning have become research hotspots. He [39] examined the impact of formative assessment on the quality of blended teaching in college English, suggesting it facilitates positive teaching development. In the context of engineering education accreditation, Li [40] developed a quality monitoring and assurance system for college English teaching, stressing its importance in enhancing the English proficiency of engineering students.

2. Preliminaries

Kandasamy [23] launched the DVNSs.

Definition 1 [23]. The DVNSs are launched:

$$CA=\left\{\left(\theta,CT_A(\theta),CIT_A(\theta),CIF_A(\theta),CF_A(\theta)\right)\middle|\theta\in\Theta\right\}. \quad (1)$$

With truth-membership $CT_A(\theta)$, $CIT_A(\theta)$ stands for indeterminacy leaning for $CT_A(\theta)$, $LIF_A(\theta)$ stands for indeterminacy leaning for $LT_A(\theta)$ and falsity-membership $CF_A(\theta)$, $CT_A(\theta),CIT_A(\theta),CIF_A(\theta),CF_A(\theta)\in[0,1]$, $0\le CT_A(\theta)+CIT_A(\theta)+CIF_A(\theta)+CF_A(\theta)\le 4$.

The DVNN is launched as $CA=\left(CT_A,CIT_A,CIF_A,CF_A\right)$, where $CT_A,CIT_A,CIF_A,CF_A\in[0,1]$, $0\le CT_A+CIT_A+CIF_A+CF_A\le 4$.

Definition 2[23]. Let $CA=\left(CT_A,CIT_A,CIF_A,CF_A\right)$ be DVNN, score value (SV) is launched:

$$SV(CA)=\frac{\left(2+CT_A+CIT_A-CIF_A-CF_A\right)}{4},\ SV(CA)\in[0,1]. \quad (2)$$

Definition 3[23]. Let $CA=\left(CT_A,CIT_A,CIF_A,CF_A\right)$ be DVNN, accuracy value (AV) is launched:

$$AV(CA)=\frac{\left(CT_A+CIT_A+CIF_A+CF_A\right)}{4},\ AV(CA)\in[0,1]\ . \quad (3)$$

The order is launched for DVNNs.

Definition 4[23]. Let $CA=\left(CT_A,CIT_A,CIF_A,CF_A\right)$ and $CB=\left(CT_B,CIT_B,CIF_B,CF_B\right)$, $SV(CA)=\frac{\left(2+CT_A+CIT_A-CIF_A-CF_A\right)}{4}$, $SV(CB)=\frac{\left(2+CT_B+CIT_B-CIF_B-CF_B\right)}{4}$, $AV(CA)=\frac{\left(CT_A+CIT_A+CIF_A+CF_A\right)}{4}$, $AV(CB)=\frac{\left(CT_B+CIT_B+CIF_B+CF_B\right)}{4}$, if $SV(CA)<SV(CB)$, $CA<CB$; if $SV(CA)=SV(CB)$, (1)if $AV(CA)=AV(CB)$, $CA=CB$; (2) if $AV(CA)<AV(CB)$, $CA<CB$.

Definition 5[23]. Let $CA=\left(CT_A,CIT_A,CIF_A,CF_A\right)$ and $CB=\left(CT_B,CIT_B,CIF_B,CF_B\right)$ be DVNNs, the operations are launched:

(1) $CA \oplus CB = \left(CT_A + CT_B - CT_A CT_B, CIT_A + CIT_B - CIT_A CIT_B, CIF_A CIF_B, CF_A CF_B\right)$;

(2) $CA \otimes CB = \left(CT_A CT_B, CIT_A CIT_B, CIF_A + CIF_B - CIF_A CIF_B, CF_A + CF_B - CF_A CF_B\right)$;

(3) $\lambda CA = \left(1-\left(1-CT_A\right)^{\lambda}, 1-\left(1-CIT_A\right)^{\lambda}, \left(CIF_A\right)^{\lambda}, \left(CF_A\right)^{\lambda}\right), \lambda > 0$;

(4) $\left(CA\right)^{\lambda} = \left(\left(CT_A\right)^{\lambda}, \left(CIT_A\right)^{\lambda}, 1-\left(1-CIF_A\right)^{\lambda}, 1-\left(1-CF_A\right)^{\lambda}\right), \lambda > 0.$

Definition 6[23]. Let $CA = \left(CT_A, CIT_A, CIF_A, CF_A\right)$ and $CB = \left(CT_B, CIT_B, CIF_B, CF_B\right)$, the DVNN Hamming distance (DVNNHD) and DVNN Euclidean distance (DVNNED) between $CA = \left(CT_A, CIT_A, CIF_A, CF_A\right)$ and $CB = \left(CT_B, CIT_B, CIF_B, CF_B\right)$ is launched:

$$\begin{aligned} &DVNNHD\left(CA, CB\right) \\ &= \frac{1}{4}\left(\left|CT_A - CT_B\right| + \left|CIT_A - CIT_B\right| + \left|CIF_A - CIF_B\right| + \left|CF_A - CF_B\right|\right) \end{aligned} \tag{4-a}$$

$$\begin{aligned} &DVNNED\left(CA, CB\right) \\ &= \sqrt{\frac{1}{4}\left(\left|CT_A - CT_B\right|^2 + \left|CIT_A - CIT_B\right|^2 + \left|CIF_A - CIF_B\right|^2 + \left|CF_A - CF_B\right|^2\right)} \end{aligned} \tag{4-b}$$

3. The DVNN-COPRAS technique for MADM

This section shows the steps of the proposed method under the DVNNs to show the strength of the proposed method. The COPRAS method is used to rank the alternatives in MADM problems. The steps of the COPRAS method under DVNNs are shown as follows:

1. Experts evaluate the criteria and alternatives.

Three experts evaluated the criteria and alternatives using the DVNNs.

2. Obtain crisp values.

We used the score function to obtain crisp values.

3. Combine crisp values into one matrix.

4. Obtain the criteria weights.

The criteria weights are determined through the one matrix with crips values.

5. Normalize the crisp values of DVNN.

$$N_{ij} = \frac{t_{ij}}{\sum_{i=1}^{m} t_{ij}} \tag{5}$$

Where $i = 1,2,\ldots,m; j = 1,2,\ldots,n$

6. Determine the values of the weighted normalized decision matrix

$$K_{ij} = w_j * N_{ij} \tag{6}$$

7. Determine the maximizing and minimizing indexes for positive (g) and negative criteria (n-g)

$$S_{+i} = \sum_{j=1}^{g} K_{ij} \tag{7}$$

$$S_{-i} = \sum_{j=g+1}^{n} K_{ij} \tag{8}$$

8. Compute the relative significant values.

$$A_i = S_{+i} + \frac{\sum_{i=1}^{m} S_{-i}}{S_{-i} \sum_{i=1}^{m} \frac{1}{S_{-i}}} \tag{9}$$

9. Rank the alternatives. Figure 1. The criteria weights.

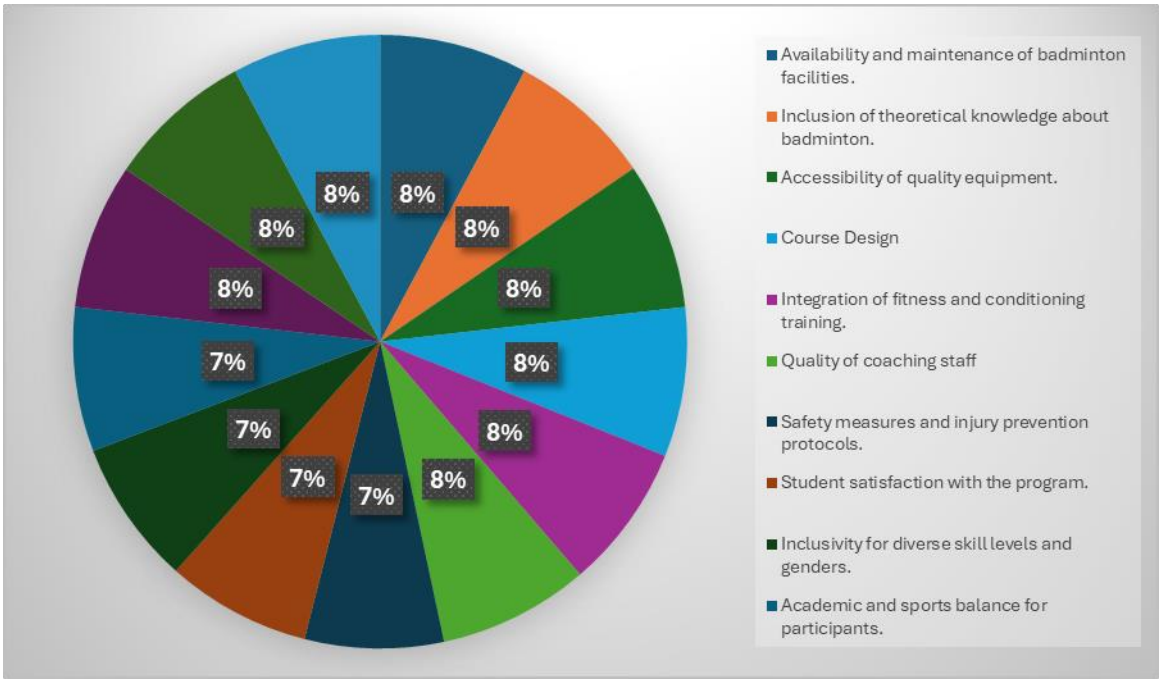


Figure 1. The criteria weights.

4. Numerical example and comparative analysis

The evaluation of college English teaching quality is a complex and multidimensional process aimed at comprehensively assessing teaching effectiveness and student learning outcomes. First, the core of the evaluation lies in course design, including the clarity of teaching objectives, the relevance of course content, and whether the selected textbooks meet students' actual needs. Secondly, the teaching methods and abilities of instructors are also crucial evaluation indicators. Excellent teachers not only need solid professional knowledge but also must employ diverse teaching techniques to stimulate students' interest and initiative in learning. Furthermore, student feedback is an essential component of evaluating teaching quality. Collecting students' opinions through surveys and discussion sessions can reveal the actual effectiveness of the teaching and identify existing issues. Students' academic performance and improvements in language skills are also key metrics; through examination results, language tests, and assessments of practical application abilities, the effectiveness of teaching can be objectively reflected. Teaching facilities and environment

significantly impact teaching quality. Modern language laboratories, multimedia equipment, and a conducive learning atmosphere can effectively enhance teaching outcomes. The support and resource investment from the institution is equally important, including support for teacher training and development, as well as encouragement for teaching innovation. Lastly, the evaluation process should be ongoing and dynamic. Regular assessments of teaching quality can help identify problems and facilitate improvements. Through continuous feedback and adjustments, the quality of college English teaching can steadily improve, providing students with a higher quality education. The CET quality evaluation is MADM. Ten potential local applied undergraduate colleges are evaluated from 13 attributes.

Table 1. The DVNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
A_1	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)
A_2	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)
A_3	(0.83, 0.32, 0.21, 0.19)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)
A_4	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)
A_5	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)
A_6	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)
A_7	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.79, 0.43, 0.31, 0.91)	(0.52, 0.53, 0.31, 0.06)	(0.61, 0.43, 0.31, 0.17)	(0.52, 0.53, 0.31, 0.06)
A_8	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)
A_9	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)
A_{10}	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
A_1	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.61, 0.43, 0.31, 0.17)
A_2	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)
A_3	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)
A_4	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)

A_5	(0.52, 0.53, 0.31, 0.06)	(0.61, 0.43, 0.31, 0.17)	(0.52, 0.53, 0.31, 0.06)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)
A_6	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)
A_7	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.83, 0.32, 0.21, 0.19)	(0.2, 0.54, 0.42, 0.55)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)
A_8	(0.61, 0.43, 0.31, 0.17)	(0.52, 0.53, 0.31, 0.06)	(0.61, 0.43, 0.31, 0.17)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)
A_9	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)
A_{10}	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
A_1	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.83, 0.32, 0.21, 0.19)
A_2	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)	(0.52, 0.53, 0.31, 0.06)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)
A_3	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.83, 0.32, 0.21, 0.19)	(0.83, 0.32, 0.21, 0.19)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)
A_4	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)
A_5	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)
A_6	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)	(0.83, 0.32, 0.21, 0.19)	(0.2, 0.54, 0.42, 0.55)
A_7	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.52, 0.53, 0.31, 0.06)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)	(0.79, 0.43, 0.31, 0.91)	(0.52, 0.53, 0.31, 0.06)
A_8	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.83, 0.32, 0.21, 0.19)
A_9	(0.79, 0.43, 0.31, 0.91)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.52, 0.53, 0.31, 0.06)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)
A_{10}	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)	(0.79, 0.43, 0.31, 0.91)	(0.83, 0.32, 0.21, 0.19)	(0.52, 0.53, 0.31, 0.06)	(0.2, 0.54, 0.42, 0.55)	(0.2, 0.54, 0.42, 0.55)	(0.61, 0.43, 0.31, 0.17)	(0.61, 0.43, 0.31, 0.17)

1. Three DVNN matrices are built based on the opinions of three experts as shown in Table 1.

2. We obtain crisp values.

3. We combine crisp values into one matrix.

4. We obtain the criteria weights as shown in Figure 1.

5. Then we used Eq. (5) to normalize the crisp values of DVNN as shown in Table 2.

6. Then we used Eq. (6) to determine the values of the weighted normalized decision matrix

7. Then we used Eqs. (7 and 8) to determine the maximizing and minimizing.

8. Then we used Eq. (9) to compute the relative significant values.

9. Then we ranked the alternatives as shown in Figure 2.

Table 2. The normalized of crisp values of DVNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
A_1	0.087986	0.115379	0.092288	0.089486	0.109792	0.116295	0.091968	0.106633	0.093833	0.078192	0.091811	0.117065	0.109281
A_2	0.094158	0.08122	0.114385	0.106701	0.096159	0.108261	0.097946	0.089719	0.08326	0.119423	0.109473	0.100175	0.093956
A_3	0.10765	0.118164	0.097487	0.109262	0.091762	0.074852	0.122471	0.11031	0.107342	0.098513	0.099548	0.103669	0.101977
A_4	0.104349	0.101745	0.080012	0.090909	0.100997	0.086411	0.103464	0.105898	0.084728	0.106759	0.105824	0.097408	0.086365
A_5	0.092579	0.08943	0.114385	0.111965	0.089856	0.110939	0.099019	0.094867	0.102937	0.101311	0.087578	0.082557	0.089086
A_6	0.106789	0.104384	0.099365	0.09788	0.117121	0.081336	0.106376	0.115752	0.1	0.101458	0.091811	0.104834	0.090089
A_7	0.091718	0.077848	0.106297	0.103429	0.087951	0.097124	0.107449	0.083395	0.11909	0.098365	0.10743	0.100175	0.115153
A_8	0.110234	0.101745	0.110919	0.095035	0.103782	0.105582	0.081392	0.093984	0.112775	0.099396	0.117647	0.111823	0.118161
A_9	0.08612	0.092215	0.108174	0.114668	0.099824	0.110939	0.097946	0.0781	0.077974	0.118392	0.111371	0.089255	0.085935
A_{10}	0.118415	0.117871	0.07669	0.080666	0.102756	0.108261	0.091968	0.121341	0.118062	0.078192	0.077507	0.09304	0.109997

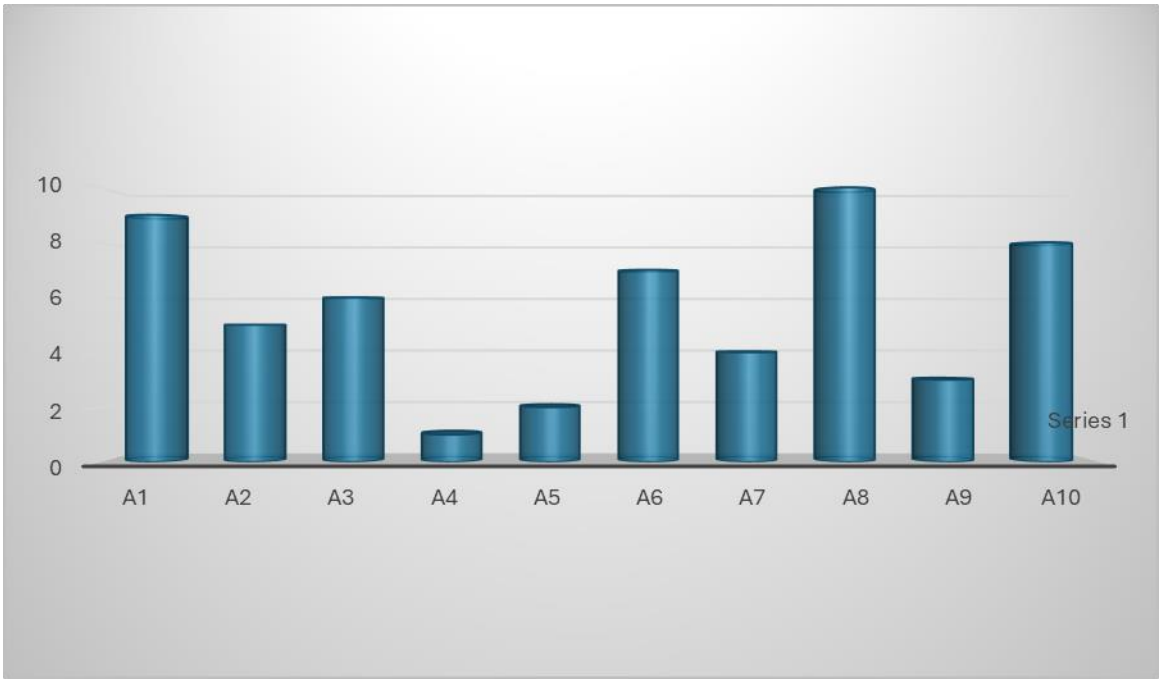


Figure 2. The rank of alternatives.

4.2 Sensitivity Analysis

This part changes the criteria weights by different cases to show the rank of alternatives under different weights. Figure 3 shows the different criteria weights. In the first case, we put all criteria with the same weights. Then we ranked the alternatives as shown in Figure 4. The results show our model is stable under different weights.

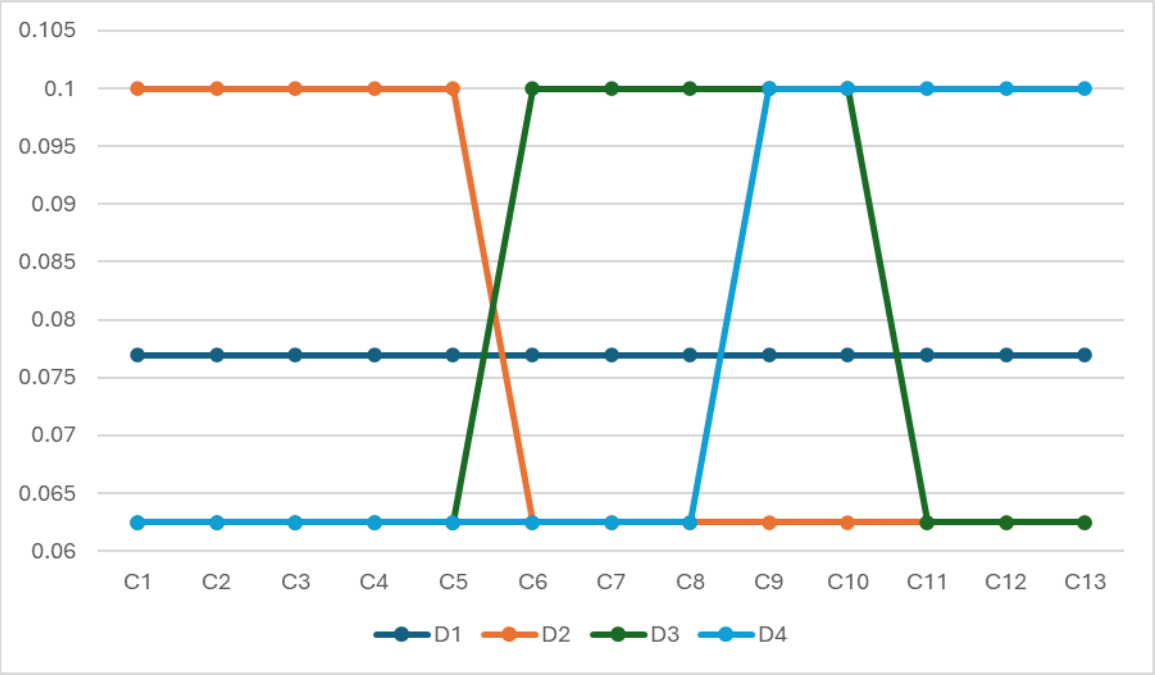


Figure 3. The different criteria weights.

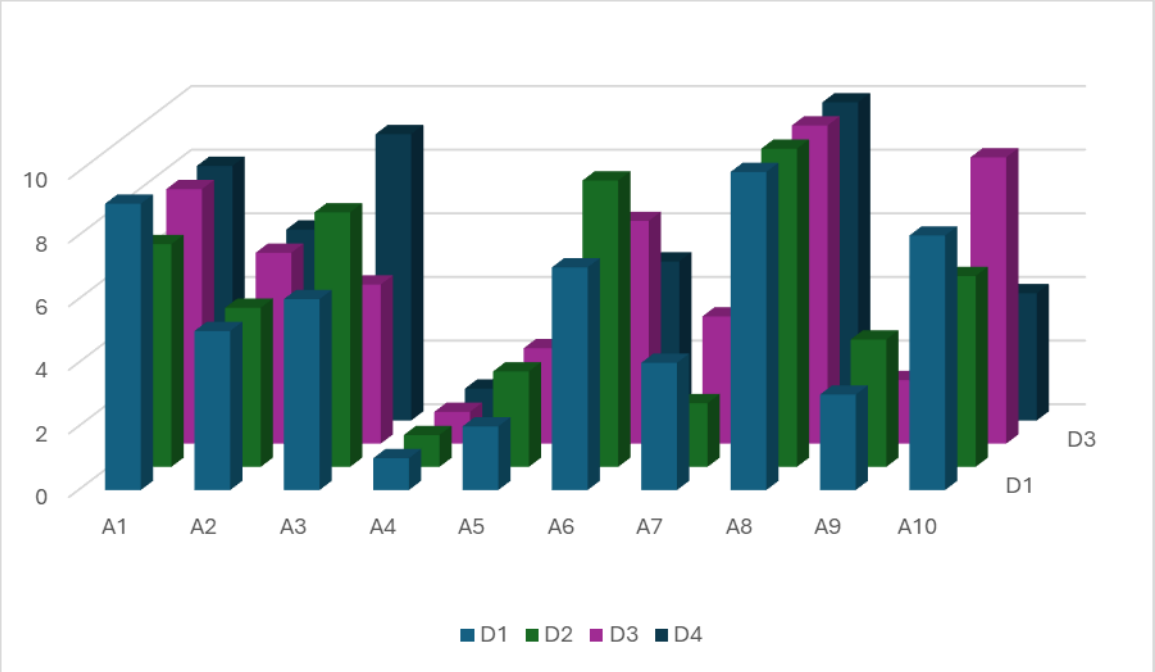


Figure 4. The rank of alternatives under different weights.

4.3. Comparative analysis

We compare our model with different MADM methods such as Taxonomy, CoCoSo, and VIKOR. We used the criteria weights from the proposed model. All methods are compared under the DVNN. Figure 5 shows the comparative ranks. We show the proposed model is strong compared to other

methods.

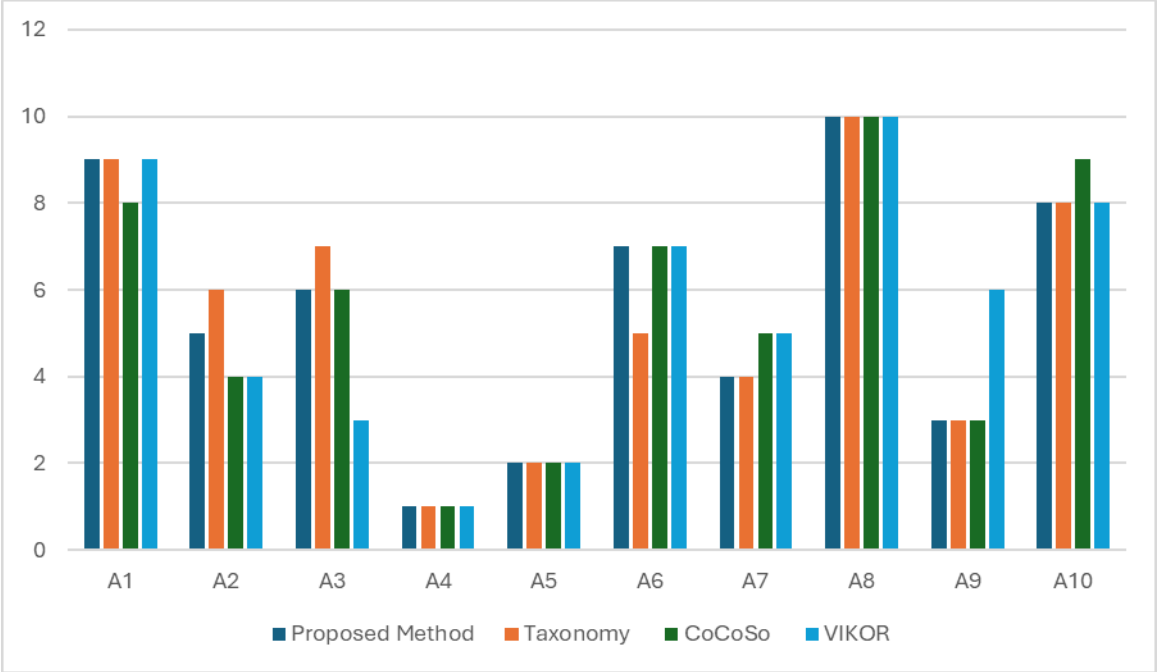


Figure 5. The comparative ranks.

5. Discussion analysis

When comparing the proposed DVNN-COPRAS technique with existing methods such as the DVNN-Taxonomy technique, DVNN-CoCoSo technique, and DVNN-VIKOR technique, the proposed method demonstrates significant advantages, though it also has some limitations. By analyzing these methods, we can better understand their disadvantages and the strengths and weaknesses of the proposed method. Some disadvantages of existing methods are outlined: The main drawback of the DVNN-Taxonomy Technique lies in its heavy reliance on subjective judgment when categorizing evaluation criteria. Although DVNN-Taxonomy can handle fuzzy and uncertain information, in complex multi-attribute decision-making problems, the process of defining classification standards may lead to inconsistent results. This is especially true when dealing with multi-dimensional, complex data, where the interrelationships between attributes may not be fully captured. While the DVNN-CoCoSo technique can address multi-attribute decision-making problems, its computational process is relatively complex and heavily dependent on the assignment of weights to evaluation criteria. This means that if the weight distribution is not reasonable, it could bias the final evaluation results. Additionally, the method's ability to handle uncertain data is limited, making it less effective in dealing with various types of uncertain information.

When comparing the proposed DVNN-COPRAS technique with existing methods three advantages of the proposed method are outlined: (1) More Effective Handling of Uncertain Data:

The DVNN-COPRAS technique, by incorporating Double-Valued Neutrosophic Numbers (DVNNs), is more effective in dealing with uncertain data. In multi-attribute decision-making processes, the introduction of DVNNs allows for more precise and robust results when handling fuzzy and uncertain information, thereby reducing information loss during the decision-making process. (2) Dual Distance Measures Enhance Decision Accuracy: The proposed method employs both Hamming distance and Euclidean distance under DVNN, which enables a more comprehensive assessment of the differences between attributes. This dual distance measure approach increases the flexibility and adaptability of the method, allowing it to better reflect the relationships between various attributes in different decision-making scenarios, ultimately improving decision accuracy. Relatively Higher Computational Efficiency: Compared to methods, the DVNN-COPRAS technique strikes a balance between computational complexity and efficiency. In multi-dimensional evaluation contexts, this method converges to reasonable results more quickly, making it particularly efficient when dealing with large datasets.

6. Conclusion

College English teaching quality evaluation is a crucial method for assessing the effectiveness of English instruction in higher education, aiming to comprehensively understand and enhance teaching quality. The evaluation typically includes aspects such as teacher proficiency, curriculum design, teaching resources, and student learning outcomes. Assessing teaching methods, classroom management, and the scientific and practical aspects of teaching content, helps teachers continually improve their strategies. In terms of curriculum design, the evaluation should focus on the systematic and cutting-edge nature of courses to ensure students acquire comprehensive and up-to-date knowledge. The richness and accessibility of teaching resources, such as textbooks, courseware, and online materials, are also key indicators. Evaluating student learning outcomes involves examining exam scores, language application skills, learning motivation, and satisfaction. By integrating these factors, college English teaching quality evaluation not only identifies issues in teaching but also provides a basis for educational reform, thereby promoting continuous improvement in teaching quality and cultivating well-rounded talents with international competitiveness. The evaluation of CET quality involves MADM. Currently, the COPRAS technique is employed to address Multi-Attribute Decision Making (MADM) problems. To handle uncertainty in this evaluation, the Dempster-Shafer Theory (DVNSs) is used as a characterization method. This study developed the DVNN-COPRAS technique, to tackle MADM challenges under DVNSs. To demonstrate the effectiveness of the proposed approach, a numerical example focusing on the evaluation of CET quality is provided. The key contributions of this study are constructed: (1) Utilizing the mean method to determine weight values under DVNSs; (2) Applying the DVNN-COPRAS technique to efficiently address MADM challenges and (3) Validating the DVNN-COPRAS method through a numerical example related to CET quality assessment.

7. Research limitations and future research directions:

Although this paper proposes an innovative method combining Double-Valued Neutrosophic Numbers with COPRAS (DVNN-COPRAS) for evaluating College English Teaching (CET) quality, its effectiveness is validated through a numerical study, several limitations exist:

(1) Method complexity: While the introduction of Double-Valued Neutrosophic Sets (DVNSs) and multiple distance measures (Hamming distance and Euclidean distance) allows for handling uncertain data, it increases the computational complexity. For practical applications, especially for decision-makers in the educational field, overly complex mathematical models might limit the method's widespread adoption.

(2)Limited empirical validation: The paper validates the proposed method through a numerical study, but the scope of empirical validation is limited. Only through real-world educational scenarios and large-scale sample data can the applicability and generalizability of the method be further confirmed.

(3) Dependence on subjective judgments: Although the double-valued neutrosophic set can deal with uncertainty, the evaluation process still depends on subjective judgment, particularly in the assignment of weights to evaluation criteria. Different evaluators may assign varying weights to the same criteria, which could lead to inconsistent results.

To address the current study’s limitations, future research can focus on the following three directions:

(1) Simplifying the model to enhance practicality: Future studies could explore ways to simplify the current DVNN-COPRAS method, or incorporate more intuitive, user-friendly tools to reduce the computational complexity and increase its applicability in real-world teaching evaluations. For instance, machine learning or other data-driven algorithms could be integrated to automatically optimize the evaluation process and minimize human intervention.

(2) Large-Scale Empirical Validation: Further research could conduct large-scale, diverse empirical studies to validate the proposed method in different educational contexts, including various types of institutions, courses, and student groups. By collecting and analyzing extensive data, researchers can ensure the method's reliability and generalizability across different settings. Additionally, real-world feedback could be used to fine-tune the model’s parameters.

(3) Incorporating Objective Data to Reduce Subjectivity: Future research could consider integrating more objective data, such as learning analytics, student performance metrics, and classroom interaction data, alongside the subjective evaluation system. By combining objective data with subjective assessments, the accuracy and consistency of the evaluation results could be further improved, minimizing biases from subjective judgment.

By addressing these research limitations and exploring the aforementioned directions, the proposed CET quality evaluation method could be more widely applied and further optimized, providing more effective and practical tools for multi-attribute decision-making in education.

References

[1] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[2] F.A. Smarandache, Unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[3] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single valued neutrosophic sets, Infinite Study, 12 (2010) 20110.

[4] S. Sudha, M. Lathamaheswari, S. Broumi, F. Smarandache, Neutrosophic fuzzy magic labeling graph with its application in academic performance of the students, Neutrosophic Sets and Systems, 60 (2023) 88-112.

[5] M. Arshad, M. Saeed, A. Ur Rahman, Interval complex single-valued neutrosophic hypersoft set with application in decision making, Neutrosophic Sets and Systems, 60 (2023) 396-419.

[6] R. Imran, K. Ullah, Z. Ali, M. Akram, An approach to multi-attribute decision-making based on single-valued neutrosophic hesitant fuzzy aczel-alsina aggregation operator, Neutrosophic Systems with Applications, 22 (2024) 43-57.

[7] M. Kamran, N. Salamat, M.S. Hameed, S.K. Khan, S. Broumi, Sine trigonometric aggregation operators with single-valued neutrosophic z-numbers: Application in business site selection, Neutrosophic Sets and Systems, 63 (2024) 285-321.

[8] B. Shil, R. Das, C. Granados, S. Das, B.C. Tripathy, Single-valued quadripartitioned neutrosophic d-ideal of d-algebra, Neutrosophic Sets and Systems, 67 (2024) 105-114.

[9] M. Luo, Z. Sun, L. Wu, Fuzzy inference quintuple implication method based on single valued neutrosophic t-representable t-norm, Neutrosophic Optimization and Intelligent Systems, 3 (2024) 8-22.

[10] M. Karak, A. Mahata, M. Rong, S. Mukherjee, S.P. Mondal, S. Broumi, B. Roy, A solution technique of transportation problem in neutrosophic environment, Neutrosophic Systems with Applications, 3 (2023) 17-34.

[11] S.F. Al-baker, I. El-henawy, M. Mohamed, Pairing new approach of tree soft with mcdm techniques: Toward advisory an outstanding web service provider based on qos levels, Neutrosophic Systems with Applications, 14 (2024) 17-29.

[12] M. Mohamed, A.A. Metwaly, M. Ibrahim, F. Smarandache, M. Voskoglou, Partnership of lean six sigma and digital twin under type 2 neutrosophic mystery toward virtual manufacturing environment: Real scenario application, Sustainable Machine Intelligence Journal, 8 (2024) (6):99-109.

[13] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single-valued neutrosophic sets, Multispace and Multistructure, 4 (2010) 410-413.

[14] M. Abouhawwash, M. Jameel, Evaluation factors of solar power plants to reduce cost under neutrosophic multi-criteria decision making model, Sustainable Machine Intelligence Journal, 2 (2023) 1-11.

[15] Z. Wang, Q. Cai, G. Wei, Enhanced todim based on vikor method for multi-attribute decision making with type-2 neutrosophic number and applications to green supplier selection, Soft Computing, (2023) https://doi.org/10.1007/s00500-00023-08768-00508.

[16] B.Q. Ning, H.J. Wang, G.W. Wei, C. Wei, Several similarity measures of probabilistic dual hesitant fuzzy sets and their applications to new energy vehicle charging station location, Alexandria Engineering Journal, 71 (2023) 371-385.

[17] N.N. Liao, Q. Cai, H. Garg, G.W. Wei, X.R. Xu, Novel gained and lost dominance score method based on cumulative prospect theory for group decision-making problems in probabilistic hesitant fuzzy environment, International Journal of Fuzzy Systems, (2023) https://doi.org/10.1007/s40815-40022-01440-40817.

[18] I. El-Henawy, S. El-Amir, M. Mohamed, F. Smarandache, Modeling influenced criteria in classifiers' imbalanced challenges based on trss bolstered by the vague nature of neutrosophic theory, Neutrosophic Sets and Systems, 65 (2024) 183-198.

[19] F. Smarandache, Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general, Neutrosophic Sets and Systems, 58 (2023) 639-643.

[20] T. Garai, S. Dalapati, F. Smarandache, Softmax function based neutrosophic aggregation operators and application in multi-attribute decision making problem, Neutrosophic Sets and Systems, 56 (2023) 213-244.

[21] S.I. Abdel Aal, Neutrosophic framework for assessment challenges in smart sustainable cities based on iot to better manage energy resources and decrease the urban environment's ecological impact, Neutrosophic Systems with Applications, 6 (2023) 9-16.

[22] J. Deng, Introduction to grey system theory, The Journal of Grey System, 1 (1989) 1-24.

[23] I. Kandasamy, Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm, Journal of Intelligent Systems, 27 (2018) 163-182.

[24] X. He, J. Wu, Attach great importance to college english teaching reform and strive to improve teaching quality, Campus English, (2014) 17-17.

[25] X. Song, J. Xia, Optimize input, strengthen output, and improve the quality of college english vocabulary teaching, Journal of Hunan University of Science and Technology, 36 (2015) 170-171.

[26] L. Lin, Exploring effective ways to improve the quality and efficiency of college english teaching, Economic Research Guide, (2015) 171-172.

[27] W. Zhao, The construction of a multiple evaluation system for college english teachers' classroom teaching quality, Journal of Anshan Normal University, 18 (2016) 51-54.

[28] Q. Gao, Research on the evaluation of teaching quality in participatory college english teaching models, Journal of Language and Literature (Foreign Language Education and Teaching), (2016) 167-168.

[29] Q. Li, Constructing an effective college english teaching model to improve teaching quality, Campus English, (2017) 11-11.

[30] S. Ma, Emphasizing innovation in college english teaching to improve teaching quality, Journal of Heihe University, 8 (2017) 146-147.

[31] W. Xu, The impact of college english teaching reform on improving the quality of college english teaching, Overseas English, (2017) 101-102.

[32] Y. Peng, Analysis of the factors affecting the quality of college english teaching, Campus English, (2018) 60-60.

[33] Y. Guo, Discussion on how to improve the quality of college english teaching, Curriculum Education Research, (2018) 108-109.

[34] W. Geng, Research on the construction of college english teaching quality evaluation system based on obe (outcome-based education), Campus English, (2019) 13-15.

[35] X. Zheng, An empirical study on the effectiveness of the chunk-based situational teaching model in improving college english oral teaching quality, Journal of Weifang University, 19 (2019) 91-94.

[36] X. Gao, L. Yan, N. Kang, Analysis of evaluation indicators for college english online teaching

quality based on spoc, Journal of Jilin Radio and TV University, (2020) 22-23.

[37] B. Lv, Practical research on improving the quality of college english teaching for international students in china: A case study of huaqiao university, Journal of Hubei University of Economics (Humanities and Social Sciences), 18 (2021) 146-149.

[38] J. Xu, The construction of the internal quality assurance system for college english teaching, Journal of Hubei Open Vocational College, 35 (2022) 160-161.

[39] F. He, The impact of formative assessment on the quality of online and offline college english teaching, Industry and Technology Forum, 22 (2023) 219-220.

[40] D. Li, Research and practice on the construction of college english teaching quality monitoring and assurance system under the background of engineering education accreditation, Campus English, (2024) 85-87.

Received: Aug 21, 2024. Accepted: Nov 28, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

A Python Framework Enhancement for Neutrosophic Topologies

Giorgio Nordo[1,*], Saeid Jafari[2] and Maikel Yelandi Leyva Vázquez[3]

[1]MIFT Department – Mathematical and Computer Science, Physical Sciences and Earth Sciences University of Messina, 98166 Sant'Agata, Messina, Italy; giorgio.nordo@unime.it

[2]Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark; jafaripersia@gmail.com

[3] Salesian Polytechnic University of Guayaquil, Guyas, Ecuador; mleyvaz@gmail.com

[*]Correspondence: giorgio.nordo@unime.it

Abstract. This paper introduces an extension to the Python Neutrosophic Sets (PYNS) framework, originally detailed in [13], with the addition of the `NSfamily` class for constructing and manipulating neutrosophic topologies. Building on existing classes like `NSuniverse` and `NSset`, the `NSfamily` class enables the definition and testing of neutrosophic families as basis and sub-basis for neutrosophic topological spaces. This extension provides tools for verifying closure properties under union and intersection, and for determining whether a given family constitutes a neutrosophic topology. Through implemented algorithms, the framework automates the generation of topologies from families of neutrosophic sets, offering an efficient tool for advancing research in neutrosophic topology. Practical applications are demonstrated with detailed examples, showcasing how this class enhances the scope and flexibility of neutrosophic modeling within the PYNS framework.

Keywords: Neutrosophic topology, Neutrosophic set, Python framework, Neutrosophic family, Neutrosophic topological basis, Neutrosophic topological sub-basis.

2020 Mathematics Subject Classification: 03E72, 54A40, 54D99, 68W99.

1. Introduction

The theory of neutrosophic sets, introduced by Smarandache in 1999 [25], generalizes both fuzzy sets [29] and Atanassov's intuitionistic fuzzy sets [1] by providing three distinct parameters—truth, indeterminacy, and falsity—for each element. This generalization offers a more flexible and nuanced model for representing uncertainty and incomplete information, making it applicable to a wide range of fields, including statistics [26] and image processing [30]. In addition, single-valued neutrosophic sets have been explored to represent degrees of truth, indeterminacy, and falsity within a simplified framework [28], with applications in graph theory and decision making [2, 12].

In recent years, neutrosophic topology has emerged as an extension of classical topological concepts to the neutrosophic domain. Key developments include the exploration of connectedness and stratification in neutrosophic topological spaces by Broumi et al. [3] and the investigation of separation axioms by Dey and Ray [4]. Additionally, foundational studies by Gallego Lupiáñez [6, 7, 16] have laid the groundwork for applying neutrosophic theory to topology, creating opportunities to model complex relationships characterized by uncertainty.

The practical application of neutrosophic sets has been greatly enhanced by the development of computational tools. El-Ghareeb introduced an open-source Python package to handle neutrosophic data [5], and Sleem et al. developed the PyIVNS tool for interval-valued neutrosophic operations [24]. In addition, Topal et al. created a Python tool for implementing operations on bipolar neutrosophic matrices, useful in applications requiring complex matrix computations [27]. Furthermore, Saranya et al. developed a C# application specifically designed to handle neutrosophic $g\alpha$-closed sets, expanding the accessibility of neutrosophic topology across different programming environments [23]. However, these tools are limited in scope and focus primarily on specific neutrosophic operations rather than providing a comprehensive and interactive framework for dealing with symbolic representations of neutrosophic sets.

The Python Neutrosophic Sets (PYNS) framework, developed by Nordo et al. [13], addresses this limitation by offering a robust, modular library for the representation and manipulation of neutrosophic sets and mappings. PYNS enables the modeling of various neutrosophic structures, such as single-valued neutrosophic mappings [9] and neutrosophic soft topological spaces [10, 11]. However, while PYNS supports essential topological operations, such as basis and sub-basis construction, it lacks a generalized framework for systematically defining and verifying neutrosophic topologies.

In this paper, we introduce the `NSfamily` class as an extension of the PYNS framework, aimed at facilitating the construction, manipulation, and verification of neutrosophic topologies. This new class draws upon theoretical work by Ozturk [14] and Salama et al. [18,20–22] on neutrosophic sets and topological structures, offering a more integrated approach for defining families of neutrosophic sets as basis and sub-basis. The `NSfamily` class supports automated generation of neutrosophic topologies, including operations to verify closure under union and intersection, providing an efficient and scalable solution for neutrosophic topological modeling.

Furthermore, `NSfamily` addresses a broad range of computational requirements in neutrosophic topology, including the ability to handle interval-valued data and complex neutrosophic structures. This flexibility makes it suitable for both theoretical research and practical applications, as demonstrated in Rabuni and Balaman's approach to neutrosophic soft topologies using Python [15], and in applications of single-valued neutrosophic ideals by Saber et al. [17].

Our contribution extends the PYNS framework with a cohesive and efficient tool for neutrosophic topological modeling, addressing the computational needs of researchers and expanding the possibilities of neutrosophic analysis in complex domains.

The structure of the paper is as follows. In Section 2, we review fundamental definitions and properties in neutrosophic topology, including closure and continuity. Section 3 details the `NSfamily` class, highlighting the core methods and their applications. Finally, we provide practical examples that illustrate the enhanced capabilities of PYNS with the `NSfamily` extension, discussing the potential impact of this work on future research in neutrosophic topology.

2. **Preliminaries**

This section provides a comprehensive overview of neutrosophic sets, neutrosophic topology, and the structure of the Python Neutrosophic Sets (PYNS) framework. This background is necessary to understand the design and functionality of the new `NSfamily` class, which extends PYNS for advanced neutrosophic topological modeling.

2.1. *Neutrosophic Sets*

The concept of neutrosophic sets, introduced by Smarandache [25], generalizes classical, fuzzy [29], and intuitionistic fuzzy sets [1] by assigning each element three independent degrees: truth, indeterminacy, and non-membership, each within the hyperreal interval $]0^-, 1^+[$ of the nonstandard real numbers. A simpler variant, the single-valued neutrosophic set [28], uses the standard interval $[0, 1]$, making it more accessible and practical for scientific and engineering applications.

Definition 2.1. [28] *Let $\mathbb{U}$ be a universal set and $A \subseteq \mathbb{U}$. A single-valued neutrosophic set (abbreviated SVN-set) over $\mathbb{U}$, denoted by $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$, is defined as:*

$$\widetilde{A} = \{(u, \mu_A(u), \sigma_A(u), \omega_A(u)) : u \in \mathbb{U}\}$$

where $\mu_A : \mathbb{U} \to I$, $\sigma_A : \mathbb{U} \to I$, and $\omega_A : \mathbb{U} \to I$ are the membership, indeterminacy, and non-membership functions of A, respectively, and $I = [0, 1]$ denotes the real unit interval. For every $u \in \mathbb{U}$, $\mu_A(u)$, $\sigma_A(u)$, and $\omega_A(u)$ are called the degree of membership, degree of indeterminacy, and degree of non-membership of u, respectively.

Definition 2.2. [25, 28] *Let $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$ be two SVN-sets over the universe set $\mathbb{U}$, we say that:*

- *$\widetilde{A}$ is a neutrosophic subset (or simply a subset) of $\widetilde{B}$ and we write $\widetilde{A} \Subset \widetilde{B}$ if, for every $u \in \mathbb{U}$, it results $\mu_A(u) \leq \mu_B(u)$, $\sigma_A(u) \leq \sigma_B(u)$ and $\omega_A(u) \geq \omega_B(u)$. We also say*

that $\widetilde{A}$ is contained in $\widetilde{B}$ or that $\widetilde{B}$ contains $\widetilde{A}$ and we write $\widetilde{B} \Supseteq \widetilde{A}$ to denote that $\widetilde{B}$ is a neutrosophic superset of $\widetilde{A}$.

- *$\widetilde{A}$ is a neutrosophically equal (or simply equal) to $\widetilde{B}$ and we write $\widetilde{A} \Subseteq \widetilde{B}$ if $\widetilde{A} \Subseteq \widetilde{B}$ and $\widetilde{B} \Subseteq \widetilde{A}$.*

Notation 2.3. *Let $\mathbb{U}$ be a set, $I = [0,1]$ the unit interval of the real numbers, for every $r \in I$, with $\underline{r}$ we denote the constant mapping $\underline{r} : \mathbb{U} \to I$ defined by $\underline{r}(u) = r$, for every $u \in \mathbb{U}$.*

Definition 2.4. [28] *Given a universe set $\mathbb{U}$:*

- *the SVN-set $\langle \mathbb{U}, \underline{0}, \underline{0}, \underline{1} \rangle$ is said to be the neutrosophic empty set over $\mathbb{U}$ and it is denoted by $\widetilde{\emptyset}$, or more precisely by $\widetilde{\emptyset}_{\mathbb{U}}$ in case it is necessary to specify the corresponding universe set*
- *the SVN-set $\langle \mathbb{U}, \underline{1}, \underline{1}, \underline{0} \rangle$ is said to be the neutrosophic absolute set over $\mathbb{U}$ and it is denoted by $\widetilde{\mathbb{U}}$.*

On neutrosophic sets, union, intersection and complement operations can be defined.

Definition 2.5. [19] *Let $\left\{ \widetilde{A}_\alpha \right\}_{\alpha \in \Lambda}$ be a family of SVN-sets $\widetilde{A}_\alpha = \langle \mathbb{U}, \mu_{A_\alpha}, \sigma_{A_\alpha}, \omega_{A_\alpha} \rangle$ over a common universe set $\mathbb{U}$, then:*

- *the neutrosophic union, denoted by $\bigcup\limits_{\alpha \in \Lambda} \widetilde{A}_\alpha$, is the neutrosophic set $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ with $\mu_A = \bigvee\limits_{\alpha \in \Lambda} \mu_{A_\alpha}$, $\sigma_A = \bigvee\limits_{\alpha \in \Lambda} \sigma_{A_\alpha}$, and $\omega_A = \bigwedge\limits_{\alpha \in \Lambda} \omega_{A_\alpha}$. In particular, the neutrosophic union of two SVN-sets $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$, denoted by $\widetilde{A} \Cup \widetilde{B}$, is the neutrosophic set defined by $\langle \mathbb{U}, \mu_A \vee \mu_B, \sigma_A \vee \sigma_B, \omega_A \wedge \omega_B \rangle$*
- *the neutrosophic intersection, denoted by $\bigcap\limits_{\alpha \in \Lambda} \widetilde{A}_\alpha$, is the neutrosophic set $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ with $\mu_A = \bigwedge\limits_{\alpha \in \Lambda} \mu_{A_\alpha}$, $\sigma_A = \bigwedge\limits_{\alpha \in \Lambda} \sigma_{A_\alpha}$, and $\omega_A = \bigvee\limits_{\alpha \in \Lambda} \omega_{A_\alpha}$. In particular, the neutrosophic intersection of two SVN-sets $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$, denoted by $\widetilde{A} \Cap \widetilde{B}$, is the neutrosophic set defined by $\langle \mathbb{U}, \mu_A \wedge \mu_B, \sigma_A \wedge \sigma_B, \omega_A \vee \omega_B \rangle$*

where $\wedge$ and $\vee$ denote the minimum and the maximum, respectively.

Definition 2.6. [25,28] *The neutrosophic complement of a SVN-set $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ over $\mathbb{U}$, denoted by $\widetilde{A}^{\complement}$, is given by: $\widetilde{A}^{\complement} = \langle \mathbb{U}, \omega_A, \underline{1} - \sigma_A, \mu_A \rangle$.*

2.2. *Neutrosophic Topology*

Neutrosophic topology generalizes classical topology by defining open and closed sets through neutrosophic set theory, enabling topologies where openness, closedness, and indeterminacy are graded. According to Salama et al [18], Serkan Karatas et al [8] and Gallego

Lupiáñez [6, 7], a neutrosophic topology is a family of neutrosophic sets which contains the empty neutrosophic set, the absolute neutrosophic set and is closed with respect to the neutrosophic union and the finite neutrosophic intersection.

Definition 2.7. *[6, 18] Let $\mathscr{T}$ be a family of neutrosophic sets over the same universe set $\mathbb{U}$, we say that $\mathscr{T}$ is a neutrosophic topology if the following four conditions hold:*

(1) $\widetilde{\emptyset} \in \mathscr{T}$

(2) $\widetilde{\mathbb{U}} \in \mathscr{T}$

(3) $\forall \mathscr{A} \subseteq \mathscr{T}$, $\bigcup \mathscr{A} \in \mathscr{T}$

(4) $\forall \widetilde{U}, \widetilde{V} \in \mathscr{T}$, $\widetilde{U} \Cap \widetilde{V} \in \mathscr{T}$

and when this occurs we also say that the pair $(\mathbb{U}, \mathscr{T})$ constitutes a neutrosophic topological space, while the elements of $\mathscr{T}$ are named neutrosophic open sets.

Although, the authors are not aware of papers in which notions such as bases and subbasis of neutrosophic topologies are introduced and treated, by simple analogy with classical topological spaces, the following general definitions and basic properties can be introduced.

Definition 2.8. *Given two nutrosophic topologies $\mathscr{T}_1$ and τ_2 over same universe set $\mathbb{U}$, we say that $\mathscr{T}_1$ is coarser (weaker or smaller) than $\mathscr{T}_2$, or equivalently, that $\mathscr{T}_2$ is finer (stronger or larger) than $\mathscr{T}_1$ if $\mathscr{T}_1 \subseteq \mathscr{T}_2$.*

So, the inclusion relation $\subseteq$ is a partial order on the set of all neutrosophic topologies over a universe set $\mathbb{U}$.

Proposition 2.9. *The intersection $\bigcap_{\alpha \in \Lambda} \mathscr{T}_\alpha$ of any family $\{\mathscr{T}_\alpha\}_{\alpha \in \Lambda}$ of neutrosophic topologies over a common universe set $\mathbb{U}$ is itself a neutrosophic topology on $\mathbb{U}$.*

Proposition 2.10. *Given a family $\mathscr{S}$ of neutrosophic subsets over a common universe set $\mathbb{U}$, there exists and is unique the smallest (minimal) neutrosophic topology $\mathscr{T}(\mathscr{S})$ on $\mathbb{U}$ containing $\mathscr{S}$.*

Definition 2.11. *Let $\mathscr{S}$ be a family of neutrosophic subsets over a common universe set $\mathbb{U}$, the neutrosophic topology $\mathscr{T}(\mathscr{S})$ of Proposition 2.10 is called the neutrosophic topology generated by the family $\mathscr{S}$.*

Proposition 2.12. *If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two families of neutrosophic sets on the same universe set $\mathbb{U}$ such that $\mathscr{S}_1 \subseteq \mathscr{S}_2$ then the corresponding generated neutrosophic topologies satisfy $\mathscr{T}(\mathscr{S}_1) \subseteq \mathscr{T}(\mathscr{S}_2)$.*

Definition 2.13. *Given a neutrosophic topological space $(\mathbb{U}, \mathscr{T})$, a family $\mathscr{B} \subseteq \mathscr{T}$ is said to be a neutrosophic basis for the neutrosophic topology $\mathscr{T}$ if every neutrosophic open set in $\mathscr{T}$*

can be expressed as a neutrosophic union of a subfamily of $\mathscr{B}$, i.e., if for each $\widetilde{U} \in \mathscr{T}$, there exists $\mathscr{A} \subseteq \mathscr{B}$ such that $\widetilde{U} = \bigcup \mathscr{A}$.

Notation 2.14. *Given a family $\mathscr{S}$ of neutrosophic sets over a common universe set $\mathbb{U}$, let us denote by $\mathscr{B}(\mathscr{S})$ (or, equivalently, by $\mathscr{S}^*$) the family of all its finite neutrosophic intersections, with the addition of the absolute neutrosophic set, i.e:*

$$\mathscr{B}(\mathscr{S}) = \left\{ \bigcap_{i=1}^{n} \widetilde{S}_i : n \in \mathbb{N}, \widetilde{S}_i \in \mathscr{S} \right\} \Cup \left\{ \widetilde{\mathbb{U}} \right\}.$$

Proposition 2.15. *If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two families of neutrosophic sets on the same universe set $\mathbb{U}$ such that $\mathscr{S}_1 \subseteq \mathscr{S}_2$ then the corresponding family of all their finite neutrosophic intersections satisfy $\mathscr{B}(\mathscr{S}_1) \subseteq \mathscr{B}(\mathscr{S}_2)$.*

Proposition 2.16. *The neutrosophic topology $\mathscr{T}(\mathscr{S})$ generated by a family of neutrosophic sets $\mathscr{S}$ over the same universe set $\mathbb{U}$ coincides with the topology $\mathscr{T}(\mathscr{B}(\mathscr{S}))$ generated by the family of all finite neutrosophic intersections $\mathscr{B}(\mathscr{S})$ of $\mathscr{S}$, i.e., we have:*

$$\mathscr{T}(\mathscr{S}) = \mathscr{T}(\mathscr{B}(\mathscr{S}))$$

and the family $\mathscr{B}(\mathscr{S})$ of all finite neutrosophic intersections forms a neutrosophic basis for $\mathscr{T}(\mathscr{S})$.

Definition 2.17. *In the situation described by Proposition 2.16, the $\mathscr{S}$ family is said to be a neutrosophic sub-basis (or a neutrosophic pre-base) of the $\mathscr{T}(\mathscr{S})$ topology.*

2.3. *The PYNS Framework*

The Python Neutrosophic Sets (PYNS) framework, developed by Nordo et al. [13], supports essential components for neutrosophic modeling. It is composed of three primary classes: `NSuniverse`, `NSset` and `NSmapping` designed to manage neutrosophic universes, individual neutrosophic sets and functions between neutrosophic sets, respectively.

The PYNS framework includes three primary classes that enable the representation and manipulation of neutrosophic structures in Python. These are:

- `NSuniverse` that defines a universe set over which neutrosophic sets are established. It supports initialization with various formats, such as lists, tuples, and strings. Key methods include:
  - `get()` - Returns the elements of the universe set
  - `cardinality()` - Returns the total count of elements
  - `isSubset(unv)` - Checks if the current universe set is a subset of another specified universe set.

- `NSset` which represents individual neutrosophic sets, where each element in the universe has associated degrees of membership, indeterminacy, and non-membership. Primary methods include:
  - `NSunion(nset)` - Computes the union of the current neutrosophic set with another specified set
  - `NSintersection(nset)` - Finds the intersection of the current set with another
  - `NScomplement()` - Generates the complement of the neutrosophic set, flipping the membership, indeterminacy, and non-membership values
  - `NSdifference(nset)` - Calculates the neutrosophic difference between the current set and another
  - `isNSdisjoint(nset)` - Verifies if the current set is disjoint from the given neutrosophic set.
- `NSmapping` that is designed to manage mappings between two neutrosophic universe sets, each instance of this class holds a dictionary to represent the mapping relations. It includes methods such as:
  - `getDomain()` - Retrieves the domain of the mapping as an `NSuniverse` object
  - `getCodomain()` - Returns the codomain as an `NSuniverse` object
  - `getMap()` - Provides access to the dictionary of element-value pairs defining the mapping
  - `setValue(u, v)` - Assigns a codomain element v to a domain element u
  - `getValue(u)` - Returns the mapping value of a given element u from the domain
  - `getFibre(v)` - Identifies all elements in the domain that map to a specified codomain value v, effectively finding the fibre of v
  - `NSimage(nset)` - Computes the neutrosophic image of a given neutrosophic set
  - `NScounterimage(nset)` - Determines the neutrosophic counterimage for the specified neutrosophic set.

In summary, the `PYNS` framework, as developed by Nordo et al. [13], provides a robust foundation for neutrosophic modeling in Python, structured around the core classes `NSuniverse`, `NSset`, and `NSmapping`. With approximately 1500 lines of code, `PYNS` supports intuitive operations on neutrosophic universes, individual sets, and mappings. Key functionalities include fundamental operations such as neutrosophic union, intersection, difference, and complement, as well as the computation of images and counterimages through mappings, all of which contribute to a flexible toolset for handling neutrosophic structures across various types of universes.

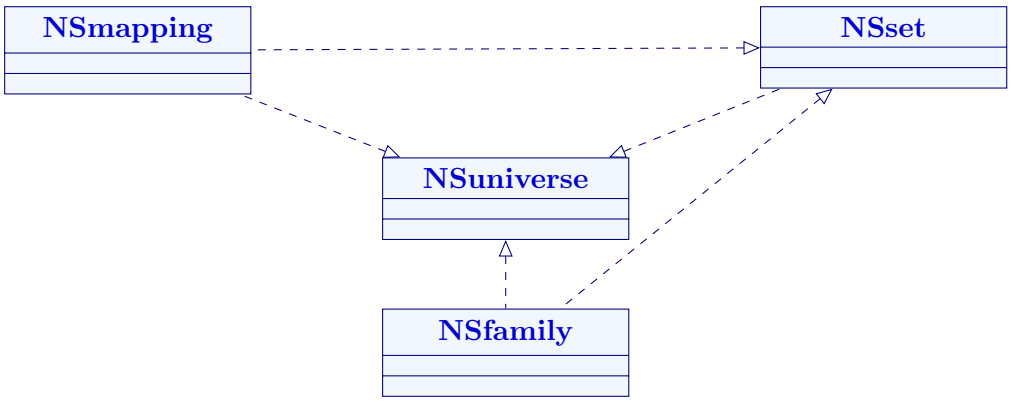
The modular design of `PYNS` not only supports interactive use, ideal for experimentation and the exploration of neutrosophic properties, but also facilitates integration into larger Python

projects due to its adaptable architecture and comprehensive, example-rich documentation. This ease of use and flexibility makes the PYNS framework a valuable open source tool for exploring the properties of neutrosophic sets, with applications spanning diverse areas of research that benefit from neutrosophic set theory.

Despite its comprehensive capabilities, PYNS currently lacks a mechanism for systematically managing collections of neutrosophic sets that could serve as bases or sub-bases in topological constructions. The ability to handle such collections is crucial for building and verifying topological structures in neutrosophic settings, which remain essential for advanced applications and theoretical explorations in neutrosophic topology. This limitation is addressed by extending the PYNS framework with the NSfamily class, presented in this paper which provides the necessary functionality to manage and interact with families of neutrosophic sets and neutrosophic topologies.

3. The NSfamily Class

The NSfamily class enhances the PYNS framework by enabling the construction and management of neutrosophic families, which serve as bases or sub-bases for topology generation. Inspired by Ozturk's theoretical work on neutrosophic topology [14] and Salama et al.'s object-oriented design principles [20,21], NSfamily provides a unified interface for creating topologies and verifying closure properties within neutrosophic environments. By adding NSfamily to the other three classes NSuniverse, NSset and NSmapping above described, the framework PYNS now supports topological operations on families of neutrosophic sets within the same universe, bridging a key gap in the framework and enabling systematic topology construction. The UML diagram below illustrates the relationships among the classes, with dashed arrows indicating "uses" relationships.



The NSfamily class manages families of neutrosophic sets stored as lists of objects of the class NSset sharing the same universe set and can be shortly described by means of its main properties and methods in the following UML class diagram.

NSfamily
__neutrosophicfamily : list of NSset
__init__(*args) : constructor with generic argument storeName() : save the name of the family as a property of the object itself setUniverse() : sets the universe of the neutrosophic sets family cardinality() : returns the number of elements of the neutrosophic family isSubset(nsfamily) : checks if is contained in that passed as parameter isSuperset(nsfamily) : checks if contains that passed as parameter union(nsfamily) : returns the union with another one intersection(nsfamily) : returns the intersection with another one isDisjoint(nsfamily) : checks if is disjoint with another one difference(nsfamily) : returns the set difference with another one complement() : returns the family of neutrosophically complemented neutrosophic sets getNSBase() : returns the neutrosophic topological base containing the current family getNSTopologyByBase() : returns the neutrosophic topology from a neutrosophic base getNSTopologyBySubBase() : returns the neutrosophic topology from the given family isNeutrosophicTopology() : checks if the family forms a neutrosophic topology

The `NSfamily` constructor initializes a family of neutrosophic sets over a shared universe. The initialization process is detailed in the following algorithm:

Constructor method of the class `NSfamily`

Function `__init__(`*args*`)`**:**
- Initialize the private property *neutrosophicfamily* as an empty list
- Get the *length* of *args*
- **if** *length* $= 0$ **then**
  - Set *universe* to `None`
- **else if** *length* $= 1$ **then**
  - Let *elem* be the first and only element in *args*
  - **if** *elem* ***is an instance of*** `NSset` **then**
    - Add *elem* to *neutrosophicfamily* and set *universe* to the universe of *elem*
  - **else if** *elem* ***is a list or tuple*** **then**
    - **foreach** *e in elem* **do**
      - **if** *e is not in neutrosophicfamily* **then**
        - Set the name of *e* and add it to *neutrosophicfamily*
    - **if** *neutrosophicfamily is not empty* **then**
      - Set *universe* to the universe of the first element in *neutrosophicfamily*
    - **else**
      - Set *universe* to `None`
- **else if** *length* > 1 **then**
  - **foreach** *e in args* **do**
    - **if** *e is not in neutrosophicfamily* **then**
      - Set the name of *e* and add it to *neutrosophicfamily*
  - Set *universe* to the universe of the first element in *args*
- Store *universe*, *neutrosophicfamily* as private properties
- Set the private property *name* to `None`

The Python code corresponding to the constructor method of the `NSfamily` class is given below.

```
from .ns_universe import NSuniverse
from .ns_set import NSset
from .ns_util import NSreplace,NSstringToDict,NSisExtDict,nameToBB,isBB
import inspect
from itertools import combinations
from functools import reduce
from time import time

class NSfamily:

    def __init__(self, *args):
        neutrosophicfamily = list()
        length = len(args)
        if length == 0:
            universe = None
        elif length == 1:
            elem = args[0]
            if type(elem) == NSset:
                neutrosophicfamily = [elem]
                universe = elem.getUniverse()
            elif type(elem) in [list ,tuple]:
                for e in elem:
                    if e not in neutrosophicfamily:
                        e.setName(e.getName())
                        neutrosophicfamily.append(e)
                if len(neutrosophicfamily) > 0:
                    universe = neutrosophicfamily[0].getUniverse()
                else:
                    universe = None
        elif length > 1:
            for e in args:
                if e not in neutrosophicfamily:
                    e.setName(e.getName())
                    neutrosophicfamily.append(e)
            universe = args[0].getUniverse()
        self.__universe = universe
        self.__neutrosophicfamily = neutrosophicfamily
        self.__name = None
```

The `NSfamily` constructor offers flexible initialization to handle various cases in creating neutrosophic set families, managing arguments through the `*args` parameter. This enables creation from individual sets, collections, or copies of existing `NSfamily` objects:

- empty family: `NSfamily()` creates an empty family with no universe, allowing elements to be added incrementally
- single neutrosophic set: `NSfamily(ns_set)` initializes a family with one set, inheriting its universe

- list or tuple of sets: `NSfamily([ns_set1, ns_set2, ...])` or or `NSfamily((ns_set1, ns_set2, ...))` accepts a collection, avoiding duplicates and setting the universe based on the first set
- multiple sets as arguments: `NSfamily(ns_set1, ns_set2, ...)` creates a family from the specified sets, with the universe derived from the first neutrosophic set, ensuring no duplicate entries
- existing `NSfamily` object: `NSfamily(existing_family)` copies an existing family, creating an independent object with the same universe.

This versatility allows the `NSfamily` class to support a broad range of initial configurations, thereby facilitating the manipulation and management of neutrosophic set families.

In order to conveniently display families of neutrosophic sets in both simplified and tabular text formats while providing a comprehensive representation, the special methods `__str__()` and `__format__()` have been implemented. These methods allow customization of the output format, especially when using the **`print`** function with f-strings for displaying `NSfamily` objects. The `__format__()` method supports the following format specifiers:

s: : simple format (default), offering a straightforward textual representation

t: : tabular format, presenting data in a structured and neatly aligned manner

l: : includes a label with the `NSfamily` object's name, if available

x: : extended format, providing additional spacing in the first column to accommodate longer names or labels.

```
def __str__(self, tabularFormat=False, label=False, extended=False):
    labelname = ""
    if label and self.__name is not None:
        labelname = f"{self.__name} = "
    if not self.__neutrosophicfamily:
        return labelname + "\u2205"
    indentation = " "*(len(labelname) + 2)
    if not tabularFormat:
        if extended:
            items = [a.__str__(tabularFormat, True, extended)
                     for a in self.__neutrosophicfamily]
            formatted_lines=[", ".join(items[i:i + 2]) for i in range(0,len(
    items),2)]
            return labelname+"{ "+f",\n{indentation}".join(formatted_lines)+"
    }\n"
        else:
            return labelname + "{ " + ", ".join(
                [a.__str__(tabularFormat, True, extended)
                 for a in self.__neutrosophicfamily]) + " }\n"
    res = labelname + "{ " + ", ".join(
        [a.__str__(tabularFormat, True, extended)
         for a in self.__neutrosophicfamily]) + " }\n"
    return res
```

```
def __format__(self, spec):
    label = "l" in spec
    extended = "x" in spec
    if "t" in spec:
        result = self.__str__(tabularFormat=True, label=label, extended=extended
    )
    else:
        result = self.__str__(tabularFormat=False, label=label, extended=
    extended)
    return result
```

Some basic methods facilitate the management of neutrosophic family properties. The `storeName()` method captures the object's name from the local context, while `getName()` retrieves it if stored. `getUniverse()` returns the associated `NSuniverse` of the neutrosophic family, and `setUniverse()` sets it, ensuring it is a valid `NSuniverse` instance. Finally, `cardinality()` provides the count of neutrosophic sets within the family. Their code is given below.

```
def storeName(self):
    frame = inspect.currentframe().f_back
    local_vars = frame.f_locals
    var_name = next((nm for nm, vl in local_vars.items()
                     if vl is self), None)
    self.__name = var_name

def getName(self):
    return self.__name

def getUniverse(self):
    return self.__universe

def setUniverse(self, universe):
    if type(universe) != NSuniverse:
        raise ValueError("The parameter's type must be NSuniverse.")
    self.__universe = universe

def cardinality(self):
    return len(self.__neutrosophicfamily)
```

Let us illustrate what just said with an example of code executed interactively in the Python console.

```
>>> from NS.pyns.ns_universe import NSuniverse
>>> from NS.pyns.ns_set import NSset
>>> from NS.pyns.ns_family import NSfamily
>>> U = NSuniverse("a,b,c")
>>> A1 = NSset(U, "(0.4,0.4,0.3), (0.1,0.1,0.1), (0.2,0.2,0.2)")
>>> A2 = NSset(U, "(0.1,0.2,0.9), (0.9,0.1,0.3), (0.5,0.3,0.4)")
>>> A1.storeName()
>>> A2.storeName()
>>> E = NSset.EMPTY(U)
>>> L = NSfamily(E, A1, A2)
```

```
>>> L.storeName()
>>> print(f"family {L:lx}")
family L = { A1 = < a/(0.4,0.4,0.3), b/(0.1,0.1,0.1), c/(0.2,0.2,0.2) >,
      A2 = < a/(0.1,0.2,0.9), b/(0.9,0.1,0.3), c/(0.5,0.3,0.4) > }
```

The following methods provide essential operations for managing and comparing neutrosophic families. These methods allow for determining subset and superset relationships, computing unions and intersections of families, checking for disjoint sets, and calculating differences and complements.

The `isSubset()` method verifies if the current neutrosophic family is contained within another, while the `__le__()` method overloads the <= operator to provide a more intuitive comparison.

```
def isSubset(self, nsfamily):
    if type(nsfamily) != NSfamily:
        raise ValueError("the parameter is not a neutrosophic family")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("the two neutrosophic families cannot be defined on
    different universe sets")
    else:
        result = True
        for e in self.__neutrosophicfamily:
            if e not in nsfamily.__neutrosophicfamily:
                result = False
                break
        return result

def __le__(self, nsfamily):
    if type(nsfamily) != NSfamily:
        raise ValueError("the second argument is not a neutrosophic family")
    return self.isSubset(nsfamily)
```

Similarly, the `isSuperset()` method checks if the current neutrosophic family contains another, with the `__ge__()` method overloading the >= operator to simplify this comparison.

```
def isSuperset(self, nsfamily):
    if type(nsfamily) != NSfamily:
        raise ValueError("the parameter is not a neutrosophic family")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("the two neutrosophic families cannot be defined on
    different universe sets")
    return nsfamily.isSubset(self)

def __ge__(self, nsfamily):
    if type(nsfamily) != NSfamily:
        raise ValueError("the second argument is not a neutrosophic family")
    return self.isSuperset(nsfamily)
```

Equality and inequality between families are determined by overloading the comparison operators == and =! through the special methods `__eq__()` and `__ne__()`.

```
def __eq__(self, nsfamily):
    if type(nsfamily) != NSfamily:
        raise ValueError("the second argument is not a neutrosophic family")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("the two neutrosophic families cannot be defined on
    different universe sets")
    equal = self.isSubset(nsfamily) and nsfamily.isSubset(self)
    return equal

def __ne__(self, nsfamily):
    if type(nsfamily) != NSfamily:
        raise ValueError("the second argument is not a neutrosophic family")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("the two neutrosophic families cannot be defined on
    different universe sets")
    different = not (self == nsfamily)
    return different
```

The `union()` method, along with its operator + overloaded by `__add__()`, combines two neutrosophic families into one, eliminating duplicates and ensuring they share the same universe.

```
def union(self, nsfamily):
    if not isinstance(nsfamily, NSfamily):
        raise ValueError("The parameter is not a neutrosophic family.")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("The two neutrosophic families cannot be defined
            on different universes.")
    combined_sets = self.__neutrosophicfamily +
            [s for s in nsfamily.__neutrosophicfamily
                    if s not in self.__neutrosophicfamily]
    return NSfamily(combined_sets)

def __add__(self, nsfamily):
    if not isinstance(nsfamily, NSfamily):
        raise ValueError("the second argument is not a neutrosophic family")
    return self.union(nsfamily)
```

The `intersection()` method identifies common elements between two families, facilitated by the `&` operator via `__and__()`, while `isDisjoint()` checks for disjointness.

```
def intersection(self, nsfamily):
    if not isinstance(nsfamily, NSfamily):
        raise ValueError("The parameter is not a neutrosophic family.")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("The two neutrosophic families cannot be defined on
    different universes.")
```

```
    common_sets = [s for s in self.__neutrosophicfamily
                if s in nsfamily.__neutrosophicfamily]
    return NSfamily(common_sets)

def __and__(self, nsfamily):
    if not isinstance(nsfamily, NSfamily):
        raise ValueError("the second argument is not a neutrosophic family")
    return self.intersection(nsfamily)

def isDisjoint(self, nsfamily):
    nsemptyfamily = NSfamily()
    nsemptyfamily.setUniverse(self.getUniverse())
    intersez = self.intersection(nsfamily)
    disjoint = intersez.cardinality() == 0
    return disjoint
```

The `difference()` method and its − operator via `__sub__()` return the elements unique to the first family, and `complement()` along with ˜ (overloaded by the special method `__invert__()`) yields the family consisting of the neutrosophic complements of the neutrosophic sets of the original family.

```
def difference(self, nsfamily):
    if not isinstance(nsfamily, NSfamily):
        raise ValueError("The parameter is not a neutrosophic family.")
    if self.getUniverse() != nsfamily.getUniverse():
        raise ValueError("The two neutrosophic families cannot be defined on
    different universes.")
    unique_sets = [s for s in self.__neutrosophicfamily
                    if s not in nsfamily.__neutrosophicfamily]
    return NSfamily(unique_sets)

def __sub__(self, nsfamily):
    if not isinstance(nsfamily, NSfamily):
        raise ValueError("The second argument is not a neutrosophic family")
    return self.difference(nsfamily)

def complement(self):
    complementary_sets = [ns_set.NScomplement()
                          for ns_set in self.__neutrosophicfamily]
    return NSfamily(complementary_sets)

def __invert__(self):
    return self.complement()
```

From Proposition 2.16 and Definition 2.17 we know that any family of neutrosophic sets over a same universe set is a neutrosophic sub-basis for the neutrosophic topology $\mathscr{T}(\mathscr{S})$ generated by $\mathscr{S}$ which has the family $\mathscr{B}(\mathscr{S})$ of all finite neutrosophic intersections of $\mathscr{S}$ as neutrosophic basis.

The method `getNSBase` returns a neutrosophic topological basis from the current neutrosophic family regarded as a sub-basis by generating all its possible neutrosophic intersections.

Method `getNSBase`

Function `getNSBase()`**:**
- Let *subbase* be `self.__neutrosophicfamily`
- Initialize an empty list *base*
- **for** $i \leftarrow 1$ **to** *`len(subbase)`* **do**
  - **foreach** *combination in `combinations(subbase, i)`* **do**
    - Compute *intersection* by applying `NSintersection` on all elements in *combination*
    - **if** *intersection_name is not empty* **then**
      - Set the name of *intersection* to *intersection_name*
    - **foreach** *s in subbase* **do**
      - **if** *intersection equals s* **then**
        - Set the name of *intersection* to name of *s*
        - **break**
    - Add *intersection* to *base*
- Convert *base* to an `NSfamily` object
- Set the universe of *base* to `self.getUniverse()`
- **return** *base*

The Python code corresponding to the method `getNSBase` is given below.

```
def getNSBase(self):
    subbase = self.__neutrosophicfamily
    base = list()
    for i in range(1, len(subbase) + 1):
        for combin in combinations(subbase, i):
            intersez = reduce(lambda x, y: x.NSintersection(y), combin)
            names = [s.getName() for s in combin if s.getName()]
            intersez_name = " ⋒ ".join(names) if names else None
            if intersez_name:
                intersez.setName(intersez_name)
            for s in subbase:
                if intersez == s:
                    intersez.setName(s.getName())
                    break
            base.append(intersez)
    base = NSfamily(base)
    base.setUniverse(self.getUniverse())
    return base
```

As an example, we show how the method described above can be used in the interactive mode by means of the Python console:

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> from pyns.ns_family import NSfamily
>>> U = NSuniverse("a,b,c")
>>> A1 = NSset(U, "(0.4,0.4,0.3), (0.1,0.1,0.1), (0.2,0.2,0.2)")
>>> A2 = NSset(U, "(0.1,0.2,0.9), (0.9,0.1,0.3), (0.5,0.3,0.4)")
>>> A1.storeName()
>>> A2.storeName()
```

```
>>> L = NSfamily(A1, A2)
>>> L.storeName()
>>> print(f"the neutrosophic family is {L:lt}")
 the neutrosophic family is L = {
 A1      |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------
 a       | 0.4            | 0.4            | 0.3            |
 b       | 0.1            | 0.1            | 0.1            |
 c       | 0.2            | 0.2            | 0.2            |
-------------------------------------------------------------
,
 A2      |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------
 a       | 0.1            | 0.2            | 0.9            |
 b       | 0.9            | 0.1            | 0.3            |
 c       | 0.5            | 0.3            | 0.4            |
-------------------------------------------------------------
>>> B = L.getNSBase()
>>> B.storeName()
>>> print(f"basis is {B:lx}")
basis is B = { A1 = < a/(0.4,0.4,0.3), b/(0.1,0.1,0.1), c/(0.2,0.2,0.2) >,
      A2 = < a/(0.1,0.2,0.9), b/(0.9,0.1,0.3), c/(0.5,0.3,0.4) >,
      A1 ⩓ A2 = < a/(0.1,0.2,0.9), b/(0.1,0.1,0.3), c/(0.2,0.2,0.4) > }
```

From Proposition 2.16 and Definition 2.13, we know that a base for a topology is a collection of sets from which every open set can be represented as a union of elements of the base. Specifically, in the context of neutrosophic topology, a neutrosophic base provides the foundation for generating all possible neutrosophic unions, which form the topology.

The method `getNSTopologyByBase` returns the neutrosophic topology derived from the current base. This is accomplished by generating all possible finite neutrosophic unions of the base sets, thus constructing the full topology.

Method `getNSTopologyByBase`

Function `getNSTopologyByBase()`**:**
- Let *base* be `self.__neutrosophicfamily`
- Initialize an empty list *topology*
- Create an *empty* neutrosophic set and add it to *topology*
- **for** $i \leftarrow 1$ **to** *len(base)* **do**
  - **foreach** *combination in combinations(base, i)* **do**
    - Compute *union* by applying `NSunion` on all elements in *combination*
    - Construct the *union_name* based on the combination's names
    - Assign *union_name* to *union*, if applicable
    - **if** *union matches any base element* **then**
      - Update *union*'s name accordingly
    - Add *union* to *topology*
- Create an *absolute* neutrosophic set from the universe and add it to *topology*
- Convert *topology* to an `NSfamily` object
- Set the universe of *topology* to `self.getUniverse()`
- **return** *topology*

The Python code corresponding to the method `getNSTopologyByBase` is given below.

```
def getNSTopologyByBase(self):
    base = self.__neutrosophicfamily
    topology = list()
    universe = self.__universe
    empty = NSset.EMPTY(universe)
    empty.setName("\u2205\u0303")
    topology.append(empty)
    for i in range(1, len(base) + 1):
        for combin in combinations(base, i):
            union = reduce(lambda x, y: x.NSunion(y), combin)
            names = [s.getName() for s in combin if s.getName()]
            union_name = " ⋓ ".join(f"({name})" if "⋒" in name else name
                                    for name in names)
            if union_name:
                union.setName(union_name)
            for b in base:
                if union == b:
                    union.setName(b.getName())
                    break
            topology.append(union)
    absolute = NSset.ABSOLUTE(universe)
    universe_name = universe.getName()
    absolute.setName(nameToBB(universe_name) if universe_name
                             and not isBB(universe_name) else universe_name)
    topology.append(absolute)
    topology = NSfamily(topology)
    topology.setUniverse(self.getUniverse())
    return topology
```

The `getNSTopologyBySubBase` method constructs the neutrosophic topology derived from a subbase by computing all possible neutrosophic unions of intersections. It first generates the neutrosophic base using the `getNSBase()` method and then forms the complete topology by applying `getNSTopologyByBase()` on this base. This process ensures the creation of a full neutrosophic topology from any given family of neutrosophic sets. The Python code corresponding to this method is shown below.

```
def getNSTopologyBySubBase(self):
    nsbase = self.getNSBase()
    nstopology = nsbase.getNSTopologyByBase()
    return nstopology
```

The `isNeutrosophicTopology` method verifies whether a neutrosophic family satisfies the axioms of a neutrosophic topology. It ensures the presence of both the empty set and the universal set in the family, and checks closure properties under union and intersection operations.

```
def isNeutrosophicTopology(self):
    family = self.__neutrosophicfamily
    universe = self.__universe
    empty = NSset.EMPTY(universe)
    if empty not in family:
        return False
    absolute = NSset.ABSOLUTE(universe)
    if absolute not in family:
        return False
    if not self.NSunionClosed():
        return False
    if not self.NSintersectionClosed():
        return False
    return True
```

The `isNeutrosophicTopology` method relies on `NSunionClosed` and `NSintersectionClosed`, both of which delegate the closure verification to the private method `__checkClosure`. This method efficiently manages the verification process by iterating over all possible combinations of sets and applying the specified operation (union or intersection) provided as a lambda function. By centralizing the core logic of closure checks, `__checkClosure` ensures consistency and efficiency in determining whether the neutrosophic family satisfies the closure properties essential for a neutrosophic topology.

```
def __checkClosure(self, operation, operation_name):
    family = self.__neutrosophicfamily
    l = len(family)
    for i in range(2, l + 1):
        for combin in combinations(family, i):
            result = reduce(operation, combin)
            if result not in family:
                return False
    return True


def NSunionClosed(self):
    return self.__checkClosure(lambda x,y: x.NSunion(y),"union")

def NSintersectionClosed(self:
    return self.__checkClosure(lambda x,y: x.NSintersection(y),"intersection")
```

The following code example executed interactively in the Python console illustrates the use of the above described methods.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> from pyns.ns_family import NSfamily
>>> U = NSuniverse("1,2,3")
>>> B1 = NSset(U, "(0.2,0.4,0.3), (0.6,0.1,0.1), (0.4,0.6,0.3)")
>>> B2 = NSset(U, "(0.3,0.2,0.9), (0.6,0.5,0.3), (0.2,0.3,0.8)")
>>> B1.storeName()
```

```
>>> B2.storeName()
>>> S = NSfamily(B1, B2)
>>> T = S.getNSTopologyBySubBase()
>>> T.storeName()
>>> print(f"topology has cardinality {T.cardinality()} and is:\n {T:tlx}")
topology has cardinality 6 and is:
 T = {
 ∅̃            |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------------
 1            | 0              | 0              | 1              |
 2            | 0              | 0              | 1              |
 3            | 0              | 0              | 1              |
-------------------------------------------------------------------

,
 B1           |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------------
 1            | 0.2            | 0.4            | 0.3            |
 2            | 0.6            | 0.1            | 0.1            |
 3            | 0.4            | 0.6            | 0.3            |
-------------------------------------------------------------------

,
 B2           |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------------
 1            | 0.3            | 0.2            | 0.9            |
 2            | 0.6            | 0.5            | 0.3            |
 3            | 0.2            | 0.3            | 0.8            |
-------------------------------------------------------------------

,
 B1 ⋒ B2      |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------------
 1            | 0.2            | 0.2            | 0.9            |
 2            | 0.6            | 0.1            | 0.3            |
 3            | 0.2            | 0.3            | 0.8            |
-------------------------------------------------------------------

,
 B1 ⋓ B2      |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------------
 1            | 0.3            | 0.4            | 0.3            |
 2            | 0.6            | 0.5            | 0.1            |
 3            | 0.4            | 0.6            | 0.3            |
-------------------------------------------------------------------

,
 Ũ            |   membership   |  indeterminacy | non-membership |
-------------------------------------------------------------------
 1            | 1              | 1              | 0              |
 2            | 1              | 1              | 0              |
 3            | 1              | 1              | 0              |
-------------------------------------------------------------------
 }
>>> T.isNeutrosophicTopology()
True
```

4. **Conclusions**

In this paper, we introduced an extension to the Python Neutrosophic Sets (PYNS) framework with the development of the `NSfamily` class. This addition allows for the efficient construction, manipulation, and analysis of neutrosophic families and their roles as bases and sub-bases in neutrosophic topological spaces. Leveraging existing PYNS classes, `NSuniverse`

and `NSset`, the new class provides robust methods for evaluating closure properties, unions, intersections, and complements, which are essential operations in topological studies.

The theoretical exploration underscored the adaptability of neutrosophic sets in addressing indeterminate and inconsistent information, a feature that differentiates them from classical set theories. The practical implementations demonstrated through detailed examples reveal the computational efficiency and flexibility of the PYNS framework, particularly in generating complete neutrosophic topologies from defined families.

Future research will aim to broaden the applicability of neutrosophic topologies, especially in decision-making scenarios characterized by high uncertainty and indeterminacy. Further enhancements to the PYNS framework are also planned to optimize performance and extend support for more complex neutrosophic operations.

This work lays a foundational framework for the integration of neutrosophic topological methods into various scientific and engineering disciplines. By providing a comprehensive toolset for the exploration and application of neutrosophic concepts, it opens new avenues for research where traditional approaches may not suffice.

The source code for the PYNS framework, including the `NSfamily` class introduced in this paper, is available at `https://github.com/giorgionordo/PythonNeutrosophicTopologies`. This repository, licensed under GPL 3.0, serves as a valuable resource for researchers and practitioners, enabling them to explore, utilize, and extend neutrosophic topologies in their projects.

Acknowledgements This research was supported by Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (G.N.S.A.G.A.) of Istituto Nazionale di Alta Matematica (INdAM) "F. Severi", Italy.

References

[1] Atanassov K.T. Intuitionistic Fuzzy Sets. *Fuzzy Sets and Systems* **20** (1), pp. 87-96, 1986.

[2] Broumi S., Talea M., Bakali A., Smarandache F. Single valued neutrosophic graphs. *Journal of New theory* **10**, pp. 86-101, 2016.

[3] Broumi S., Smarandache F., Bakali A. Connectedness and Stratification of Single-Valued Neutrosophic Topological Spaces. *Neutrosophic Sets and Systems*, **36**, pp. 77-84, 2020.

[4] Dey S., Ray G.C., Separation Axioms in Neutrosophic Topological Spaces. *Neutrosophic Systems with Applications* **2**, 2023.

[5] El-Ghareeb H.A. Novel Open Source Python Neutrosophic Package. *Neutrosophic Sets and Systems* **25**, pp. 136-160, 2019.

[6] Gallego Lupiáñez F., On Neutrosophic Topology. *Kybernetes*, 2008.

[7] Gallego Lupiáñez F., On Neutrosophic Sets and Topology. *Procedia Computer Science* **120** (1), pp. 975–982, 2017.

[8] Karatas S., Kuru C. Neutrosophic Topology. *Neutrosophic Sets and Systems* **13**, pp. 90-95, 2016.

[9] Latreche A., Barkat O., Milles S., Ismail F. Single valued neutrosophic mappings defined by single valued neutrosophic relations with applications. *Neutrosophic Sets and Systems* **32**, pp. 203-220, 2020.

[10] Mehmood A., Nadeem F., Nordo G., Zamir M., Park C.,Kalsoom H., Jabeen S., Khan M.Y. Generalized neutrosophic separation axioms in neutrosophic soft topological spaces. *Neutrosophic Sets and Systems* **32** (1), pp. 38–51, 2020.

[11] Mehmood A., Nadeem F., Park C., Nordo G., Kalsoom H., Rahim Khan M., Abbas N. Neutrosophic soft alpha-open set in neutrosophic soft topological spaces. *Journal of Algorithms and Computation* **52**, pp. 37–66, 2020.

[12] Mondal K, Pramanik S. Neutrosophic Decision Making Model of School Choice. *Neutrosophic Sets and Systems* **7** (1), pp. 62-68, 2015.

[13] Nordo G., Jafari S., Mehmood A., Basumatary B. A Python Framework for Neutrosophic Sets and Mappings. *Neutrosophic Sets and Systems* **65**, pp. 199-236, 2024.

[14] Ozturk T.Y., Some Structures on Neutrosophic Topological Spaces. *Applied Mathematics and Nonlinear Sciences*, pp. 1–12, 2020.

[15] Mershia Rabuni J.J., Balaman N., Computation of Neutrosophic Soft Topology using Python. *Neutrosophic Sets and Systems* **60**, pp. 548-569, 2023.

[16] Ray G.C., Dey S. Neutrosophic point and its neighbourhood structure *Neutrosophic Sets and Systems* **43**, pp. 1-13, 2021.

[17] Saber Y., Alsharari F., Smarandache F. On Single-Valued Neutrosophic Ideals in Şostak Sense. *Symmetry* **12** (2):193, 2020.

[18] Salama A.A., Alblowi S., Neutrosophic Set and Neutrosophic Topological Spaces. *International Journal of Mathematical Archive*, 2012.

[19] Salama A.A., Alagamy H. Neutrosophic Filters. *International Journal of Computer Science Engineering and Information Technologt Research* **3** (1), pp. 307-312, 2013.

[20] Salama A., El-Ghareeb H.A.. Manie A.M., Smarandache F. Introduction to develop some software programs for dealing with neutrosophic sets. *Neutrosophic Sets and Systems* **3**, pp. 51-52, 2019.

[21] Salama A., Abd el-Fattah M., El-Ghareeb H.A.. Manie A.M. Design and Implementation of Neutrosophic Data Operations Using Object Oriented Programming. *International Journal of Computer Application* **4** (5), pp. 163-175, 2014.

[22] Salama A.A., Smarandache F., Kromov V. Neutrosophic Closed Set and Neutrosophic Continuous Functions. *Neutrosophic Sets and Systems* **4**, pp. 4-8, 2014.

[23] Saranya S., Vigneshwaran M., Jafari S. C# Application to Deal with Neutrosophic $g\alpha$-Closed Sets In Neutrosophic Topology. *Application and Applied Mathematics* **15** (1), pp. 226-239, 2020.

[24] Sleem A., Abdel-Baset M., El-henawy I. PyIVNS: A python based tool for Interval-valued neutrosophic operations and normalization. *SoftwareX* **12**, pp. 1-7, 2020.

[25] Smarandache F. A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. Rehoboth: American Research Press, 1999.

[26] Smarandache F. Introduction to Neutrosophic Statistics. USA: Sitech & Education Publishing, 2014.

[27] Topal S., Broumi S., Bakali A., Talea M., Smarandache F. A Python Tool for Implementations on Bipolar Neutrosophic Matrices. *Neutrosophic Sets and Systems* **28**, pp. 138-161, 2019.

[28] Wang H., Smarandache F., Zhang Y.Q., Sunderraman R. Single Valued Neutrosophic Sets. *Technical Sciences and Applied Mathematics*, pp. 10-14, 2012.

[29] Zadeh L.A. Fuzzy Sets, *Information and Control* **8** (3), pp. 338-353, 1965.

[30] Zhang M., Zhang L. Cheng H.D. A neutrosophic approach to image segmentation based on watershed method. *Signal Processing* **90**, pp. 1510-1517, 2010.

Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Triangular Fuzzy Neutrosophic Model for Assessing and Optimizing Educational Quality in University Badminton Initiatives: A comprehensive study

Yi Zheng[1], Jun Chen[2]*

[1.]College of Physical Education and Health, Wenzhou University, Wenzhou, 325035, Zhejiang, China (E-mail: 18957725115@163.com)

[2.]Wenzhou University of Technology, Wenzhou, 325000, Zhejiang, China

*Corresponding author, E-mail: 20210574@wzut.edu.cn; 13819720353@163.com

Abstract: The evaluation of teaching quality in public badminton classes at general universities primarily considers students' learning outcomes, teachers' teaching abilities, class organization, and the appropriateness of the teaching content. Evaluation criteria include students' mastery of skills, participation, interest development, and teachers' teaching methods, classroom management, and interaction effectiveness. By combining quantitative and qualitative approaches, the evaluation aims to assess teaching quality, promote improvements, enhance students' physical fitness and sports skills, and support teachers' professional development and improvement of teaching standards. The teaching quality evaluation of public badminton classes at general universities is a multi-attribute decision-making (MADM). In this paper, a novel technique is introduced that combines triangular fuzzy neutrosophic numbers and PROMETHEE II (TFNN-PROMETHEE II) to address this issue. The TFNN-PROMETHEE II method leverages the properties of TFNN- PROMETHEE II and triangular fuzzy neutrosophic sets (TFNSs) to solve MADM problems effectively. A detailed numerical example is provided to demonstrate the application of the TFNN-PROMETHEE II technique in evaluating the teaching quality of public badminton classes at general universities. Additionally, several comparisons are made to highlight the advantages of this approach. The primary contributions of this paper are outlined: (1) The TFNN-PROMETHEE II technique is developed within the framework of TFNSs; (2) sensitivity analysis was conducted to show the stability of the rank of alternatives; (3) The technique is specifically applied to evaluate the teaching quality of public badminton classes at general universities; (4) Comparative analysis demonstrates that the TFNN-PROMETHEE II technique is also effective in assessing the teaching quality of public badminton classes at general universities.

Keywords: MADM; TFNSs; PROMETHEE II technique; TFNN-PROMETHEE II approach; teaching quality evaluation

1. Introduction

The evaluation of teaching quality in public badminton courses at ordinary universities is a comprehensive and ongoing process aimed at improving classroom effectiveness and student satisfaction. The evaluation primarily focuses on four aspects: the teacher's teaching ability, the design of the course content, the adequacy of teaching facilities, and the students' learning outcomes. Firstly, the teacher's teaching ability directly impacts the quality of the class. Teachers need not only a solid grasp of badminton techniques but also scientific teaching methods. In the classroom, teachers should adjust their teaching strategies flexibly according to the different levels and characteristics of the students. By combining demonstrations and guidance, they can enhance students' interest and participation. Additionally, the teacher's ability to organize the class is crucial, as good classroom management ensures that every student can learn efficiently within the limited time. Secondly, the design of the course content determines the students' learning outcomes. A well-designed badminton course should cover basic techniques, common tactics, and competition rules. Throughout the teaching process, the difficulty of the course should gradually increase, allowing students to have clear goals and achievements at each stage. The course should not only focus on skill training but also on fostering students' teamwork and competitive spirit. Meanwhile, the quality of teaching facilities and the environment is also an important factor. Adequate badminton courts and equipment ensure the smooth progress of the class, and good facility conditions can enhance students' interest and enthusiasm for learning. The school should regularly inspect and maintain the courts and equipment to provide a safe and comfortable learning environment for students. Finally, students' learning outcomes are a crucial measure of teaching quality. After the teaching period, observing students' skill improvement and collecting their feedback can help understand the actual effectiveness of the course and identify any shortcomings. Student satisfaction not only reflects their level of recognition on the course but also provides valuable references for teachers' improvements. In conclusion, the evaluation of teaching quality in public badminton courses at ordinary universities is not only an assessment of current teaching outcomes but also an important means of driving continuous improvement in teaching. Through scientific evaluation and ongoing enhancement, teachers can continually improve the quality of the course, providing students with richer learning experience.

MADM is an important branch of decision science, primarily used to address complex decision problems that involve multiple evaluation criteria or attributes[1-3]. In real life, many decision-making problems not only involve a single objective or attribute but often require consideration of several independent or interrelated factors simultaneously. For example, when a company selects a supplier, it needs to evaluate not only the price but also factors such as quality, delivery time, and service level. Such problems are typically solved using MADM methods [4]. The core of MADM is to evaluate alternatives based on different attributes and, through certain decision-making methods, select the optimal solution or rank the alternatives. Common MADM methods include the weighted sum method, Analytic Hierarchy Process (AHP), fuzzy set theory, and the TOPSIS method. These methods usually require assigning weights to each attribute to reflect their relative importance in the decision-making process. MADM methods are widely applied in various fields such as engineering, management, economics,

environmental decision-making, and educational evaluation. They help decision-makers systematically analyze complex problems and can handle uncertainty and ambiguity, providing more comprehensive and rational bases for decision-making. In the era of information and big data, the application of MADM continues to expand, becoming a crucial tool for addressing complex decision problems. The evaluation of teaching quality for public badminton classes at general universities is a MADM problem. TFNSs [5] have been introduced as an effective approach to represent uncertain information in this context. TFNSs are an extension of fuzzy sets, incorporating the concept of indeterminacy and offering a more flexible framework for dealing with uncertainty in decision-making. Here are two key advantages of TFNS: (1) Better handling of uncertainty and indeterminacy: Unlike traditional fuzzy sets that only account for membership degrees, TFNS also consider non-membership and indeterminacy degrees. This allows for a more comprehensive representation of uncertainty, particularly in complex situations where information is incomplete, ambiguous, or contradictory. The triangular form further simplifies the representation of these degrees, making it easier to model real-world problems where precise values are difficult to obtain. (2) Enhanced computational efficiency: The triangular representation in TFNS reduces computational complexity compared to other forms of fuzzy neutrosophic sets (such as trapezoidal or general fuzzy neutrosophic sets). The triangular structure requires fewer parameters (only three points: lower, middle, and upper bounds), allowing for faster calculations in decision-making processes, while still effectively capturing the range of uncertainty, indeterminacy, and membership. In previous studies, several techniques have employed PROMETHEE II [6-10] and entropy [11] separately to determine the optimal solution.

Herein, this study introduces the use of TFNSs as a robust solution to handle the inherent uncertainties between attributes and alternatives in MADM processes. Specifically, the TFNSs, a specialized form of neutrosophic set, is employed to manage this complexity. In real-world MADM scenarios, the variability and unpredictability of initial data often make it challenging for DMs to accurately convey their assessments[12]. Traditional methods struggle with capturing the nuances of vague, incomplete, or contradictory information. To address this, Smarandache's neutrosophic sets (NSs) [13] were developed, offering a more flexible framework that can simultaneously represent degrees of truth, indeterminacy, and falsity. This triad enables a more comprehensive representation of ambiguous data, far beyond the capabilities of classical and fuzzy sets. By leveraging this expanded framework, the study applies the PROMETHEE II method within the context of TFNSs, offering a more nuanced approach to evaluating and ranking alternatives. This integration of PROMETHEE II with TFNSs allows for a more precise reflection of the complexities within the decision-making process, ensuring better-informed and more reliable outcomes in uncertain environments. To address this gap, the triangular fuzzy neutrosophic number PROMETHEE II (TFNN-PROMETHEE II) technique is proposed, utilizing both PROMETHEE II and TFNSs. This TFNN-PROMETHEE II method is designed to tackle the MADM problem. A numerical example is provided to demonstrate its application in evaluating the teaching quality of public badminton classes at general universities, and comparative analysis is conducted to highlight the advantages of this approach.

Some significant Highlights of the work

(1) To introduce the TFNN-PROMETHEE II technique within the framework of TFNSs.

(2) To apply the TFNN-PROMETHEE II technique for MADM problems under TFNSs.

(3) To implement the TFNN-PROMETHEE II technique specifically for the evaluation of teaching quality in public badminton classes at general universities.

(4) Through several comparative analyses, it is shown that the TFNN-PROMETHEE II technique is highly effective for this purpose.

2. Literature review

The evaluation of teaching quality in public badminton courses at regular universities should be based on multiple dimensions to ensure the improvement of both course effectiveness and student experience. First, the scientific and reasonable design of the teaching content is fundamental. The course should balance technical skills, physical fitness, and tactical training, progressively enhancing students' badminton abilities. Second, the teaching methods employed by the instructor are crucial. Emphasis should be placed on making the lessons engaging and interactive, stimulating students' interest in learning, while also providing individualized instruction to meet the needs of students at different skill levels. Third, students' learning outcomes and mastery of skills are central to the evaluation. The effectiveness of the teaching can be assessed through students' performance in practice, technical progress, and their understanding of badminton rules. Additionally, student participation and course satisfaction are important reference indicators. A comprehensive evaluation across these dimensions provides valuable insights for improving teaching, thereby enhancing the overall quality of public badminton courses in universities. In 2017, Sun [14] emphasized the importance of core strength training in badminton teaching at universities. He pointed out that traditional badminton teaching overly focused on training activities while neglecting theoretical guidance, and that core strength training could effectively enhance students' athletic abilities and improve teaching outcomes. The following year, Wang [15] explored the construction and innovation of badminton teaching methods in universities. By analyzing current teaching issues, he proposed innovative teaching suggestions aimed at improving teaching quality and promoting the development of badminton. Yu [16] studied badminton teaching methods in public physical education courses, suggesting that teachers should break away from traditional methods to enhance student interest and, in turn, improve course quality. Luo [17] investigated the innovation of teaching and training methods in university badminton courses, analyzing the popularity of badminton in universities and proposing suggestions for improving both teaching and training methods to enhance teaching quality. Simultaneously, Zhang [18] examined the application of dynamic layered teaching methods in badminton courses, noting that this approach could cater to different student needs and proposed solutions to the constraints in its implementation. In 2020, Guan [19] studied the restructuring of badminton courses in physical education programs under the background of MOOCs. He pointed out the conflict between MOOCs and physical practice courses and suggested that combining MOOCs with flipped classrooms could break the limitations of time and space, thereby improving teaching quality. Following that, in 2021, Cao [20] explored the innovation of badminton teaching models in the "Internet+" era. She proposed a blended teaching model that integrates online and offline elements, supported by big data, to enhance interactivity and flexibility in badminton teaching. By 2022, Li [21]

conducted practical research on the flipped classroom in university badminton teaching, analyzing its feasibility and proposing recommendations for optimizing the teaching scheme. In the same year, Fu [22] constructed a "MOOC + Flipped Classroom" teaching model for university badminton courses, analyzing its value in curriculum reform and offering application suggestions to improve teaching quality. In 2023, You [23] studied the application of the OMO (Online-Merge-Offline) teaching model in university badminton courses, analyzing both its advantages and limitations, and proposed optimization strategies to improve teaching quality. Finally, in 2024, Ye and Tang [24] explored the current state of badminton teaching in universities under the background of physical education integration and proposed strategies for improvement, while also outlining future development directions to promote the comprehensive development of badminton teaching in universities.

3. Preliminaries

Biswas et al. [5] initiated the TFNSs.

Definition 1[5]. Let X be a fixed set, the TFNSs Z is initiated:

$$Z=\left\{\left(\theta,A(\theta),B(\theta),C(\theta)\right)\middle|\,\theta\in\Theta\right\} \tag{1}$$

where $A(\theta),B(\theta),C(\theta)\in[0,1]$ initiated truth-membership, indeterminacy membership and falsity-membership which are initiated through triangular fuzzy numbers.

$$A(\theta)=\left(A^{L}(\theta),A^{M}(\theta),A^{U}(\theta)\right),0\le A^{L}(\theta)\le A^{M}(\theta)\le A^{U}(\theta)\le 1 \tag{2}$$

$$B(\theta)=\left(B^{L}(\theta),B^{M}(\theta),B^{U}(\theta)\right),0\le B^{L}(\theta)\le B^{M}(\theta)\le B^{U}(\theta)\le 1 \tag{3}$$

$$C(\theta)=\left(C^{L}(\theta),C^{M}(\theta),C^{U}(\theta)\right),0\le C^{L}(\theta)\le C^{M}(\theta)\le C^{U}(\theta)\le 1 \tag{4}$$

For convenience, we let $Z=\left\{\left(A^{L},A^{M},A^{U}\right),\left(B^{L},B^{M},B^{U}\right),\left(C^{L},C^{M},C^{U}\right)\right\}$ be an TFNN, $0\le A^{U}+B^{U}+C^{U}\le 3$.

Definition 2[5]. There are three TFNNs $Z_1=\left\{\left(A_1^{L},A_1^{M},A_1^{U}\right),\left(B_1^{L},B_1^{M},B_1^{U}\right),\left(C_1^{L},C_1^{M},C_1^{U}\right)\right\}$, $Z_2=\left\{\left(A_2^{L},A_2^{M},A_2^{U}\right),\left(B_2^{L},B_2^{M},B_2^{U}\right),\left(C_2^{L},C_2^{M},C_2^{U}\right)\right\}$ and $Z=\left\{\left(A^{L},A^{M},A^{U}\right),\left(B^{L},B^{M},B^{U}\right),\left(C^{L},C^{M},C^{U}\right)\right\}$, the operation laws of TFNNs are initiated:

$$(1)\ Z_1 \oplus Z_2 = \begin{Bmatrix} \left(A_1^L + A_2^L - A_1^L A_2^L, A_1^M + A_2^M - A_1^M A_2^M, A_1^U + A_2^U - A_1^U A_2^U\right), \\ \left(B_1^L B_2^L, B_1^M B_2^M, B_1^U B_2^U\right), \left(C_1^L C_2^L, C_1^M C_2^M, C_1^U C_2^U\right) \end{Bmatrix};$$

$$(2)\ Z_1 \otimes Z_2 = \begin{Bmatrix} \left(A_1^L A_2^L, A_1^M A_2^M, A_1^U A_2^U\right), \\ \left(B_1^L + B_2^L - B_1^L B_2^L, B_1^M + B_2^M - B_1^M B_2^M, B_1^U + B_2^U - B_1^U B_2^U\right), \\ \left(C_1^L + C_2^L - C_1^L C_2^L, C_1^M + C_2^M - C_1^M C_2^M, C_1^U + C_2^U - C_1^U C_2^U\right) \end{Bmatrix};$$

$$(3)\ \lambda Z = \begin{Bmatrix} \left(1-\left(1-A^L\right)^\lambda, 1-\left(1-A^M\right)^\lambda, 1-\left(1-A^U\right)^\lambda\right), \\ \left(\left(B^L\right)^\lambda, \left(B^M\right)^\lambda, \left(B^U\right)^\lambda\right), \\ \left(\left(C^L\right)^\lambda, \left(C^M\right)^\lambda, \left(C^U\right)^\lambda\right) \end{Bmatrix}, \lambda > 0;$$

$$(4) Z^\lambda = \begin{Bmatrix} \left(\left(A^L\right)^\lambda, \left(A^M\right)^\lambda, \left(A^U\right)^\lambda\right), \\ \left(1-\left(1-B^L\right)^\lambda, 1-\left(1-B^M\right)^\lambda, 1-\left(1-B^U\right)^\lambda\right), \\ \left(1-\left(1-C^L\right)^\lambda, 1-\left(1-C^M\right)^\lambda, 1-\left(1-C^U\right)^\lambda\right) \end{Bmatrix}, \lambda > 0.$$

From Definition 2, the operation laws for TFNNs have some properties.

$$(1)\ Z_1 \oplus Z_2 = Z_2 \oplus Z_1; \tag{5}$$

$$(2) Z_1 \otimes Z_2 = Z_2 \otimes Z_1, \left(\left(Z_1\right)^{\lambda_1}\right)^{\lambda_2} = \left(Z_1\right)^{\lambda_1 \lambda_2}; \tag{6}$$

$$(3)\ \lambda\left(Z_1 \oplus Z_2\right) = \lambda Z_1 \oplus \lambda Z_2, \left(Z_1 \otimes Z_2\right)^\lambda = \left(Z_1\right)^\lambda \otimes \left(Z_2\right)^\lambda; \tag{7}$$

$$(4)\ \lambda_1 Z_1 \oplus \lambda_2 Z_1 = \left(\lambda_1 + \lambda_2\right) Z_1, \left(Z_1\right)^{\lambda_1} \otimes \left(Z_1\right)^{\lambda_2} = \left(Z_1\right)^{\left(\lambda_1 + \lambda_2\right)}. \tag{8}$$

Definition 3[5]. Let $Z = \left\{\left(A^L, A^M, A^U\right), \left(B^L, B^M, B^U\right), \left(C^L, C^M, C^U\right)\right\}$ be TFNN, the fuzzy score functions (FSF) and fuzzy accuracy functions (FAF) are initiated:

$$FSF\left(Z\right) = \frac{1}{12}\begin{bmatrix} 8 + \left(A^L + 2A^M + A^U\right) - \left(B^L + 2B^M + B^U\right) \\ -\left(C^L + 2C^M + C^U\right) \end{bmatrix}, FSF\left(Z\right) \in \left[0,1\right] \tag{9}$$

$$FAF\left(Z\right) = \frac{1}{4}\left[\left(A^L + 2A^M + A^U\right) + \left(B^L + 2B^M + B^U\right)\right], FAF\left(Z\right) \in \left[0,1\right] \tag{10}$$

For Z_1 and Z_2, in line with the Definition 3, then

(1) *if* $FSF(Z_1) \prec FSF(Z_2),\ Z_1 \prec Z_2;$
(2) *if* $FSF(Z_1) \succ FSF(Z_2),\ Z_1 \succ Z_2;$
(3) *if* $FSF(Z_1) = FSF(Z_2), FAF(Z_1) \prec FAF(Z_2),\ Z_1 \prec Z_2;$
(4) *if* $FSF(Z_1) = FSF(Z_2), FAF(Z_1) \succ FAF(Z_2), Z_1 \succ Z_2;$
(5) *if* $FSF(Z_1) = FSF(Z_2), FAF(Z_1) = FAF(Z_2), Z_1 = Z_2.$

4. Proposed TFNN-PROMETHEE II approach for MADM

This section outlines the steps of the PROMETHEE II method used to solve a decision-making problem within the MADM process. The steps are as follows:

Step 1: Identify criteria and alternatives

A team of experts and decision-makers is formed to define the criteria and alternatives relevant to the MADM problem.

Step 2: Conduct interviews and surveys

Experts participate in interviews and complete questionnaires designed for the PROMETHEE II method. These experts, equipped with the necessary knowledge and experience, evaluate the criteria and alternatives. To enhance the evaluation, they use Triangular Fuzzy Number Networks (TFNNs).

Step 3. The decision matrix is built

Step 4. Normalize decision matrix is computed as:

$U_{ij} = \frac{e_{ij} - \min e_{ij}}{\max e_{ij} - \min e_{ij}}$ for positive criteria (11)

$U_{ij} = \frac{\max e_{ij} - e_{ij}}{\max e_{ij} - \min e_{ij}}$ for negative criteria (12)

Where $i = 1,2, \ldots., m; j = 1,2, \ldots, n$

Step 5. Compute the alternative difference of i^{th} alternative with respect to other alternatives.

Step 6. Compute the criteria weights.

The criteria weights are determined using the mean method.

Step 7. Compute the preference function.

$P_j(x, y) = 0\ if\ U_{xj} \leq U_{yj}$ (13)

$P_j(x, y) = (U_{xj} - U_{yj})\ if\ U_{xj} > U_{yj}$ (14)

Step 8. Compute the combined preference functions.

$$CP = \frac{\left(\sum_{j=1}^{n} w_j P_j(x,y)\right)}{\sum_{j=1}^{n} w_j} \quad (15)$$

Step 9. Compute the leavening and entering outranking flows

$$Q^+ = \frac{1}{m-1}\sum_{i=i}^{m} CP \quad (16)$$

$$Q^- = \frac{1}{m-1}\sum_{j=i}^{n} CP \quad (17)$$

Step 10. Commute the net outranking

$$Q = Q^+ - Q^- \quad (18)$$

Step 11. Rank the alternatives.

5. Application for Triangular Fuzzy Neutrosophic Sets

The evaluation of teaching quality in public badminton classes at general universities is a systematic process of analyzing and assessing the teaching process and outcomes, aimed at ensuring the effectiveness of teaching activities and the holistic development of students. As an essential part of university physical education, badminton classes not only enhance students' physical health but also foster their interest in sports and promote lifelong exercise habits. Therefore, evaluating the teaching quality of these classes is of great significance. First, the core of teaching quality evaluation lies in the overall planning and implementation of the teaching process. The effective delivery of badminton classes depends on whether the instructor can design a reasonable teaching plan based on the actual situation of the students. The scientific, structured, and practical design of the curriculum directly affects whether students can grasp the basic skills and tactics of badminton within the limited class time. Additionally, the ability of teachers to adapt their teaching plans to meet the needs of students at different levels is a crucial aspect of evaluating teaching quality. Second, the organization and management of classroom teaching are key factors affecting teaching quality. Badminton classes typically require a high level of court utilization and active student participation, making the teachers' organizational and leadership abilities vital. Good classroom management helps increase student engagement and learning efficiency, ensuring that the teaching content is effectively conveyed. Moreover, how the teacher handles unexpected situations in class and establishes effective interaction with students are also important elements in assessing teaching quality. In addition, the professional competence and teaching attitude of the teacher directly influence the quality of teaching. Badminton classes require instructors to not only possess solid

professional knowledge and skills but also have rich teaching experience and the ability to tailor their instruction to individual students. The teacher's attitude, sense of responsibility, and commitment to the course directly affect students' satisfaction and their enthusiasm for learning. Whether the teacher demonstrates sufficient passion and patience in teaching and whether they can stimulate students' interest in learning are important factors to consider in the evaluation of teaching quality. Lastly, the evaluation of teaching quality should also take into account the overall effectiveness of the course. The goal of badminton classes is not only to impart sports skills but also to cultivate students' interest in sports, enhance teamwork, and improve their mental resilience. Therefore, the ultimate goal of teaching quality evaluation is to determine whether the course has achieved its expected educational outcomes and whether it has contributed to the comprehensive development of the students. In summary, the evaluation of teaching quality in public badminton classes at general universities is a comprehensive and systematic process. It requires attention to the rationality of the teaching plan and course implementation, as well as an assessment of the teacher's ability and attitude. The ultimate aim is to ensure that students gain well-rounded benefits from the class, both physically and mentally. The teaching quality evaluation of public badminton classes at general universities is MADM. Nine general universities are assessed with 12 attributes as shown in Table 1.

Step 1. The List of criteria is shown in Table 1.

Step 2. Three experts are evaluated the criteria and alternatives with expertise and knowledge in this kind of problem.

Step 3. Table 2 shows the decision matrix with TFNNS. Then we applied the score function to obtain crisp values. Then we combined these numbers into one matrix.

Step 4. Eq. (11) is used to normalize the TFNNSs as shown in Table 3. All criteria in this study are positive.

Step 5. Then we compute the alternative difference of i^{th} alternative with respect to other alternatives.

Step 6. Criteria weights are computed as shown in Table 1.

Step 7. Then we used Eqs. (13 and 14) to compute the preference function

Step 8. Then we compute the combined preference functions by using Eq. (15) as shown in Table 4.

Step 9. Then we used Eqs. (16 and 17) to compute the leavening and entering outranking flows

Step 10. Then we used Eq. (18) to commute to the net outranking

Step 11. Rank the alternatives as shown in Figure 1.

Table 1. List of criteria.

Criteria	Weights
C1: Student Satisfaction	0.083362
C2: Program Accessibility and Inclusiveness	0.08387
C3: Community Engagement and Social Impact	0.081413
C4: Outreach Programs	0.080989
C5: Sustainability and Long-Term Development	0.082218
C6: Integration with Academic Goals	0.082684
C7: Student Performance and Development	0.085225
C8: Coaching and Instruction Quality	0.086623
C9: Facilities and Equipment	0.085691
C10: Promotion of Health and Fitness	0.078744
C11: Collaboration and Networking	0.08243
C12: Program Curriculum and Content	0.08675

Table 2. The TFNNs with three experts.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_2	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
A_3	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_4	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
A_5	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
A_6	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_7	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)

A_8	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_9	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_{10}	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_2	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
A_3	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_4	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
A_5	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
A_6	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
A_7	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_8	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_9	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_{10}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}

A_1	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_2	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
A_3	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_4	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_5	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_6	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
A_7	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)
A_8	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_9	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
A_{10}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)

Table 3. The normalized TFNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	1	0.645161	0	0.604167	0.797619	0.461538	0.157895	0.965517	0	0
A_2	0.27907	0.935484	0.375	0.208333	0.452381	1	0.657895	0.241379	0.95	0.473684
A_3	0.232558	0	0.125	0.1875	0.785714	0.551282	0.434211	0.977011	0	0.157895
A_4	0.813953	0.967742	0.777778	0	0.452381	0.115385	0.5	0.666667	0.3	1
A_5	0.232558	0.129032	0.625	0.395833	0.285714	0	0	0.54023	0.483333	0.807018
A_6	0.325581	0.387097	1	0.5625	0.369048	0.74359	0.236842	0	0	0.666667
A_7	0.55814	1	0.736111	1	0	0.307692	1	0.287356	0.283333	0.77193
A_8	0.465116	0	0	0.3125	1	0.153846	0.473684	1	0.1	0.473684
A_9	0	0.580645	0.930556	0	0	0.230769	0.986842	0.517241	1	0.315789
A_{10}	1	0.645161	0	0.604167	0.797619	0.461538	0.157895	0.965517	0	0

Table 4. The combined preference functions.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
A_1	0	0.260721	0.054575	0.234778	0.307019	0.163959	0.314889	0.092409	0.256965	0
A_2	0.226381	0	0.091129	0.187809	0.283979	0.122258	0.233768	0.155815	0.101445	0.226381
A_3	0.249176	0.320069	0	0.310821	0.161532	0.274662	0.433381	0.159324	0.269514	0.249176
A_4	0.234084	0.221454	0.115526	0	0.241896	0.18677	0.207761	0.185802	0.143473	0.234084
A_5	0.307019	0.283979	0.161532	0.241896	0	0.194788	0.312991	0.212181	0.210212	0.307019
A_6	0.219607	0.212246	0.13571	0.243113	0.194788	0	0.268957	0.191119	0.210648	0.219607
A_7	0.220589	0.173808	0.144481	0.114156	0.312991	0.119009	0	0.187324	0.097156	0.220589
A_8	0.170724	0.26847	0.043038	0.264811	0.212181	0.213786	0.35994	0	0.251675	0.170724
A_9	0.374047	0.252866	0.191995	0.261249	0.210212	0.272082	0.308537	0.290441	0	0.374047
A_{10}	0	0.260721	0.054575	0.234778	0.307019	0.163959	0.314889	0.092409	0.256965	0

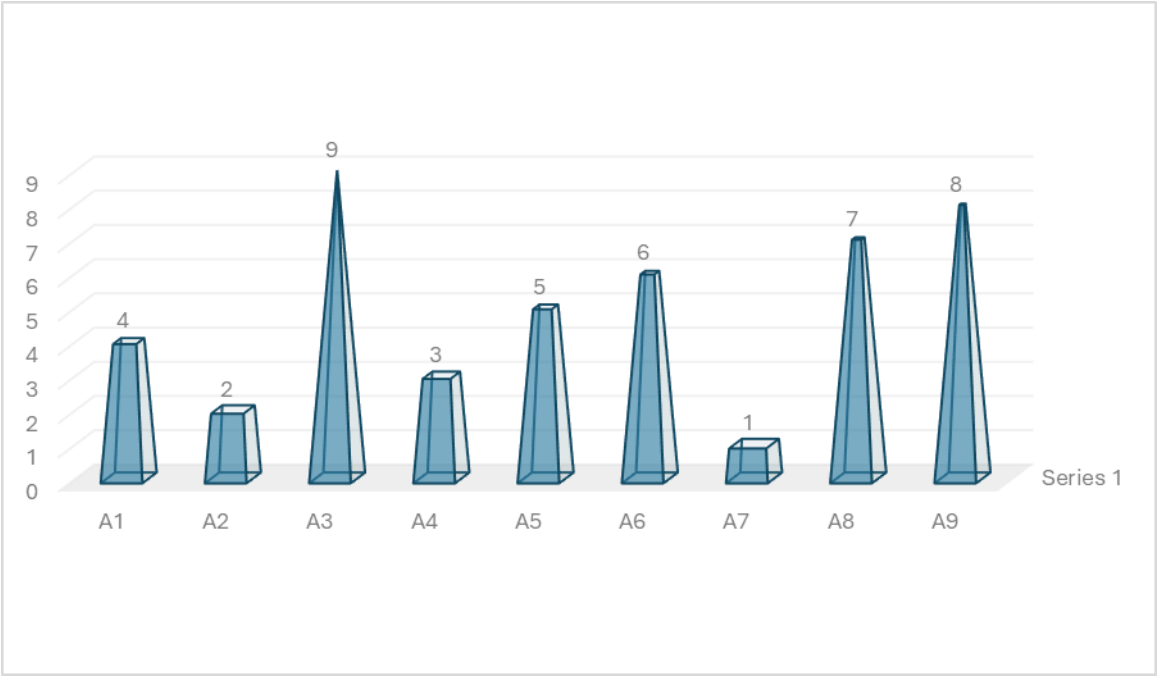


Figure 1. The rank of alternatives.

5.1 Sensitivity analysis

This section changes the criteria weights by different values as shown in Figure 2. We put all criteria with equal weights. Then we put the first criterion with 0.2 and the second criterion with 0.1 weights. Then we applied the proposed method to show the rank of alternatives as shown in Figure 3. The results show the rank of alternatives is stable under different weights.

Figure 2. Different criteria weights.

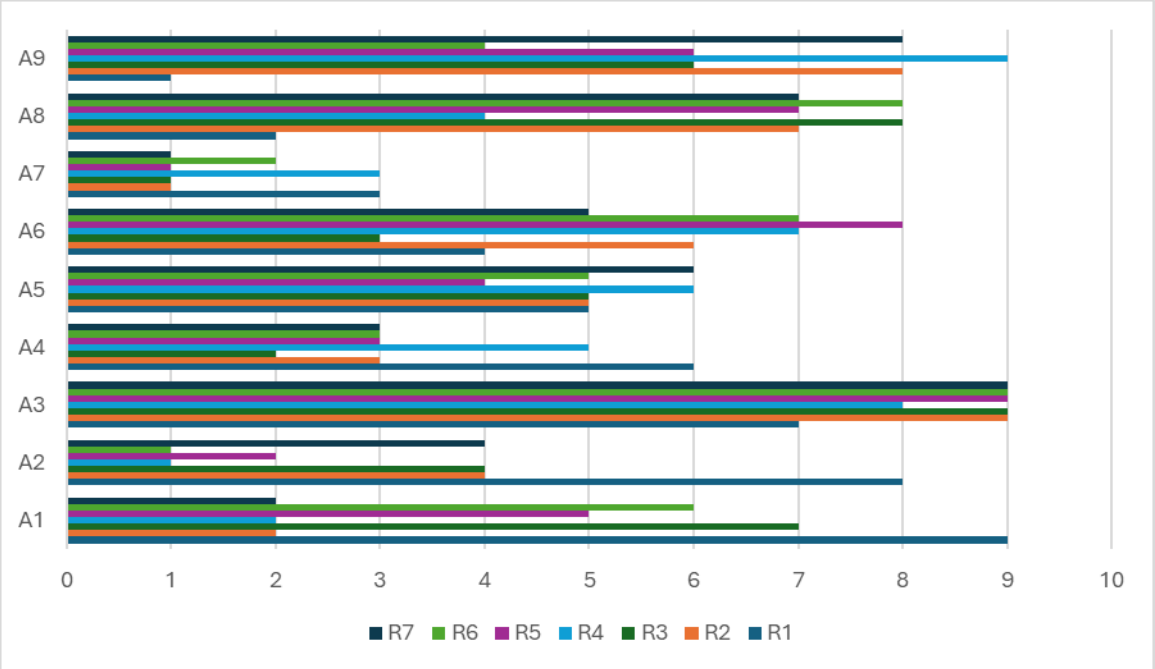


Figure 3. Different ranks of alternatives.

5.2. Comparative analysis among TFNSs

This section presents a comparative analysis of different MADM methods, including WASPAS, MABAC, and EDAS, to demonstrate the robustness of the proposed method. These methods were applied using the same weighting criteria as the proposed approach and were implemented within the framework of Triangular Fuzzy Number Networks (TFNNs). The ranking of alternatives is

illustrated in Figure 4. The results indicate that the proposed method exhibits superior performance compared to the other MADM methods.

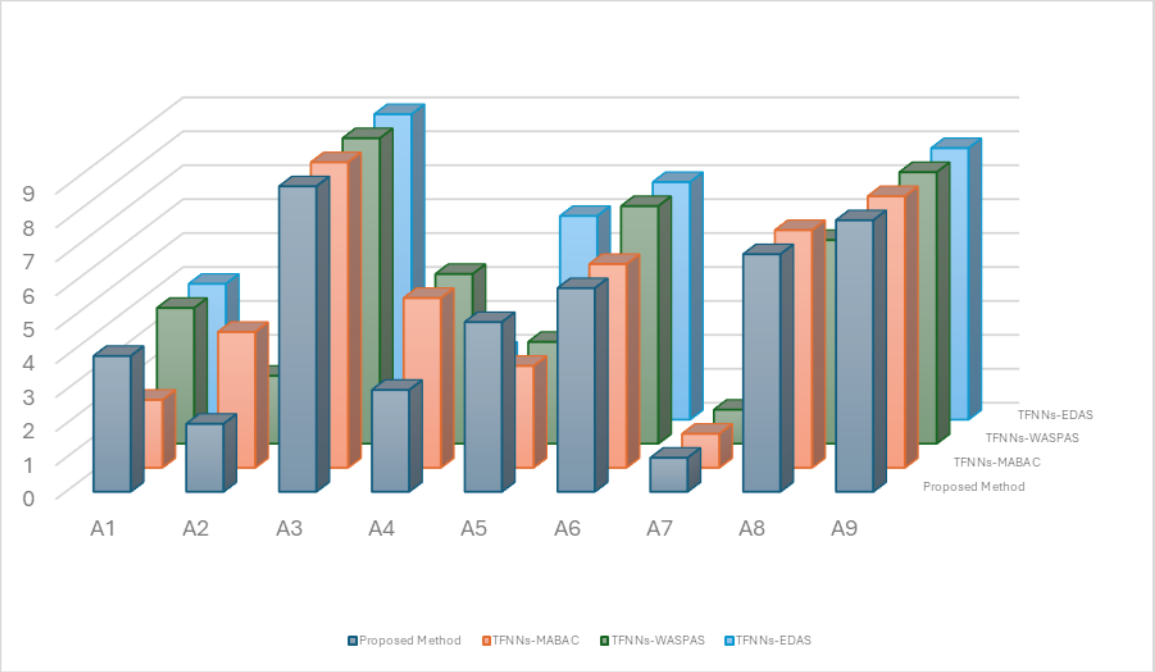


Figure 4. Comparative analysis results.

6. Results Discussion and Interpretation

In comparison with existing methods, the TFNN-PROMETHEE II approach proposed in this paper demonstrates notable advantages, alongside certain limitations. Established methods such as TFNN-EDAS, TFNN-WASPAS, and TFNN-MABAC, though widely used in MADM, have the following drawbacks:

Limited ability to handle fuzziness and uncertainty: These methods are less effective in managing complex fuzzy and uncertain data.

Strong reliance on weight allocation: The dependence on decision-makers' preferences can introduce bias, potentially compromising the reliability of evaluations.

High computational complexity: While these methods perform well in specific scenarios, their computational processes are intricate. This complexity poses challenges for large-scale decision-making problems, making them less efficient and slower in generating results.

In contrast, the TFNN-PROMETHEE II method offers three key advantages:

Enhanced capability to handle fuzziness: By integrating the strengths of triangular fuzzy neutrosophic numbers and PROMETHEE II, this approach effectively manages highly uncertain decision-making problems.

6.1 Managerial Implications of TFNN-PROMETHEE II in Teaching Quality Evaluation

The TFNN-PROMETHEE II method offers several key managerial benefits for university administrators:

By handling uncertainty and providing detailed data, this approach allows managers to make informed decisions about curriculum changes, teacher training, and resource allocation.

The method helps identify areas where instructors need improvement, enabling targeted professional development and improving teaching effectiveness.

Student Engagement: Including student feedback ensures that teaching methods can be adjusted to boost participation and satisfaction.

By assessing class organization and resource utilization, administrators can optimize the use of facilities and equipment, leading to better learning experiences.

This method ensures transparent and objective evaluations, fostering a culture of accountability among teachers and students.

Insights from the evaluation can guide improvements in course content and teaching strategies, aligning them better with student needs.

Comparing results with other institutions helps identify best practices, driving continuous improvements in teaching quality.

6.2 Implications for Practice

The findings from this study have several practical implications for the evaluation of teaching quality in public badminton classes at general universities. The use of the TFNN-PROMETHEE II method allows university administrators and instructors to make more informed decisions regarding course design, teaching methods, and resource allocation.

Teachers can benefit from the insights into their teaching effectiveness, enabling them to focus on areas that need improvement, such as student engagement and classroom management. This approach also helps improve the overall learning experience by tailoring teaching methods to better meet students' needs and preferences.

For administrators, the method provides a clearer understanding of how resources, such as equipment and court space, are being utilized, helping them make more efficient decisions. Additionally, by incorporating student feedback into the evaluation, universities can continually optimize their programs, leading to higher levels of student satisfaction and better learning outcomes.

7. Conclusion

The evaluation of teaching quality in public badminton classes at general universities primarily focuses on teaching effectiveness, student learning outcomes, teacher competence, and course implementation. First, students' mastery of techniques and their level of participation are key to the evaluation, with the emphasis on whether students can grasp basic badminton skills and tactics within the limited course time. Second, the teacher's teaching methods, classroom management

skills, and ability to guide students directly impact on the quality of instruction. The ability of the teacher to tailor instruction to individual needs and effectively manage the classroom is an important evaluation criterion. Additionally, the curriculum design and efficient use of teaching resources, including court and equipment allocation, also affect teaching outcomes. Finally, student feedback and satisfaction provide critical insights for assessing teaching quality. Students' opinions and suggestions help continually optimize course content and methods, ensuring that teaching objectives are met and promoting course improvement. The teaching quality evaluation of public badminton classes at general universities is MADM. In this paper, a PROMETHEE II technique under TFNSs is introduced based on the traditional PROMETHEE II method. First, TFNSs are outlined. Then, by combining the fuzzy PROMETHEE II technique with TFNS information, the TFNN-PROMETHEE II technique is developed for MADM under TFNSs. Finally, a numerical example is provided to evaluate the teaching quality of public badminton classes at general universities, and several comparisons are made to demonstrate the advantages of the TFNN-PROMETHEE II technique with TFNSs. The key contributions of this paper are as follows: (1) The TFNN-PROMETHEE II technique is developed under TFNSs; (2) the TFNN-PROMETHEE II technique is applied to MADM under TFNSs; (3) the TFNN-PROMETHEE II technique is proposed for evaluating the teaching quality of public badminton classes at general universities; and (4) through several efficient comparisons, the TFNN-PROMETHEE II technique is shown to be effective for teaching quality evaluation in public badminton classes.

References

[1] M. Mohamed, Toward smart logistics: Hybrization of intelligence techniques of machine learning and multi-criteria decision-making in logistics 5.0, Multicriteria Algorithms with Applications, 1 (2023) 42-57.

[2] M. Mohamed, A.A. Metwaly, M. Ibrahim, F. Smarandache, M. Voskoglou, Partnership of lean six sigma and digital twin under type 2 neutrosophic mystery toward virtual manufacturing environment: Real scenario application, Sustainable Machine Intelligence Journal, 8 (2024) (6):99-109.

[3] S.F. Al-baker, I. El-henawy, M. Mohamed, Pairing new approach of tree soft with mcdm techniques: Toward advisory an outstanding web service provider based on qos levels, Neutrosophic Systems with Applications, 14 (2024) 17-29.

[4] M. Mohamed, A. Elmor, F. Smarandache, A.A. Metwaly, An efficient superhypersoft framework for evaluating llms-based secure blockchain platforms, Neutrosophic Sets and Systems, 72 (2024) 1-21.

[5] P. Biswas, S. Pramanik, B.C. Giri, Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making, Neutrosophic Sets and Systems, 12 (2016) 20-40.

[6] W.S. Wu, Probabilistic linguistic promethee i and ii methods for evaluation of the reform scheme of postgraduate innovation and entrepreneurship education talent training mode under the big data environment, Mathematical Problems in Engineering, 2022 (2022) 10.

[7] L.Z. Tong, J.D. Wang, Z.M. Pu, Sustainable supplier selection for smes based on an extended promethee ii approach, Journal of Cleaner Production, 330 (2022) 18.

[8] Z.F. Li, Q. Pan, D.Y. Wang, P.D. Liu, An extended promethee ii method for multi-attribute group decision-making under q-rung orthopair 2-tuple linguistic environment, International Journal of Fuzzy Systems, 24 (2022) 3039-3056.

[9] Z.M. Liu, X.Y. Wang, N.N. Sun, L. Li, D. Wang, P.D. Liu, Fmea using the normalized projection-based todim-promethee ii model for blood transfusion, International Journal of Fuzzy Systems, 23 (2021) 1680-1696.

[10] R.V. Rao, B.K. Patel, Decision making in the manufacturing environment using an improved promethee method, International Journal of Production Research, 48 (2010) 4665-4682.

[11] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[12] M. Mohamed, K.M. Sallam, Leveraging neutrosophic uncertainty theory toward choosing biodegradable dynamic plastic product in various arenas, Neutrosophic Systems with Applications, 5 (2023) 1-9.

[13] F.A. Smarandache, Unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[14] C. Sun, Exploration of the importance of core strength training in university badminton teaching, Sporting Goods and Technology, (2017) 185-186.

[15] S. Wang, Construction and innovation of university badminton teaching methods, Contemporary Sports Technology, 8 (2018) 66-68.

[16] F. Yu, Research on badminton teaching methods in public physical education in universities, Sporting Goods and Technology, (2018) 156-157.

[17] X. Luo, Research on innovation in university badminton teaching and training methods, Sports Fashion, (2019) 184.

[18] H. Zhang, Research on the application of dynamic layered teaching method in university badminton courses, Sporting Goods and Technology, (2019) 127-128.

[19] H. Guan, Teaching reconstruction of university badminton courses in sports programs under the background of moocs, Sports Science and Technology, 41 (2020) 106-107, 109.

[20] G. Cao, Research on the innovation of university badminton teaching mode from the perspective of "internet+", Contemporary Sports Technology, 11 (2021) 74-77.

[21] J. Li, Practical research on flipped classroom in university badminton teaching, Youth Sports, (2022) 107-108.

[22] X. Fu, Research on the construction of "mooc + flipped classroom" teaching mode in university badminton courses, Contemporary Sports Technology, 12 (2022) 63-65.

[23] M. You, Research on the application and optimization of the omo model in university badminton teaching, Sports Fashion, (2023) 149-151.

[24] Y. Ye, W. Tang, The current situation and countermeasures of badminton teaching in universities under the background of physical education integration, Western Quality Education, 10 (2024) 90-93.

Received: Aug 22, 2024. Accepted: Nov 27, 2024

Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices

K. Radhika[1], T. Harikrishnan[2], R. Ambrose Prabhu[3], P.Tharaniya[4], M.John peter[5], M.Anandhkumar[6]

[1]Department of Mathematics, Rajah Serfoji Government College (Autonomous) (Affiliated to Bharathidasan University), Thanjavur - 613005, Tamilnadu, India.

radhi9791222789@gmail.com

[2]Department of Mathematics, Faculty of Science and Humanities, SRM Institute of Science and Technology, Ramapuram, Tamilnadu, India

mokshihari2009@gmail.com

[3]Department of Mathematics, Rajalakshmi Institute of Technology, Poonamalle, Chennai, Tamilnadu, India

bennyamb0457@gmail.com

[4]Department of Mathematics, Rajalakshmi Institute of Technology, Poonamalle, Chennai, Tamilnadu, India

tharamanisharmi@gmail.com

[5] Department of Mathematics, Panimalar Engineering College, Chennai – 600123, Tamil Nadu, India.

johnpmath@gmail.com

[6]Department of Mathematics, IFET College of Engineering (Autonomous), Villupuram, Tamilnadu, India.

anandhkumarmm@gmail.com

Abstract – In this paper, we present equivalent characterizations of k-kernel symmetric (k-KS) Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs). Additionally, we establish the necessary and sufficient conditions for the Schur complement (SC) within a k-KS QPNFM to be k-symmetric. The study also offers equivalent characterizations of both KS and k-KS QPNFMs. A few fundamental examples of KS QPNFMs are provided to clarify these concepts. It is shown that although k-symmetry implies k-KS, the converse does not necessarily hold. Several fundamental properties of k-KS QPNFMs are also derived. Finally, decision-making model utilizing QPNSMs has been successfully developed and validated through its application to real-world problems.

Keywords: QPNFM, Schur Complement, KS, k-KS.

1. Introduction

In the growing domain of fuzzy matrix theory, the Schur complement has emerged as a powerful mathematical tool, particularly for solving systems of linear equations, matrix decomposition, and optimization problems. The Schur complement has been widely applied in various branches of linear algebra and matrix theory, enabling the simplification of complex matrix

structures by partitioning them into smaller submatrices and providing insights into their properties. When extended to fuzzy matrices, the Schur complement offers valuable methods for handling uncertainty and vagueness in system modeling, which is especially crucial in real-world applications where precise information is often unavailable. In recent years, the integration of neutrosophic logic into fuzzy matrices has further advanced the capacity to manage uncertainty. Neutrosophic fuzzy matrices introduce three distinct components—truth (T), falsity (F), and indeterminacy (I)—which allow for a more nuanced representation of uncertain or incomplete data. The k-kernel symmetric matrices, another essential class of matrices in fuzzy theory, provide structured approaches for modeling symmetry and balance within systems. These matrices maintain symmetry in relation to a kernel, offering valuable properties in matrix decomposition and transformation.

The concept of quadri partitioning within neutrosophic fuzzy matrices introduces a novel method for dividing the matrix into four distinct partitions, each of which can represent various aspects of the system being modeled. This partitioning method enhances the flexibility of matrix operations, enabling more efficient computation and improved structural analysis, especially in systems that exhibit complex or multidimensional relationships. The Schur complement in the context of k-kernel symmetric quadri partitioned neutrosophic fuzzy matrices represents a fusion of these powerful mathematical tools. It aims to extend the traditional Schur complement to this new matrix class, allowing for more sophisticated handling of uncertainty and system decomposition. The combination of neutrosophic logic, kernel symmetry, and quadri partitioning offers a unique framework for solving complex systems with multiple layers of uncertainty, making it particularly useful in fields such as decision-making, artificial intelligence, and network theory, where incomplete or conflicting information is common. In this study, we explore the application of the Schur complement in k-kernel symmetric quadri partitioned neutrosophic fuzzy matrices, aiming to extend its utility to more intricate and uncertain systems. This research contributes to the broader understanding of matrix theory by introducing novel theoretical frameworks and practical methodologies for solving systems with high degrees of uncertainty and complexity.

The growing complexity of real-world systems has posed significant challenges in modeling uncertainty and imprecision. Traditional set theory, which relies on binary logic, often falls short in capturing the ambiguity present in natural phenomena. To address these limitations, Lotfi A. Zadeh introduced the concept of fuzzy sets in 1965, which allowed for partial membership, offering a more flexible framework for uncertainty modeling [1]. This revolutionary idea laid the foundation for fuzzy logic and its widespread applications in control systems, decision-making, and artificial intelligence.Since Zadeh's pioneering work, several extensions and generalizations of fuzzy sets have emerged, each aimed at refining the ability to handle various forms of uncertainty. Atanassov's intuitionistic fuzzy sets, introduced in 1983, provided an additional degree of freedom by incorporating both membership and non-membership functions [2]. This model was soon followed by Smarandache's introduction of neutrosophic sets, which generalized intuitionistic fuzzy sets by adding an indeterminacy function to capture even more complex uncertainties [3]. These advancements have been crucial in various fields, such as medical diagnosis, pattern recognition, and engineering systems.

As fuzzy set theory evolved, so did the need for algebraic structures capable of representing and processing fuzzy data. Matrix theory emerged as a powerful tool for this purpose, enabling the structured manipulation of fuzzy information. One of the early contributions to this area was made by Kim and Roush, who generalized the concept of fuzzy matrices in 1980 [4]. This work opened up new possibilities for representing relationships between fuzzy quantities in matrix form, making it easier to perform operations such as addition, multiplication, and inversion on fuzzy data. Following this development, several researchers contributed to expanding the theory and

applications of fuzzy matrices. Meenakshi's work in 2008 on fuzzy matrix theory significantly advanced the understanding of how matrices can be used in fuzzy logic and decision-making systems [5]. Matrix algebra provided a robust mathematical foundation for fuzzy models, and studies such as those by Hill and Waters [6] and Baskett and Katz [7] further explored the properties of κ-real and κ-Hermitian matrices, as well as EPr matrices. These investigations were essential in understanding the behavior of complex matrix operations, which are widely used in systems theory and optimization problems.

In parallel, research on special classes of matrices such as κ-EP matrices and their Schur complements gained traction. Meenakshi and Krishnamoorthy made substantial contributions in this domain, examining the properties and applications of κ-EP matrices and Schur complements [8-10]. Their findings have had significant implications for the study of matrix decompositions and applications in numerical analysis, signal processing, and optimization. Moreover, the exploration of secondary matrix structures, such as secondary κ-kernel symmetric matrices, introduced new directions for research in fuzzy mathematics. These matrices, first studied by Meenakshi and D. Jaya Shree, extend the concept of symmetry in matrices to fuzzy environments, allowing for more nuanced analysis of fuzzy systems [10-11]. Secondary symmetric matrices, including secondary skew-symmetric and orthogonal matrices, were further examined by An Lee [13], contributing to the understanding of how symmetry properties can influence matrix behavior in fuzzy systems.

More recent developments have focused on interval-valued fuzzy matrices, which account for uncertainty by allowing membership values to be represented as intervals rather than precise numbers. Shyamal and Pal's work in this area demonstrated the utility of interval-valued fuzzy matrices in dealing with imprecise data [12, 20]. Building on this, Meenakshi and Kalliraja introduced regular interval-valued fuzzy matrices, adding further refinement to the representation and manipulation of uncertain information [21]. The introduction of neutrosophic sets by Smarandache added another layer of complexity to fuzzy matrix theory. Neutrosophic fuzzy matrices, which incorporate indeterminacy in addition to membership and non-membership values, have gained considerable attention in recent years. Anandhkumar and colleagues made notable contributions to this field, exploring generalized symmetric neutrosophic fuzzy matrices [23] and secondary κ-column symmetric neutrosophic fuzzy matrices [24]. These studies have significantly expanded the applicability of fuzzy matrices in fields such as image processing, decision analysis, and communication systems.

Neutrosophic matrices have also been applied to new areas of research, including the study of pseudo-similarity in neutrosophic fuzzy matrices, as explored by Anandhkumar et al. in 2023 [27]. Their findings offer new perspectives on how similarity and dissimilarity can be quantified in fuzzy environments, providing valuable insights for applications in data mining, pattern recognition, and machine learning. Additionally, various inverse operations on neutrosophic fuzzy matrices, examined in [28], have opened up new possibilities for solving complex systems of equations in fuzzy environments, with potential applications in artificial intelligence, economics, and systems theory. As the field continues to grow, the intersection of matrix theory and fuzzy set theory remains a rich area of research. Ongoing studies on the properties of matrices, such as secondary κ-range symmetric fuzzy matrices [26] and interval-valued secondary κ-range symmetric neutrosophic fuzzy matrices [25], continue to push the boundaries of what is possible with fuzzy systems. Punithavalli et al [35] has discussed Reverse Sharp and Left-T Right-T Partial Ordering On IFM. These advancements highlight the importance of fuzzy matrix theory in addressing modern challenges in various scientific and engineering disciplines.

1.1 Literature Review

The concept of fuzzy sets, first introduced by Lotfi A. Zadeh in 1965, revolutionized the field of mathematical modeling by allowing for degrees of membership, thus accommodating uncertainty and imprecision in real-world systems [1]. Zadeh's work laid the foundation for fuzzy logic, which has since become essential in diverse applications, from control systems and decision-making to artificial intelligence and data analysis. Fuzzy set theory addressed the limitations of classical set theory, where membership was strictly binary. By allowing partial membership, fuzzy sets provided a more nuanced approach to modeling uncertainty.

Further developments in fuzzy set theory led to the introduction of intuitionistic fuzzy sets (IFS) by Atanassov in 1983 [22]. Intuitionistic fuzzy sets extended Zadeh's fuzzy sets by introducing an additional degree of uncertainty through a non-membership function, complementing the membership function. Unlike traditional fuzzy sets, which only consider the degree of membership, intuitionistic fuzzy sets account for both membership and non-membership, along with a degree of hesitation or indeterminacy.

The theory of intuitionistic fuzzy sets has found applications in areas such as decision-making, pattern recognition, and medical diagnosis [2]. For instance, intuitionistic fuzzy sets are particularly useful in situations where experts are unsure about the degree to which an element belongs to a set, allowing for a more comprehensive representation of uncertainty. Although the initial work focused on theoretical aspects, subsequent research has expanded into applications, such as multi-criteria decision-making and image processing [2, 12].

In 2005, Smarandache introduced neutrosophic sets, a generalization of intuitionistic fuzzy sets that incorporated an additional indeterminacy function [3]. Neutrosophic sets allow for the representation of truth, falsity, and indeterminacy independently, making them more flexible for handling complex uncertainty. This development addressed limitations in intuitionistic fuzzy sets, where the sum of membership and non-membership functions had to equal 1. Neutrosophic sets relaxed this restriction, allowing for more comprehensive modeling of uncertain, inconsistent, and incomplete information.

Neutrosophic sets have since been applied to a wide range of fields, including decision-making, optimization, and image processing. The flexibility provided by the independent consideration of truth, falsity, and indeterminacy has made neutrosophic sets particularly valuable in environments with high uncertainty and ambiguity [23]. More recent work has focused on integrating neutrosophic sets with other mathematical frameworks, such as matrices, to improve their applicability in complex systems.

Matrix theory has long been a fundamental tool in various branches of mathematics, and its extension to fuzzy environments has expanded its utility. Fuzzy matrices represent relationships between fuzzy quantities in a structured way, enabling matrix operations such as addition, multiplication, and inversion to be applied to uncertain data [4]. One of the early works in this area was Kim and Roush's 1980 exploration of generalized fuzzy matrices [4]. Their work established the foundational concepts needed for the manipulation of fuzzy matrices, setting the stage for further research.

Subsequent developments in fuzzy matrix theory include A. R. Meenakshi's work in 2008, which contributed significantly to the application of fuzzy matrices in decision-making and optimization [5]. Meenakshi's work extended the theory of fuzzy matrices to include new classes of matrices, such as κ-EP matrices, which have important applications in system theory, numerical analysis, and signal processing [8-10].
Research on the Schur complement of κ-EP matrices by Meenakshi and Krishnamoorthy has been particularly influential, as it addresses the properties and behavior of matrices under certain transformations [9]. The Schur complement plays a vital role in the decomposition of matrices and has applications in solving systems of linear equations, making it an essential tool in fields such as optimization, control theory, and econometrics.

One significant extension of fuzzy matrix theory is the development of interval-valued fuzzy matrices, which allow for the representation of uncertainty in the form of intervals rather than precise membership values. This approach was introduced by Shyamal and Pal in 2006 [12], who demonstrated the advantages of interval-valued fuzzy matrices in handling data with a high degree of imprecision. By allowing membership values to vary within a range, interval-valued fuzzy matrices offer a more flexible and realistic representation of uncertainty in real-world problems, where exact probabilities or membership degrees may be difficult to obtain.

Meenakshi and Kalliraja further developed this concept by introducing regular interval-valued fuzzy matrices, which added an additional layer of structure and consistency to the representation of uncertainty [21]. These matrices have found applications in areas such as decision analysis and pattern recognition, where handling imprecise data is critical.The introduction of neutrosophic sets into matrix theory led to the development of neutrosophic fuzzy matrices, which incorporate the three components of neutrosophic logic: truth, falsity, and indeterminacy [23]. This extension allows matrices to model even more complex forms of uncertainty, where the degree of indeterminacy plays a significant role.

Anandhkumar and colleagues have been at the forefront of research on neutrosophic fuzzy matrices, exploring several new classes of matrices, such as generalized symmetric neutrosophic fuzzy matrices and secondary κ-column symmetric neutrosophic fuzzy matrices [23-25]. These studies have highlighted the potential of neutrosophic fuzzy matrices in fields such as image processing, decision support systems, and cryptography, where uncertainty and indeterminacy are inherent.In addition to traditional and neutrosophic fuzzy matrices, researchers have investigated the properties of secondary matrices, which introduce a new level of symmetry to matrix theory. Secondary symmetric, secondary skew-symmetric, and secondary orthogonal matrices, first explored by An Lee in 1976 [13], have unique properties that make them useful in specialized mathematical and engineering applications. These matrices have been further studied in fuzzy environments, particularly in the context of κ-EP matrices and κ-kernel symmetric matrices [10, 11].

Recent advances in the field have focused on the application of fuzzy and neutrosophic matrices in modern technologies such as artificial intelligence, machine learning, and blockchain. The work of Anandhkumar et al. on pseudo-similarity in neutrosophic fuzzy matrices [27] and various inverse operations on these matrices [28] has opened up new possibilities for their use in data mining, pattern recognition, and cryptographic systems. Additionally, the exploration of time-varying fuzzy matrices and their applications in dynamic systems presents a promising area of future research. Despite these advancements, gaps remain in the practical application of fuzzy and neutrosophic matrices, particularly in terms of computational efficiency and scalability for large-scale systems. Moreover, the unification of various fuzzy matrix frameworks, such as intuitionistic, neutrosophic, and interval-valued matrices, into a single, cohesive structure remains an open research question.

1.2 Novelties

In recent years, significant advancements have emerged in the field of fuzzy matrix theory, particularly with the integration of neutrosophic logic, which extends the traditional fuzzy set framework to handle more complex forms of uncertainty. One of the most notable innovations is the development of neutrosophic fuzzy matrices, which incorporate the three elements of truth, falsity, and indeterminacy, offering a more comprehensive approach to modeling uncertainty in real-world systems [23]. This innovation has paved the way for the creation of new matrix classes, such as secondary κ-column symmetric neutrosophic fuzzy matrices and generalized symmetric neutrosophic fuzzy matrices [24, 23]. These novel matrix structures enable the representation of intricate data relationships, particularly in fields like communication systems and cryptography, where uncertainty is pervasive.

Additionally, the introduction of interval-valued neutrosophic matrices has further enhanced the ability to model uncertain data through ranges of values, making them particularly useful in decision-making and machine learning applications [25]. Recent research also explores the pseudo-similarity of neutrosophic matrices, providing new methods to analyze relationships between matrices that share approximate similarities, a significant leap for pattern recognition and clustering techniques [27]. Moreover, the extension of inverse operations in neutrosophic matrices [28] and the application of these matrices in cryptography and blockchain systems represent cutting-edge developments, highlighting the vast potential of neutrosophic logic in securing and managing decentralized systems. These advancements not only expand the theoretical foundations of fuzzy matrices but also demonstrate their growing relevance in modern technologies, offering robust tools for handling complex uncertainty across various scientific and engineering domains.

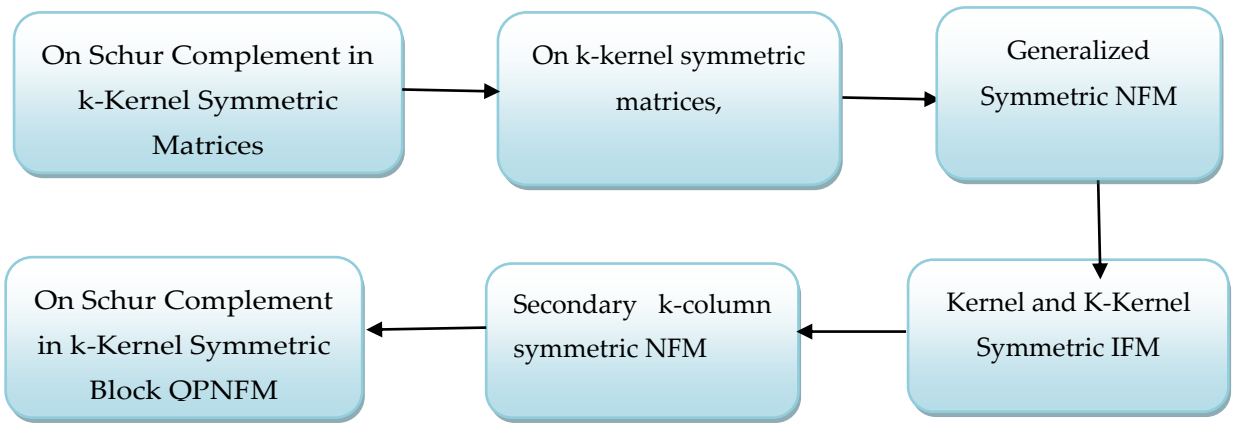
1.3 Research Gap:

Meenakshi, Jayasree [10,17] and Anandhkumar et al [23,24,30] introduced the concepts of k-kernel symmetric matrices and the Schur complement in the context of k-KS matrices.

Building on their work, we have applied these principles to Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs). In this study, we examine several results and extend these concepts to QPNFMs. First, we present equivalent characterizations for the Schur complement within k-KSQPNFMs and provide equivalent conditions that QPNFMs must satisfy to exhibit kernel symmetry (KS). Additionally, we explore the relationship between KS and k-KS in this framework.

Table:1 Review of the Extension of QPNFM.

Ref	**Journal Name**	**Authors Name**	**Extension of NFM.**	**Year**
[10]	Int. Journal of Math. Analysis	A. R. Meenakshi and D. Jaya Shree	On Schur Complement in k-Kernel Symmetric Matrices,	1989
[17]	International Journal of Mathematics and Mathematical Sciences	AR.Meenakshi and D.Jaya Shree,	On k-kernel symmetric matrices,	2009
[23]	Neutrosophic Sets and Systems	M.Anandhkumar et al	Generalized Symmetric NFM	2023
[30]	TWMS J. App. and Eng. Math	G. Punithavalli, and M. Anandhkumar,	Kernel and K-Kernel Symmetric IFM	2024
[24]	Neutrosophic Sets and Systems	M. Anandhkumar et al	Secondary k-column symmetric NFM	2024
Proposed	Neutrosophic Sets and Systems	K. Radhika et al	On Schur Complement in k-Kernel Symmetric Block QPNFM	2024


On Schur Complement in k-Kernel Symmetric Matrices
On k-kernel symmetric matrices,
Generalized Symmetric NFM
Kernel and K-Kernel Symmetric IFM
Secondary k-column symmetric NFM
On Schur Complement in k-Kernel Symmetric Block QPNFM

1.3.1 Validation and Comparison of QPNFM with Existing Models

The Quadripolar Neutrosophic Fuzzy Matrix (QPNFM) demonstrates significant advancements when compared to existing models, such as traditional fuzzy matrices and intuitionistic fuzzy matrices (IFM), especially in handling uncertainty and indeterminacy. Fuzzy matrices rely on a single membership value to represent truth, making them suitable for well-defined problems but often inadequate in contexts involving ambiguity or conflicting information. Similarly, while IFMs extend this framework by introducing truth and falsity components, they lack a robust mechanism to explicitly account for indeterminacy.

QPNFM addresses these limitations by incorporating quadripolar neutrosophic elements, enabling the simultaneous representation of truth, falsity, and indeterminacy as independent parameters. This added dimension allows QPNFM to model complex relational data with greater precision, particularly in scenarios characterized by incomplete or vague information. For example, in social network analysis, where relationships between entities may not be clearly defined, QPNFM provides a more nuanced representation compared to traditional fuzzy matrices.

Moreover, QPNFM outperforms existing models in practical case studies by effectively capturing and processing high degrees of indeterminacy. While this added complexity increases computational requirements, the trade-off is justified by the enhanced capability to derive meaningful insights and support decision-making. Through empirical validation, QPNFM has demonstrated superior accuracy and reliability in modeling and analyzing uncertain systems, establishing itself as a comprehensive and effective alternative to traditional and intuitionistic fuzzy matrices.

1.4 Notation

let $(Q^T, Q^C, Q^I, Q^F)^T$ transpose of (Q^T, Q^C, Q^I, Q^F),

$R((Q^T, Q^C, Q^I, Q^F))$ Row space of (Q^T, Q^C, Q^I, Q^F)

$N((Q^T, Q^C, Q^I, Q^F))$ Null space of (Q^T, Q^C, Q^I, Q^F)

$(Q^T, Q^C, Q^I, Q^F)^+$ Moore-Penrose inverse of (Q^T, Q^C, Q^I, Q^F)

$C((Q^T, Q^C, Q^I, Q^F))$ column space of (Q^T, Q^C, Q^I, Q^F)

QPNFM = Quadri Partitioned Neutrosophic fuzzy matrices,
KSNFM=Kernel symmetric Neutrosophic fuzzy matrices,
KS = Kernel symmetric,
RS = Range symmetric,

PM =Permutation matrices.

1.5 The structure of the article is as follows.

In section 1, We present intraduction, Literature Review, Novelties, Research Gap, Notation.

In section 2, We present some elementary definitions and findings.

In section 3, we provided k-kernel symmetric QPNFM.

In section 4, we introduced schur complement in k-kernel symmetric QPNFM.

In section 5, Application of Fuzzy Quadripartitioned Neutrosophic Soft Matrix in Medical Decision Making Problem

2.Definitions and Theorems

Definition : 2.1 Let Q =< (Q^T, Q^C, Q^I, Q^F) > ∈ (QPNFM)n then Q is called as kernel or null if

N (Q^T, Q^C, Q^I, Q^F) = {x ∈ $\left(QPNFM\right)_{1n}$ / x (Q^T, Q^C, Q^I, Q^F) = (0,0,1,1) } = (0,0,1,1).

Example: 2.1 Consider a QPNFM

$$Q = (Q^T, Q^C, Q^I, Q^F) = \begin{bmatrix} <0.5,0.6,0.7,0.4> & <0.3,0.3,0.7,0.9> & <0.9,0.4,0.3,0.1> \\ <0.2,0.6,0.4,0.8> & <0.6,0.1,0.7,0.6> & <0.1,0.6,0.7,0.2> \\ <0.9,0.6,0.7,0.4> & <0.4,0.6,0.7,0.8> & <0.2,0.4,0.7,0.1> \end{bmatrix}$$

Choose, $x = \left[<a,b,c,d>,<e,f,g,h>,<i,j,k,l>\right] \in \left(QPNFM\right)_{1n}$

Such that $x(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Therefore, $N(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Definition 2.2 A QPNFM $(Q^T, Q^C, Q^I, Q^F) \in (QPNFM)_n$ is said to be k-kernel symmetric if

$N((Q^T, Q^C, Q^I, Q^F)) = N(K(Q^T, Q^C, Q^I, Q^F)^T K)$.

Definition 2.3 For QPNFM $(Q^T, Q^C, Q^I, Q^F) \in (QPNFM)_n$ is kernel symmetric if

$N(Q^T, Q^C, Q^I, Q^F) = N((Q^T, Q^C, Q^I, Q^F)^T)$.

Example: 2.2 Consider a QPNFM

$$Q = (Q^T, Q^C, Q^I, Q^F) = \begin{bmatrix} <0.7,0.6,0.2,0.4> & <0.6,0.3,0.5,0.9> & <0.1,0.4,0.4,0.1> \\ <0.3,0.6,0.4,1> & <0.6,0.1,0.2,0.6> & <0.4,0.6,0.7,0.2> \\ <0.2,0.6,0.5,0.4> & <0.4,0.6,0.5,0.8> & <0.2,0.4,0.7,0.1> \end{bmatrix}$$

Choose, $x = \left[<a,b,c,d>,<e,f,g,h>,<i,j,k,l>\right] \in \left(QPNFM\right)_{1n}$

Such that $x(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Therefore, $N(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

$$Q^T = (Q^T, Q^C, Q^I, Q^F)^T = \begin{bmatrix} <0.7,0.6,0.2,0.4> & <0.3,0.6,0.4,1> & <0.2,0.6,0.5,0.4> \\ <0.6,0.3,0.5,0.9> & <0.6,0.1,0.2,0.6> & <0.4,0.6,0.5,0.8> \\ <0.1,0.4,0.4,0.1> & <0.4,0.6,0.5,0.8> & <0.2,0.4,0.7,0.1> \end{bmatrix}$$

Choose, $x = [<a,b,c,d>, <e,f,g,h>, <i,j,k,l>] \in (QPNFM)_{1n}$

Such that $x(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Therefore, $N(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Therefore, $N(Q^T, Q^C, Q^I, Q^F) = N((Q^T, Q^C, Q^I, Q^F)^T)$.

Definition 2.4 For (Q^T, Q^C, Q^I, Q^F), $(R^T, R^C, R^I, R^F) \in$ QPNFMn, (Q^T, Q^C, Q^I, Q^F) is k-similar to (R^T, R^C, R^I, R^F) if there exists a permutation matrix (PM) P such that $(Q^T, Q^C, Q^I, Q^F) = (KP^TK)\ (R^T, R^C, R^I, R^F)P$.

Theorem 2.1 For A ∈ Fn, the following subsequence are equivalent:

(i) A is KS,

(ii) PAP^T is KS for some PM P,

(iii) There exists a PM such that $PAP = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ with det D > 0

3. k-kernel symmetric Quadri Partitioned Neutrosophic fuzzy matrices

Consider (Q^T, Q^C, Q^I, Q^F) to be a QPNFM. If (Q^T, Q^C, Q^I, Q^F) is a part of (QPNFM)n is known as k-kernel symmetric QPNFM if N((Q^T, Q^C, Q^I, Q^F)) is more significant than N(K$(Q^T, Q^C, Q^I, Q^F)^T$K). Matrices are essential in a variety of areas of research in engineering and science. The conventional matrix theory must address issues with a wide range of uncertainty. Let $QPNFM_{mn}$ indicates the set of every m×n QPNFM over the QPNF algebra (QPNF)n. We denote a solution Z of the equation $(Q^T, Q^C, Q^I, Q^F)Z(Q^T, Q^C, Q^I, Q^F) = (Q^T, Q^C, Q^I, Q^F)$ by $(Q^T, Q^C, Q^I, Q^F)^-$.

For a complex matrix Q subdivided in the form $Q = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (R^T, R^C, R^I, R^F) \\ (S^T, S^C, S^I, S^F) & (T^T, T^C, T^I, T^F) \end{bmatrix}$.

Schur complement of (Q^T, Q^C, Q^I, Q^F) in Q denoted by $(T^T, T^C, T^I, T^F) - (S^T, S^C, S^I, S^F)\ (Q^T, Q^C, Q^I, Q^F)^-\ (R^T, R^C, R^I, R^F)$. This is known as the QPNFM generalized Schur complement. If (Q^T, Q^C, Q^I, Q^F) Z (Q^T, Q^C, Q^I, Q^F) = (Q^T, Q^C, Q^I, Q^F) has a solution, (Q^T, Q^C, Q^I, Q^F) is considered regular. A solution Z to the equation (Q^T, Q^C, Q^I, Q^F) Z (Q^T, Q^C, Q^I, Q^F) = (Q^T, Q^C, Q^I, Q^F) is defined by $(Q^T, Q^C, Q^I, Q^F)^-$ and is referred to as a generalized inverse, or generalized inverse of (Q^T, Q^C, Q^I, Q^F).

A QPNFM (Q^T, Q^C, Q^I, Q^F) is range symmetric if R (Q^T, Q^C, Q^I, Q^F) = R $(Q^T, Q^C, Q^I, Q^F)^T$ and kernel symmetric if N (Q^T, Q^C, Q^I, Q^F) = N $(Q^T, Q^C, Q^I, Q^F)^T$. For QPNFM $(Q^T, Q^C, Q^I, Q^F) \in$ QPNFMn, (Q^T, Q^C, Q^I, Q^F) is range symmetric, that is, R (Q^T, Q^C, Q^I, Q^F) = R $(Q^T, Q^C, Q^I, Q^F)^T$ implies N (Q^T, Q^C, Q^I, Q^F) = N $(Q^T, Q^C, Q^I, Q^F)^T$ but converse needs not be true.

Definition: 3.1 Let $R = <[R_T, R_C, R_I, R_F]>$ be a QPNFM, if R [$[R_T, R_C, R_I, R_F]$] = R $[[R_T, R_C, R_I, R_F]^T]$ then $R = <[R_T, R_C, R_I, R_F]>$ is called as RS.

Example: 3.1 Consider an QPNFM

$$[R_T, R_C, R_I, R_F] = \begin{bmatrix} <0.4,0,1,0.6> & <1,0,1,1> & <0.2,0.3,0.4,0.7> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.7> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

Here, R $[R_T, R_C, R_I, R_F]$ = R $[[R_T, R_C, R_I, R_F]^T]$ Range symmetric

Therefore N $[R_T, R_C, R_I, R_F]$ = N$[[R_T, R_C, R_I, R_F]^T]$ kernel symmetric

Therefore the above example both Range symmetric and kernel symmetric QPNFM.

The following matrices does not satisfy the range symmetric condition

$$[R_T, R_C, R_I, R_F] = \begin{bmatrix} <0.4,0,1,0.7> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.4,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$$[R_T, R_C, R_I, R_F]^T = \begin{bmatrix} <0.4,0,1,0.7> & <1,0,1,1> & <0.2,0.4,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$[(0.4,0,1,0.7)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)] \in R([R_T, R_C, R_I, R_F])$,

$[(0.4,0,1,0.7)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)] \notin R([R_T, R_C, R_I, R_F]^T)$,

$[(1,0,1,1)\ (1,0,1,1)\ (1,0,1,1)] \in R[R_T, R_C, R_I, R_F]$,

$[(1,0,1,1)\ (1,0,1,1)\ (1,0,1,1)] \in R([R_T, R_C, R_I, R_F]^T)$,

$[(0.2,0.4,0.4,0.6)\ (1,0,1,1)\ (1,0,1,1)] \in R([R_T, R_C, R_I, R_F])$,

$[(0.2,0.4,0.4,0.6)\ (1,0,1,1)\ (1,0,1,1)] \notin R([R_T, R_C, R_I, R_F]^T)$,

Here, R $[R_T, R_C, R_I, R_F]$ $\neq$ R $[[R_T, R_C, R_I, R_F]$T] not Range symmetric

Therefore N $[R_T, R_C, R_I, R_F]$ = N$[[R_T, R_C, R_I, R_F]$T] kernel symmetric

Remark: 3.1 The above example shows that a kernel-symmetric QPNFM is not necessarily range-symmetric. Consequently, every range-symmetric QPNFM is also kernel-symmetric.

Remark 3.2. In particular, when κ(i)= i for each i = 1 to n, the associated PM K reduces to the identity matrix reduces to N((Q^T, Q^C, Q^I, Q^F)) = N((Q^T, Q^C, Q^I, Q^F))^T , that is, (Q^T, Q^C, Q^I, Q^F) is kernel symmetric. If (Q^T, Q^C, Q^I, Q^F) is symmetric, then (Q^T, Q^C, Q^I, Q^F) is k-KS for all transpositions k in S_n.

Further, (Q^T, Q^C, Q^I, Q^F) is k-Symmetric implies it is k-KS, for (Q^T, Q^C, Q^I, Q^F) = K(Q^T, Q^C, Q^I, Q^F) ^{T}K automatically implies N((Q^T, Q^C, Q^I, Q^F)) = N(K(Q^T, Q^C, Q^I, Q^F) ^{T}K).

The opposite need not be true, though.

Therefore, $(Q^T, Q^C, Q^I, Q^F) \neq K(Q^T, Q^C, Q^I, Q^F)^T K$

But, $N((Q^T, Q^C, Q^I, Q^F)) = N(K(Q^T, Q^C, Q^I, Q^F)^T K) = (0,0,1,1)$

Therefore, (Q^T, Q^C, Q^I, Q^F) is not k-symmetric. For this (Q^T, Q^C, Q^I, Q^F) , since (Q^T, Q^C, Q^I, Q^F) has no zero rows and no zero columns. $N(K(Q^T, Q^C, Q^I, Q^F)^T K) = (0,0,1,1)$.

Hence (Q^T, Q^C, Q^I, Q^F) is k-KS, but (Q^T, Q^C, Q^I, Q^F) is not k-symmetric.

Theorem 3.1 Let $(R^T, R^C, R^I, R^F) = \begin{bmatrix} (T^T, T^C, T^I, T^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$ where (T^T, T^C, T^I, T^F) is r × r QPNFM with no zero rows and no zero columns, then the following equivalent conditions hold:

(i) (R^T, R^C, R^I, R^F) is k-KS

(ii) $N(R^T, R^C, R^I, R^F)^T = N\left((R^T, R^C, R^I, R^F)K\right)^T$

(iii) $K = \begin{bmatrix} K_1 & (0,0,1,1) \\ (0,0,1,1) & K_2 \end{bmatrix}$ where K_1 and K_2 are QPNFPM of order r and n-r, respectively,

Proof: Since (T^T, T^C, T^I, T^F) has no zero rows and no zero columns $N(T^T, T^C, T^I, T^F) = N(T^T, T^C, T^I, T^F)^T = (0,0,1,1).$

Therefore, $N(R^T, R^C, R^I, R^F) = N(R^T, R^C, R^I, R^F)^T \neq (0,0,1,1)$ and (R^T, R^C, R^I, R^F) is KS.

Now we will show that (i),(ii) and (iii) . (R^T, R^C, R^I, R^F) is k-KS⇔ $N(R^T, R^C, R^I, R^F) = N\left((R^T, R^C, R^I, R^F)K\right)^T$.

Choose z =[0 y] with all element of $y \neq (0,0,1,1)$ and subdivided in conformity with that

of $(R^T, R^C, R^I, R^F) = \begin{bmatrix} (T^T, T^C, T^I, T^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$

Clearly,

$$z \in N(R^T, R^C, R^I, R^F) = N(R^T, R^C, R^I, R^F)^T = N\left((R^T, R^C, R^I, R^F)K\right)^T.$$

Let us subdivided K as $K = \begin{bmatrix} K_1 & K_3 \\ K^T{}_3 & K_2 \end{bmatrix}$

Then

$$K(R^T, R^C, R^I, R^F) = \begin{bmatrix} K_1 & K_3 \\ K^T{}_3 & K_2 \end{bmatrix} \begin{bmatrix} (T^T, T^C, T^I, T^F)^T & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$$

$$= \begin{bmatrix} K_1(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \\ K^T{}_3(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \end{bmatrix}$$

$$z = [0 \ y] \in N(R^T, R^C, R^I, R^F) = N\left(K(R^T, R^C, R^I, R^F)^T\right)$$

$$\Rightarrow [0 \ y] \begin{bmatrix} K_1(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \\ K^T{}_3(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \end{bmatrix} = (0,0,1,1)$$

$$\Rightarrow yK^T{}_3(T^T, T^C, T^I, T^F)^T = (0,0,1,1)$$

Since $N(T^T, T^C, T^I, T^F)^T = (0,0,1,1)$, it follows that $yK^T{}_3 = (0,0,1,1)$.

Since all element of y $\neq (0,0,1,1)$ under max-min arrangement $yK^T{}_3 = (0,0,1,1)$ this implies $K^T{}_3 = (0,0,1,1) \Rightarrow K_3 = (0,0,1,1)$.

Therefore $K = \begin{bmatrix} K_1 & (0,0,1,1) \\ (0,0,1,1) & K_2 \end{bmatrix}$

Thus, (iii) holds, conversely, if (iii) holds, then

$$K(R^T,R^C,R^I,R^F)^T=\begin{bmatrix} K_1(T^T,T^C,T^I,T^F)^T & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix},$$

$$N\left(K(R^T,R^C,R^I,R^F)^T\right)=N\left((R^T,R^C,R^I,R^F)\right)$$

Thus (i)⇔(ii)⇔(iii) holds.

Theorem 3.2. For $(Q^T,Q^C,Q^I,Q^F)\in$ Fn and k = k_1k_2 . Then the subsequence are equivalent:

(i) (Q^T,Q^C,Q^I,Q^F) is k-KS of rank r,

(ii) (Q^T,Q^C,Q^I,Q^F) is k-similar to a diagonal block matrix

$\begin{bmatrix} (T^T,T^C,T^I,T^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$ with det $(T^T,T^C,T^I,T^F)>(0,0,1,1)$,

(iii) (Q^T,Q^C,Q^I,Q^F) = $KGLG^T$ and L ∈ F_r with det L > $(0,0,1,1)$ and $G^TG = I_r$.

Proof. (i) ⇔(ii) (Q^T,Q^C,Q^I,Q^F) is k-KS ⇔ K (Q^T,Q^C,Q^I,Q^F) is KS

$$\Leftrightarrow PK(Q^T,Q^C,Q^I,Q^F)P^T=\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix} \text{ with det}$$

$(E^T,E^C,E^I,E^F)>(0,0,1,1)$, for some Permutation matrix P

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=KP^T\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)K\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix} K_1 & (0,0,1,1) \\ (0,0,1,1) & K_2 \end{bmatrix}\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix} K_1(E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix}(T^T,T^C,T^I,T^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}P$$

Thus (Q^T,Q^C,Q^I,Q^F) is k-similar to a diagonal block matrix $\begin{bmatrix}(T^T,T^C,T^I,T^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}$, where $(T^T,T^C,T^I,T^F)=K_1(E^T,E^C,E^I,E^F)$ and

det $(T^T,T^C,T^I,T^F)>(0,0,1,1)$.

However, (ii) $\Leftrightarrow$ (iii)

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix}K_1(E^T,E^C,E^I,E^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=K\begin{bmatrix}P_1^T & P_3^T\\ P_2^T & P_4^T\end{bmatrix}\begin{bmatrix}K_1 & (0,0,1,1)\\ (0,0,1,1) & K_2\end{bmatrix}\begin{bmatrix}(T^T,T^C,T^I,T^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}\begin{bmatrix}P_1 & P_2\\ P_3 & P_4\end{bmatrix}$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=K\begin{bmatrix}P_1^T\\ P_2^T\end{bmatrix}K_1(T^T,T^C,T^I,T^F)\begin{bmatrix}P_1 & P_2\end{bmatrix}$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=KGLG^T,\ G=\begin{bmatrix}P_1^T\\ P_2^T\end{bmatrix},G^T=\begin{bmatrix}P_1 & P_2\end{bmatrix},L=K_1(T^T,T^C,T^I,T^F)\in(QPNFM)_r$$

$$G^TG=\begin{bmatrix}P_1 & P_2\end{bmatrix}\begin{bmatrix}P_1^T\\ P_2^T\end{bmatrix}=P_1P_1^T+P_2P_2^T=I_r,L\in(QPNFM)_r$$

4. Schur Complement in k-Kernel QPNFM

Consider block QPNFM of the following form throughout $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$ with respect to this subdividing a SC of (Q^T,Q^C,Q^I,Q^F) in M is a QPNFM of the form $M/(Q^T,Q^C,Q^I,Q^F)=(T^T,T^C,T^I,T^F)-(S^T,S^C,S^I,S^F)(Q^T,Q^C,Q^I,Q^F)^-(R^T,R^C,R^I,R^F)$ .where (Q^T,Q^C,Q^I,Q^F) and (T^T,T^C,T^I,T^F) are square QPNFM. Here M/ (Q^T,Q^C,Q^I,Q^F) is a QPNFM iff $(T^T,T^C,T^I,T^F)\geq(S^T,S^C,S^I,S^F)(Q^T,Q^C,Q^I,Q^F)^-(R^T,R^C,R^I,R^F)$ that is

$(T^T,T^C,T^I,T^F)=(T^T,T^C,T^I,T^F)+(S^T,S^C,S^I,S^F)(T^T,T^C,T^I,T^F)^{-}(R^T,R^C,R^I,R^F)$. A subdivided QPNFM M of the form $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$ is k-KS then it is not true in general that a SC of (Q^T,Q^C,Q^I,Q^F) in M, $M/(Q^T,Q^C,Q^I,Q^F)$ is k-KSQPNFM. Here the both if and only if conditions for $M/(Q^T,Q^C,Q^I,Q^F)$ to be k-KS are obtained.

Theorem 4.1 Let M be a matrix of the form $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$ with $N((Q^T,Q^C,Q^I,Q^F))\subseteq N((R^T,R^C,R^I,R^F))$ and $N(M/(Q^T,Q^C,Q^I,Q^F))\subseteq N((S^T,S^C,S^I,S^F))$, then the subsequent statement are equivalent.

(i) M is k – KSQPNFM with $k = k_1k_2$.

(ii) (Q^T,Q^C,Q^I,Q^F) is k -KS, $M/(Q^T,Q^C,Q^I,Q^F)$ is k -KS,

$N((Q^T,Q^C,Q^I,Q^F)^T)\subseteq N((S^T,S^C,S^I,S^F)^T)$ and $N((M/(Q^T,Q^C,Q^I,Q^F))^T)\subseteq N((R^T,R^C,R^I,R^F)^T)$

(iii) Both the matrices $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (0,0,1,1)\\ (S^T,S^C,S^I,S^F) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}$ and

$M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (0,0,1,1) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}$ are k- kernel symmetric.

Proof:(i)⇒ (ii):

To prove (Q^T,Q^C,Q^I,Q^F) is k is k-KS, $M/(Q^T,Q^C,Q^I,Q^F)$ is k-KS

Let $y_1\in N((Q^T,Q^C,Q^I,Q^F))$ and $y_2\in N(M/(Q^T,Q^C,Q^I,Q^F))$.

Hence $y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1)$ and $y_2(M/(Q^T,Q^C,Q^I,Q^F))=(0,0,1,1)$

Define $y=[y_1\quad y_2]$

we claim that $yM = [y_1\ y_2]\begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F) \\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F) \end{bmatrix} = (0,0,1,1)$

Since $N(M/(Q^T,Q^C,Q^I,Q^F)) \subseteq N(S^T,S^C,S^I,S^F)$,

$y_2(M/(Q^T,Q^C,Q^I,Q^F) = (0,0,1,1)$

Implies, $y_2(S^T,S^C,S^I,S^F) = (0,0,1,1)$

$N((Q^T,Q^C,Q^I,Q^F)) \subseteq N((R^T,R^C,R^I,R^F))$,

$y_1(Q^T,Q^C,Q^I,Q^F) = (0,0,1,1) \Rightarrow y_1(R^T,R^C,R^I,R^F) = (0,0,1,1)$

Hence, $y_1(Q^T,Q^C,Q^I,Q^F) + y_2(S^T,S^C,S^I,S^F) = (0,0,1)$ and

$y_1(R^T,R^C,R^I,R^F) + y_2(T^T,T^C,T^I,T^F) = (0,0,1,1)$.

Therefore yM $= (0,0,1,1)$ i.e) $y \in N(M)$.

Since M is k - KS, $N(M) = N(KM^TK)$

Then, $yKM^TK = (0,0,1,1)$

$$[y_1\ y_2]\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} (Q^T,Q^C,Q^I,Q^F)^T & (R^T,R^C,R^I,R^F)^T \\ (S^T,S^C,S^I,S^F)^T & (T^T,T^C,T^I,T^F)^T \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = (0,0,1,1)$$

$\Rightarrow y_1K(Q^T,Q^C,Q^I,Q^F)^TK + y_2K(R^T,R^C,R^I,R^F)^TK = (0,0,1,1)$

$\Rightarrow y_1K(Q^T,Q^C,Q^I,Q^F)^TK = (0,0,1,1)$ and $y_2K(R^T,R^C,R^I,R^F)^TK = (0,0,1,1)$ and

$\Rightarrow y_1K(S^T,S^C,S^I,S^F)^TK + y_2K(T^T,T^C,T^I,T^F)^TK = (0,0,1,1)$

$\Rightarrow y_1K(S^T,S^C,S^I,S^F)^TK = (0,0,1)$ and $y_2K(T^T,T^C,T^I,T^F)^TK = (0,0,1,1)$

Hence

$y_1 \in N[K(Q^T,Q^C,Q^I,Q^F)K], y_2 \in N[K(R^T,R^C,R^I,R^F)^TK]$ and

$y_2 \in N[K(T^T,T^C,T^I,T^F)K]$

Since $y_1 \in N[(Q^T,Q^C,Q^I,Q^F)]$ and $y_2 \in N[M/(Q^T,Q^C,Q^I,Q^F)]$ it follows that

$N[(Q^T,Q^C,Q^I,Q^F)] \subseteq N[K(Q^T,Q^C,Q^I,Q^F)K]$,

$$N\left[M/(Q^T,Q^C,Q^I,Q^F)\right]\subseteq N\left[K(R^T,R^C,R^I,R^F)^TK\right]$$

$$N\left[M/(Q^T,Q^C,Q^I,Q^F)\right]\subseteq N\left[K(T^T,T^C,T^I,T^F)^TK\right]\text{implies}$$

$$N\left[M/(Q^T,Q^C,Q^I,Q^F)\right]\subseteq N\left[K(M/(Q^T,Q^C,Q^I,Q^F))K\right]$$

Likewise, it may be demonstrated that

$$N\left[K(Q^T,Q^C,Q^I,Q^F)K\right]\subseteq N\left[(Q^T,Q^C,Q^I,Q^F)\right]$$

Thus (Q^T,Q^C,Q^I,Q^F) is k-KS

Since, $y_1\in N\left[K(S^T,S^C,S^I,S^F)K\right]$ and (Q^T,Q^C,Q^I,Q^F) is k-KS

$$N\left[(Q^T,Q^C,Q^I,Q^F)\right]=N\left[K(Q^T,Q^C,Q^I,Q^F)^TK\right]\subseteq N\left[K(S^T,S^C,S^I,S^F)^TK\right]$$

$$N\left[(Q^T,Q^C,Q^I,Q^F)^T\right]\subseteq N\left[(S^T,S^C,S^I,S^F)^T\right]$$

$$M/(Q^T,Q^C,Q^I,Q^F)=(T^T,T^C,T^I,T^F)-(S^T,S^C,S^I,S^F)(Q^T,Q^C,Q^I,Q^F)^-$$

(R^T,R^C,R^I,R^F).

Thus (i) implies (ii) holds

(ii) implies (iii)

$$M_1=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (0,0,1,1)\\ (S^T,S^C,S^I,S^F) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}\text{and}$$

$$M_1=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (0,0,1,1) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}$$

are k-KS.

Let $y\in N(M_1)$. Partition y conformity with that M_1 as $y=[y_1\ y_2]$ then,

$$[y_1\ y_2]\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (0,0,1,1)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}=(0,0,1,1)$$

$$y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1),\ y_2(S^T,S^C,S^I,S^F)=(0,0,1,1),$$

$$\Rightarrow y_2\left[M/(Q^T,Q^C,Q^I,Q^F)\right]=(0,0,1,1)$$

Since (Q^T, Q^C, Q^I, Q^F) and $M/(Q^T, Q^C, Q^I, Q^F)$ are k-KS

$$y_1 \in N\left[(Q^T, Q^C, Q^I, Q^F)\right] = N\left[K(Q^T, Q^C, Q^I, Q^F)^T K\right]$$

$$\Rightarrow y_1 K(Q^T, Q^C, Q^I, Q^F)^T K = (0,0,1,1)$$

$$y_2 \in N\left[M/(Q^T, Q^C, Q^I, Q^F)\right]^T = N\left[K\left(M/(Q^T, Q^C, Q^I, Q^F)\right)^T K\right]$$

$$\Rightarrow y_2\left[K\left(M/(Q^T, Q^C, Q^I, Q^F)\right)^T K\right] = (0,0,1,1)$$

Since, $N\left[(Q^T, Q^C, Q^I, Q^F)^T\right] \subseteq N\left[(S^T, S^C, S^I, S^F)^T\right]$

$$N\left[K(Q^T, Q^C, Q^I, Q^F)^T K\right] \subseteq N\left[K(S^T, S^C, S^I, S^F)^T K\right]$$

$$\Rightarrow y_1 K(S^T, S^C, S^I, S^F)^T K = (0,0,1,1)$$

Now by using $y_1 K(Q^T, Q^C, Q^I, Q^F)^T K = (0,0,1,1), y_1 K(S^T, S^C, S^I, S^F)^T K = (0,0,1,1)$

and $y_2 K\left[M/(Q^T, Q^C, Q^I, Q^F)\right]^T K = (0,0,1,1)$ it can be verified that

$$[y_1\ y_2]\begin{bmatrix} K(Q^T, Q^C, Q^I, Q^F)^T K & K(S^T, S^C, S^I, S^F)K \\ (0,0,1,1) & K(M/(Q^T, Q^C, Q^I, Q^F)^T K \end{bmatrix} = (0,0,1,1)$$

Thus $N\left[M_1\right] \subseteq N\left[KM_1{}^T K\right]$.

$$N\left[KM_1{}^T K\right] \subseteq N\left[M_1\right]$$

Hence Therefore, $N\left[KM_1{}^T K\right] = N\left[M_1\right]$

Hence M_1 is k-KS.

Similarly, it may be demonstrated that M_2 is k-KS.

Thus (ii)⇒ (iii) holds.

(iii)⇒ (i) M_1 is k - KS $\Rightarrow N(M_1) = N(KM^T K)$

M_2 is k- KS⇒ $N(M_2) = N(KM^T K)$

To prove, M is k-KS that is $N(M) = N(KM^T K)$.

Let $y \in N(M) \Rightarrow yM = 0$.

M as y = $[y_1\ y_2]$ then,

$$[y_1\ y_2]\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}=(0,0,1,1)$$

$$y_1(Q^T,Q^C,Q^I,Q^F)+y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$$

$$\Rightarrow y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1), y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$$

$$y_1(R^T,R^C,R^I,R^F)+y_2(T^T,T^C,T^I,T^F)=(0,0,1,1)$$

$$\Rightarrow y_1(R^T,R^C,R^I,R^F)=(0,0,1,1), y_2(T^T,T^C,T^I,T^F)=(0,0,1,1)$$

From the definition of M/ (Q^T,Q^C,Q^I,Q^F) = (T^T,T^C,T^I,T^F) $-(S^T,S^C,S^I,S^F)$

$(Q^T,Q^C,Q^I,Q^F)^-$ (R^T,R^C,R^I,R^F)

We have, $y_2(T^T,T^C,T^I,T^F)=(0,0,1,1), y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$

$$\Rightarrow y_2\left(M/(Q^T,Q^C,Q^I,Q^F)\right)=(0,0,1,1)$$

$y_1(Q^T,Q^C,Q^I,Q^F)+y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$ and $y_2\left(M/(Q^T,Q^C,Q^I,Q^F)\right)=(0,0,1,1)$

And $y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1),$

$$y_1(R^T,R^C,R^I,R^F)+y_2\left[M/(Q^T,Q^C,Q^I,Q^F)\right]=(0,0,1,1)$$

$$y\in N(M_1)\Rightarrow y\in N(KM_1{}^TK)$$

$$y\in N(M_2)\Rightarrow y\in N(KM_2{}^TK)$$

Hence, $y\in N(KM^TK)$.

$$N(M)\subseteq N(KM^TK)$$

Similarly, $N(M)\supseteq N(KM^TK)$

$$N(M) = N(KM^T K)$$

Therefore M is k-KSNFM.

Theorem 4.2 Let M be a QPNFM of the form

$$M = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (R^T, R^C, R^I, R^F) \\ (S^T, S^C, S^I, S^F) & (T^T, T^C, T^I, T^F) \end{bmatrix}$$ with $N((Q^T, Q^C, Q^I, Q^F)^T) \subseteq$

$N((S^T, S^C, S^I, S^F)^T)$ and $N(\left(M/(Q^T, Q^C, Q^I, Q^F)\right)^T \subseteq N((R^T, R^C, R^I, R^F)^T)$, then the subsequence are equivalent.

(i) M is $k-$ KSQPNFM with $k = k_1k_2$.

(ii) (Q^T, Q^C, Q^I, Q^F) is k -KS, $M/(Q^T, Q^C, Q^I, Q^F)$ is k -KS, $N(Q^T, Q^C, Q^I, Q^F) \subseteq$

$$N(R^T, R^C, R^I, R^F) \quad \text{and} \quad N((M/(Q^T, Q^C, Q^I, Q^F)) \subseteq N(S^T, S^C, S^I, S^F)$$

(iii) Both the matrices $M = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (0,0,1,1) \\ (S^T, S^C, S^I, S^F) & M/(Q^T, Q^C, Q^I, Q^F) \end{bmatrix}$ and

$M = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (R^T, R^C, R^I, R^F) \\ (0,0,1,1) & M/(Q^T, Q^C, Q^I, Q^F) \end{bmatrix}$ are k- kernel symmetric.

Proof: This theorem is directly supported by Theorem (4.1) and the observation that M is k-KS⇔ M^T is k - KS.

Theorem 4.3 Let M be a QPNFM of the form $M = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (R^T, R^C, R^I, R^F) \\ (S^T, S^C, S^I, S^F) & (T^T, T^C, T^I, T^F) \end{bmatrix}$

with $N((Q^T, Q^C, Q^I, Q^F)) \subseteq N((R^T, R^C, R^I, R^F)^T)$ and $N(\left(M/(Q^T, Q^C, Q^I, Q^F)\right) \subseteq$

$N((R^T, R^C, R^I, R^F))$, then the following are equivalent.

(i) M is $k-$ KSQPNFM with $k = k_1k_2$.

(ii) (Q^T, Q^C, Q^I, Q^F) is k -KS, $M/(Q^T, Q^C, Q^I, Q^F)$ is k -KS,

(iii) The matrices $M = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (0,0,1,1) \\ (S^T, S^C, S^I, S^F) & M/(Q^T, Q^C, Q^I, Q^F) \end{bmatrix}$ is k- KS.

Remark 4.1 It is crucial to consider the condition that is placed on M in Theorems 4.1 and 4.2. The example that follows serves to illustrate this.

Example:4.1 Let us consider a QPNFM

$$M=\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\end{bmatrix} \text{ and}$$

$$M=\begin{bmatrix} & v_1 & v_2 & v_3 & v_4\\ v_1 & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\ v_2 & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\ v_3 & (1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (1,1,0,0)\\ v_4 & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\end{bmatrix}$$

$$K=\begin{bmatrix}(0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$$

$$(Q^T,Q^C,Q^I,Q^F)=\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},(R^T,R^C,R^I,R^F)=\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},$$

$$(S^T,S^C,S^I,S^F)=\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},\ (T^T,T^C,T^I,T^F)=\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}$$

For this M, since M has ho zero rows and no zero columns N(M) = (0,0,1,1).

$N(KM^TK)$ = (0,0,1,1).Thus N(M) = $N(KM^TK) \Rightarrow$ M is k- KS.

$(Q^T,Q^C,Q^I,Q^F)^-=\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}$ is a g-inverse, with respect to $(Q^T,Q^C,Q^I,Q^F)^-$

$M/(Q^T,Q^C,Q^I,Q^F) = (T^T,T^C,T^I,T^F) - (S^T,S^C,S^I,S^F)\ (Q^T,Q^C,Q^I,Q^F)^-$

(R^T,R^C,R^I,R^F).

$$M/(Q^T,Q^C,Q^I,Q^F) =\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix} -\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}$$

$$\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},$$

$$M/(Q^T,Q^C,Q^I,Q^F) = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) \\ (1,1,0,0) & (1,1,0,0) \end{bmatrix},$$

$M/(Q^T,Q^C,Q^I,Q^F)$ is k - KS,

Since $N(M/(Q^T,Q^C,Q^I,Q^F)) = N(K(M/(Q^T,Q^C,Q^I,Q^F))^T K) = (0,0,1,1)$.

(Q^T,Q^C,Q^I,Q^F) is k - KS, since $N(M) = N(KM^T K) = (0,0,1,1)$ for all K.

$N(\ (Q^T,Q^C,Q^I,Q^F)\ ) \subseteq N(\ (R^T,R^C,R^I,R^F)\ )$ and $N(\ (Q^T,Q^C,Q^I,Q^F)^T\ ) \subseteq N((S^T,S^C,S^I,S^F)^T)$.

Here, $N(M/(Q^T,Q^C,Q^I,Q^F)) = (0,0,1,1) = N((M/(Q^T,Q^C,Q^I,Q^F))^T)$,

$N((S^T,S^C,S^I,S^F)) = (0,0,1,1)$, $N((R^T,R^C,R^I,R^F)^T) = (0,0,1,1)$

$N(M/(Q^T,Q^C,Q^I,Q^F))$ contained in $N((S^T,S^C,S^I,S^F)^T)$ and $N((M/(Q^T,Q^C,Q^I,Q^F))^T$ contained in $N((R^T,R^C,R^I,R^F)^T)$.

Further

$$M_1 = \begin{bmatrix} (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \end{bmatrix} \text{ and}$$

$$K = \begin{bmatrix} (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \end{bmatrix}$$

$N(M_1) = (0,0,1,1)$.

$$KM_1^T K = \begin{bmatrix} (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \end{bmatrix}$$

$$\begin{bmatrix} (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) \\ (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1) \end{bmatrix} \begin{bmatrix} (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \end{bmatrix}$$

$$KM_1{}^T K = \begin{bmatrix} (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1) \\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \end{bmatrix}$$

$$N(KM_1{}^T K) = \left\{(0,0,1,1),(0,0,1,1),(0,0,1,1),(Q^T,Q^C,Q^I,Q^F) : (Q^T,Q^C,Q^I,Q^F) \in (QPNFM)\right\}$$

$\Rightarrow M_1$ is not k- Kernel Symmetric.

$$M_2 = \begin{bmatrix} (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1) \\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \end{bmatrix}$$

M_2 is not k- Kernel Symmetric.

Remark:4.2 For a KSQPNFM M of the form $M = \begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F) \\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F) \end{bmatrix}$

with $k = k_1k_2$ following are equivalent.

$N($ (Q^T,Q^C,Q^I,Q^F) $) \subseteq N($ (R^T,R^C,R^I,R^F) $)$ and $N($M/ (Q^T,Q^C,Q^I,Q^F) $) \subseteq N((S^T,S^C,S^I,S^F))$,

$N($ $(Q^T,Q^C,Q^I,Q^F)^T$ $) \subseteq N($ $(S^T,S^C,S^I,S^F)^T$ $)$, and $N((M/$ (Q^T,Q^C,Q^I,Q^F) $)^T) \subseteq N((R^T,R^C,R^I,R^F)^T)$

Example:4.2 Let us consider a QPNFM

$$M = \begin{bmatrix} (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (1,1,0,0) \end{bmatrix} \text{ and}$$

$$K = \begin{bmatrix} (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \end{bmatrix}$$

$$N(M)=\left\{(0,0,1,1),(0,0,1,1),(Q^T,Q^C,Q^I,Q^F),(0,0,1,1):(Q^T,Q^C,Q^I,Q^F)\in(QPNFM)\right\}$$

$$KM^TK=\begin{bmatrix}(0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (1,1,0,0)\\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1) & (0,0,1) & (1,1,0,0)\\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (1,1,0,0)\end{bmatrix}\begin{bmatrix}(0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$KP^TK=\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\\ (0,0,1,1) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$N(KM^TK)=(0,0,1,1)$$

$$N(M)\neq N(KM^TK)$$

Therefore M is not k - KS.

$$\text{P}/(Q^T,Q^C,Q^I,Q^F)=\begin{bmatrix}(0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

Here

$$N(Q^T,Q^C,Q^I,Q^F)\subseteq N(R^T,R^C,R^I,R^F),N(M/(Q^T,Q^C,Q^I,Q^F))\subseteq N(S^T,S^C,S^I,S^F)$$

But $N(Q^T,Q^C,Q^I,Q^F)^T$ is not contained $N(S^T,S^C,S^I,S^F)^T$.

$N(M/(Q^T,Q^C,Q^I,Q^F))^T$ is not contained $N(R^T,R^C,R^I,R^F)^T$.

5. Application of Fuzzy QuadripartitionedNeutrosophic Soft Matrix in Medical Decision Making Problem.

The process of medical decision-making is among the most complex, as it requires comprehensive knowledge of a patient's medical history, the symptoms they are experiencing, and the treatments they have undergone under the guidance of medical experts. Often, the information provided by patients can be incomplete, unclear, or ambiguous, making it challenging for healthcare professionals to accurately diagnose the condition. This uncertainty may lead to misdiagnosis and improper treatment. To address this, experts should compile a thorough list of the patient's

symptoms and continuously monitor their health status from the onset. In cases involving complex or atypical symptoms, a panel of specialists may be required for an accurate diagnosis and effective treatment.

While biopsies are widely regarded as a definitive diagnostic tool, they are invasive and come with certain risks. It is essential to develop methodologies that can minimize the need for such procedures, especially for patients who are not at significant risk. Given the inherent uncertainty in medical treatment, the application of Fuzzy Quadripartition Neutrosophic Soft Sets (FQNSSs) is proposed. This approach offers a high-precision decision-making framework, which is crucial for accurate medical investigations. Below, we outline an algorithm to apply FQNSSs in the context of medical diagnosis.

Algorithm:

Step 1- Input the FQNSSs (M,S) and (N,D) called the patient- symptom and symptom-disease sets, respectively, and write their corresponding matrices P,Q.

Step 2- Compute $P*Q$ and $P\nabla*Q$.

$$P*Q =< \max\left(\min(T^P,T^Q)\right), \max\left(\min(C^P,C^Q)\right), \min\left(\max(U^P,U^Q)\right), \min\left(\max(F^P,F^Q)\right) >$$

$$P\nabla*Q =< \max\left\{\frac{T^A+T^B}{2}\right\}, \max\left\{\frac{C^A+C^B}{2}\right\}, \min\left\{\frac{U^A+U^B}{2}\right\}, \min\left\{\frac{F^A+F^B}{2}\right\} >$$

Step 3- Compute $S\left(P*Q\right)$ and $S\left(P\nabla*Q\right)$

$$S\left(P*Q\right) =< \frac{T^{(P*Q)}+C^{(P*Q)}-U^{(P*Q)}-F^{(P*Q)}}{2} >.$$

$$S\left(P\nabla*Q\right) =< \frac{T^{(P*Q)}+C^{(P*Q)}-U^{(P*Q)}-F^{(P*Q)}}{2} >$$

Step 4- Find the total score T_S = $S\left(P*Q\right)+S\left(P\nabla*Q\right)$

Step 5- Identify the maximum total score T_S for each patient P_i and conclude that the patient P_i is surely suffering from the disease D_j.

5.1 Example

Suppose there are three patients denoted by the set P ={ p_1, p_2, p_3 } with symptoms denoted by the set S = e_1: Fewer; e_2: Muscle aches, e_3: Fatigue; Let the possible diseases denoted by d_1 = Dengue fewer ; d_2 = Malaria, d_3 = Typhoid. Let the FQNSS (M,S), S over P and Q is given by

$$P = \begin{pmatrix} <0.4,0.3,0.5,0.6> & <0.2,0.4,0.3,0.4> & <0.3,0.6,0.4,0.5> \\ <0.2,0.4,0.7,0.3> & <0.4,0.6,0.5,0.4> & <0.7,0.2,0.1,0.2> \\ <0.5,0.4,0.1,0.5> & <0.6,0.7,0.3,0.8> & <0.5,0.3,0.6,0.7> \end{pmatrix}$$

$$Q=\begin{pmatrix} <0.5,0.4,0.7,0.2> & <0.2,0.5,0.6,0.5> & <0.4,0.5,0.6,0.7> \\ <0.6,0.2,0.4,0.4> & <0.6,0.4,0.7,0.2> & <0.6,0.3,0.4,0.5> \\ <0.2,0.6,0.4,0.6> & <0.5,0.6,0.4,0.5> & <0.4,0.2,0.4,0.6> \end{pmatrix}$$

$$P*Q=\begin{pmatrix} <0.4,0.6,0.4,0.4> & <0.3,0.6,0.4,0.4> & <0.4,0.3,0.4,0.5> \\ <0.2,0.4,0.4,0.3> & <0.5,0.4,0.4,0.4> & <0.4,0.4,0.4,0.5> \\ <0.6,0.4,0.4,0.5> & <0.6,0.5,0.6,0.5> & <0.6,0.4,0.4,0.7> \end{pmatrix}$$

$$P\nabla*Q=\begin{pmatrix} <0.45,0.6,0.35,0.4> & <0.4,0.6,0.4,0.3> & <0.4,0.4,0.35,0.45> \\ <0.5,0.4,0.25,0.25> & <0.6,0.5,0.25,0.3> & <0.55,0.45,0.25,0.4> \\ <0.6,0.45,0.35,0.35> & <0.6,0.55,0.35,0.5> & <0.6,0.5,0.35,0.6> \end{pmatrix}$$

$$S\left(P*Q\right)=\begin{pmatrix} 0.1 & 0.05 & -0.1 \\ -0.05 & 0.05 & -0.05 \\ 0.05 & 0 & -0.05 \end{pmatrix}$$

$$S\left(P\nabla*Q\right)=\begin{pmatrix} 0.15 & 0.15 & 0 \\ 0.2 & 0.27 & 0.17 \\ 0.17 & 0.15 & 0.07 \end{pmatrix}$$

$$T_S=S\left(P*Q\right)+S\left(P\nabla*Q\right)=\begin{pmatrix} & d_1 & d_2 & d_3 \\ p_1 & 0.25 & 0.2 & -0.1 \\ p_2 & 0.15 & 0.32 & 0.12 \\ p_3 & 0.22 & 0.15 & 0.02 \end{pmatrix}$$

From the given matrix T_S, it is evident that patients p_1, p_2, and p_3 are diagnosed with diseases d_1, d_2, and d_2, respectively. Furthermore, no patient has been identified with disease d_3.

Conclusion: 6

This work presents theorems that describe the properties of k-kernel symmetry (k-KS) and the Schur complement in the context of k-KSQPNFMs. We introduce the concepts of kernel symmetry (KS) and k-KS QPNFMs, exploring various properties and providing examples to illustrate these findings. Several equivalent characterizations of kernel symmetric and k-KSNFMs are discussed, along with fundamental examples that enhance the understanding of KSQPNFMs. It is shown that while k-symmetry implies k-KS, the reverse implication does not hold. Future work will aim to prove additional properties related to the generalized inverses of the Schur complement and k-kernel symmetric QPNFMs.

An algorithm has been developed to facilitate real-world decision-making within the FQNSM framework. To demonstrate its practical utility, the algorithm has been successfully applied to a

medical diagnosis problem. Future research could explore extending this work by incorporating interval-based FQNSM theory, where the truth, contradiction, ignorance, and false membership degrees are represented as intervals rather than crisp values. Additionally, the proposed approach has potential applications in areas such as game theory, similarity measures, risk management, and group decision-making problems.

References

[1]. Zadeh L.A., Fuzzy Sets, Information and control.,(1965),,8, pp. 338-353.

[2]. K.Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Physica-Verlag, 1999.

[3]. Smarandache,F, Neutrosophic set, a generalization of the intuitionistic fuzzy set. Int J Pure Appl Math.; .,(2005),.24(3):287–297.

[4]. K. H. Kim and F. W. Roush, "Generalized fuzzy matrices," Fuzzy Sets and Systems, vol. 4, no. 3, pp. 293–315, 1980.

[5]. A. R. Meenakshi, Fuzzy Matrix: Theory and Applications, MJP, Chennai, India, 2008.

[6]. R. D. Hill and S. R. Waters, "On κ-real and κ-Hermitian matrices," Linear Algebra and Its Applications, vol. 169, pp. 17–29, 1992.

[7]. T. S. Baskett and I. J. Katz, "Theorems on products of EPr matrices," Linear Algebra and Its Applications, vol. 2, pp. 87–103, 1969.

[8]. A. R. Meenakshi and S. Krishnamoorthy, "On κ-EP matrices," Linear Algebra and Its Applications, vol. 269, pp. 219–232, 1998.

[9]. Meenakshi AR and Krishnamoorthy S, On Schur complement in k-EP matrices, Indian J.Pure appl.math (2002) 1889-1902.

[10]. A. R. Meenakshi and D. Jaya Shree On Schur Complement in k-Kernel Symmetric Matrices, Int. Journal of Math. Analysis, Vol. 4, 2010, no. 7, 331 – 339.

[11]. D.Jaya shree , Secondary κ-Kernel Symmetric Fuzzy Matrices, Intern. J. Fuzzy Mathematical Archive Vol. 5, No. 2, 2014, 89-94 ISSN: 2320 –3242 (P), 2320 –3250 ,Published on 20 December 2014.

[12]. A. K. Shyamal and M. Pal, Interval valued Fuzzy matrices, Journal of Fuzzy Mathematics 14(3) (2006), 582-592.

[13]. An Lee, Secondary Symmetric, Secondary Skew Symmetric, Secondary Orthogonal Matrices, Period Math, Hungary, 7 (1976) 63-76.

[14]. C.Antonio and B.Paul, Properties of the eigen vectors of persymmetric matrices with applications to communication theory, IEEE Trans. Comm., 24 (1976) 804 –809.

[15]. AR.Meenakshi and S.Krishanmoorthy, On Secondary k-Hermitian matrices, Journal of Modern Science, 1 (2009) 70-78.

[16]. AR. Meenakshi, S.Krishnamoorthy and G.Ramesh, On s-k-EP matrices, Journal of Intelligent System Research, 2(2008) 93-100.

[17]. AR.Meenakshi and D.Jaya Shree, On k-kernel symmetric matrices, International Journal of Mathematics and Mathematical Sciences, 2009, Article ID 926217, 8 Pages.

[18]. D.Jaya shree , Secondary κ-Kernel Symmetric Fuzzy Matrices, Intern. J. Fuzzy Mathematical Archive Vol. 5, No. 2, 2014, 89-94 ISSN: 2320 –3242 (P), 2320 –3250 ,Published on 20 December 2014.

[19]. AR.Meenakshi and D.Jaya Shree, On K -range symmetric matrices, Proceedings of the National conference on Algebra and Graph Theory, MS University, (2009), 58- 67.

[20]. A. K. Shyamal and M. Pal, Interval valued Fuzzy matrices, Journal of Fuzzy Mathematics 14(3) (2006), 582-592.

[21]. A. R. Meenakshi and M. Kalliraja, Regular Interval valued Fuzzy matrices, Advance in Fuzzy Mathematics 5(1) (2010), 7-15.

[22]. Atanassov K., , Intuitionistic Fuzzy Sets, Fuzzy Sets and System. (1983), 20, pp. 87- 96.

[23]. M.Anandhkumar; G.Punithavalli; T.Soupramanien; Said Broumi, Generalized Symmetric Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 57,2023, 57, pp. 114–127.

[24]. Anandhkumar, M.; G. Punithavalli; and E. Janaki. "Secondary k-column symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 64, 1 (2024).

[25]. Anandhkumar, M.; G. Punithavalli; R. Jegan; and Said Broumi. "Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 61, 1 (2024).

[26]. D. Jaya Shree, Secondary κ-range symmetric fuzzy matrices, Journal of Discrete Mathematical Sciences and Cryptography 21(1):1-11,2018

[27]. Anandhkumar, M., Kamalakannan, V., Chithra, S.M., Said, B., "Pseudo Similarity of Neutrosophic Fuzzy matrices", International Journal of Neutrosophic Science, Vol. 20, No. 04, PP. 191-196, 2023.

[28]. Anandhkumar, M., Kanimozhi, B., Chithra, S.M., Kamalakannan, V., Said, B., "On various Inverse of Neutrosophic Fuzzy Matrices", International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023.

[29]. Anandhkumar, M., Harikrishnan, T., Chithra, S.M., ...Kanimozhi, B., Said, B. "Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices"International Journal of Neutrosophic Science, 2023, 21(4), pp. 135–145.

[30]. G. Punithavalli, M. Anandhkumar, Kernel and K-Kernel Symmetric Intuitionistic Fuzzy Matrices,TWMS J. App. and Eng. Math. V.14, N.3, 2024, pp. 1231-1240.

[31]. M. Anandhkumar, T. Harikrishnan, S. M. Chithra, V. Kamalakannan, B. Kanimozhi, Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices, International Journal of Neutrosophic Science, Vol. 23, no. 2, 2024, pp. 286-295.

[32]. M.Anandhkumar, B.Kanimozhi, S.M. Chithra, V.Kamalakannan, Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic fuzzy matrices, Mathematical Modelling of Engineering Problems, 2023, 10(4), pp. 1427–1432.

[33]. M. Anandhkumar, H. Prathab, S. M. Chithra, A. S. Prakaash, A. Bobin, Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, vol. 23, no. 4, 2024, pp. 23-28.

[34]. Anandhkumar, M.; A. Bobin; S. M. Chithra; and V. Kamalakannan. "Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 70, 1 (2024).

[35]. G. Punithavalli , M. Anandhkumar, Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices, TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1772-1783.

Received: Aug 23, 2024. Accepted: Nov 25, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

DVNN-CoCoSo: A Hybrid Neutrosophic Model for Evaluating Social and Economic Impacts of Sports Events

Changshun Bai, Pute Lan*
Physical Education Department of Inner Mongolia Medical University, Hohhot, 010010, Inner Mongolia, China (E-mail: 18104853118@163.com)
*Corresponding author E-mail: 20120171@immu.edu.cn

Abstract

The evaluation of the social benefits of major sports events is a multifaceted process, focusing on assessing the positive impacts such events bring to a host city and society. These benefits include promoting public health, fostering national pride and social cohesion, boosting local economies, and improving infrastructure. This evaluation process can be viewed as a multiple-attribute decision-making (MADM) problem. This study introduces a novel double-valued neutrosophic number CoCoSo (DVNN-CoCoSo) technique for MADM, incorporating DVNN Hamming distance (DVNNHD) and DVNN Logarithmic distance (DVNNLD). This method effectively handles the uncertainty inherent in evaluating social benefits. A numerical example of social benefits evaluation for major sports events illustrates the practicality and efficacy of the DVNN-CoCoSo technique. The study's major contributions include the development of the DVNN-CoCoSo technique for MADM, the consideration of objective weights through entropy techniques, and the application of this novel method to the social benefits evaluation of major sports events.

Keywords: Multiple-attribute decision-making (MADM); double-valued neutrosophic sets (DVNSs); CoCoSo technique; social benefits evaluation of major sports events

1. Introduction and Literature Review

The evaluation of the social benefits of major sports events refers to the systematic analysis and assessment of the impact these events have on various aspects of society. Its primary goal is to

measure the contributions of the event to the hosting city and society in areas such as economy, culture, environment, and public health. Firstly, these events can promote the development of urban infrastructure, improving transportation, venues, and facilities, thereby enhancing the living conditions of local residents. Secondly, sports events boost the city's brand image and international influence, attracting more tourists and investments, which in turn stimulates related industries such as tourism, hospitality, and dining. The evaluation of social benefits also includes cultural impact, such as raising public awareness of physical fitness, strengthening social cohesion, and increasing community participation. Additionally, hosting such events can improve publicity and cultivate a sense of social responsibility. Through a scientific evaluation system, both quantitative and qualitative analyses can be conducted to ensure the long-term positive effects of the event on society. This evaluation also provides a basis for the planning and management of future sports events. Effective social benefit evaluations help governments, businesses, and the public better understand and leverage the comprehensive benefits brought by major sports events. The earliest research was conducted by Zhai and Zhou [1], who explored the evaluation methods of social benefits for host cities of major sports events. The study developed an indicator system with 3 primary indicators, 10 secondary indicators, and 44 tertiary indicators, covering the impacts of events on urban infrastructure, economic development, social stability, and cultural life. The research employed methods such as literature analysis, expert interviews, surveys, and logical analysis, and used the Analytic Hierarchy Process (AHP) to determine the weight of each indicator. This research provided a scientific basis for evaluating the social benefits of major sports events and offered essential references for future event planning and management. In 2011, Du and Zhang [2] further examined the social benefits of major sports events from three dimensions: ecological environment, social spirit, and material culture. The paper emphasized that if the government strictly adhered to event environmental requirements and focused on reusing sports facilities and promoting the role of sports stars as role models, major events could positively impact the local ecological environment, social civilization, and balanced development of urban and rural sports facilities. This study provided a multi-dimensional reference for assessing social benefits in future major sports events. In 2020, Li, Zhang and Lü [3] focused on evaluating the social benefits of meteorological services in major

sports events. Using the 2014 Nanjing Youth Olympic Games as a case study, they summarized the technical methods for such evaluations. By selecting research objects, applying social survey methods, and designing evaluation indicators, they proposed how to meet service demands during events, collect evaluation data, and conduct phased assessments. This research provided a framework for effectively evaluating meteorological services in future major sports events.

MADM is a method used to solve problems where choices must be made based on multiple, often conflicting, attributes or criteria[4-6]. This approach is widely applied in complex decision-making scenarios, helping decision-makers select the optimal option or rank alternatives. Each alternative is evaluated based on a set of attributes or criteria, which may involve both quantitative and qualitative data, and typically carry different levels of importance[7-10]. To make a decision, weights are usually assigned to each attribute to reflect its relative significance. By combining the scores of each attribute with its corresponding weight, an overall score for each alternative is calculated, enabling comparison and evaluation. Common MADM methods include Analytic Hierarchy Process (AHP), TOPSIS, and Fuzzy Comprehensive Evaluation. These methods utilize different mathematical models and algorithms to help decision-makers handle uncertainty, ambiguity, and preferences in a more scientific manner [11-14]. In practice, MADM is widely used in areas such as project evaluation, investment selection, resource allocation, and supplier selection. It effectively enhances the systematization, rationality, and scientific basis of decision-making, ensuring that final decisions better align with actual needs and preferences. The social benefits evaluation of major sports events is regarded as the defined MADM. Recently, the CoCoSo [15] and entropy [16] has been used to cope with MADM. The DVNSs [17] are used as a technique for characterizing fuzzy information during the social benefits evaluation of major sports events. Furthermore, many techniques employed CoCoSo technique [15] and entropy [16] separately to manage the MADM. Until now, no or few techniques have been administrated on entropy technique [16] and CoCoSo [15] under DVNSs. Therefore, the DVNN-CoCoSo model is founded to manage the MADM. Finally, numerical example for social benefits evaluation of major sports events and comparative analysis is administrated to validate the DVNN-CoCoSo model. The major research motivation of this work is managed: (1) the novel MADM is put forward based on CoCoSo and

entropy technique under DVNSs; (2) The objective weights are considered through entropy technique; (3) The new MADM technique based on DVNN-CoCoSo technique is proposed for social benefits evaluation of major sports events; (4) numerical example for social benefits evaluation of major sports events and comparative analysis are employed to prove the DVNN-CoCoSo model.

The structure of this paper is as follows: Section 1 provides the introduction and literature review. Section 2 introduces the DVNSs. In Section 3, the DVNN-CoCoSo technique is applied for MADM. Section 4 presents numerical examples for evaluating blended teaching quality decisions and includes a comparative analysis. Finally, the conclusions and future work are discussed in Section 5.

2. Preliminaries and Definitions

Double-valued neutrosophic sets (DVNSs), introduced by Kandasamy [17], are an extension of traditional neutrosophic sets, where each element is characterized by two independent values for truth, indeterminacy, and falsity. This dual representation provides a more comprehensive framework for handling uncertainty and vagueness in data. DVNSs have been applied in various fields, including clustering algorithms and graph-based problems, demonstrating their effectiveness in complex decision-making situations.

Definition 1 [17].

The DVNSs ZA in Θ as:

$$ZA=\left\{\left(\theta,ZT_A(\theta),ZIT_A(\theta),ZIF_A(\theta),ZF_A(\theta)\right)\middle|\theta\in\Theta\right\}. \tag{1}$$

where $ZT_A(\theta)$ is truth-membership, $ZIT_A(\theta)$ is indeterminacy leaning towards $ZT_A(\theta)$, $ZIF_A(\theta)$ is indeterminacy leaning towards $ZF_A(\theta)$, $ZF_A(\theta)$ is falsity-membership, $ZT_A(\theta),ZIT_A(\theta),ZIF_A(\theta),ZF_A(\theta)\in[0,1]$, $0\le ZT_A(\theta)+ZIT_A(\theta)+ZIF_A(\theta)+ZF_A(\theta)\le 4$

The DVNN is expressed as $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$, where $ZT_A,ZIT_A,ZIF_A,ZF_A\in[0,1]$, $0\le ZT_A+ZIT_A+ZIF_A+ZF_A\le 4$.

Definition 2 [17].

Let $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$ be the DVNN, the score value is:

$$SV\left(ZA\right)=\frac{\left(2+ZT_A+ZIT_A-ZIF_A-ZF_A\right)}{4},\ SV\left(RA\right)\in\left[0,1\right]. \tag{2}$$

Definition 3[17].

Let $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$ be DVNN, the accuracy value is controlled:

$$AV\left(ZA\right)=\frac{\left(ZT_A+ZIT_A+ZIF_A+ZF_A\right)}{4},\ AV\left(ZA\right)\in\left[0,1\right]\ . \tag{3}$$

The order for DVNNs is managed.

Definition 4[17].

Let $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$ and $ZB=\left(ZT_B,ZT_B,ZIF_B,ZF_B\right)$, $SV\left(ZA\right)=\frac{\left(2+ZT_A+ZIT_A-ZIF_A-ZF_A\right)}{4}$, $SV\left(ZB\right)=\frac{\left(2+ZT_B+ZIT_B-ZIF_B-ZF_B\right)}{4}$, $AV\left(ZA\right)=\frac{\left(ZT_A+ZIT_A+ZIF_A+ZF_A\right)}{4}$, $AV\left(ZB\right)=\frac{\left(ZT_B+ZIT_B+ZIF_B+ZF_B\right)}{4}$, if $SV\left(ZA\right)<SV\left(ZB\right)$, $ZA<ZB$; if $SV\left(ZA\right)=SV\left(ZB\right)$, (1)if $AV\left(ZA\right)=AV\left(ZB\right)$, $ZA=ZB$; (2) if $AV\left(ZA\right)<AV\left(ZB\right)$, $ZA<ZB$.

Definition 5 [17].

Let $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$ and $ZB=\left(ZT_B,ZT_B,ZIF_B,ZF_B\right)$ be two DVNNs, the operations are illustrated below as:

(1) $ZA\oplus ZB=\left(ZT_A+ZT_B-ZT_AZT_B,ZIT_A+ZIT_B-ZIT_AZIT_B,ZIF_AZIF_B,ZF_AZF_B\right)$;

(2) $ZA\otimes ZB=\left(ZT_AZT_B,ZIT_AZIT_B,ZIF_A+ZIF_B-ZIF_AZIF_B,ZF_A+ZF_B-ZF_AZF_B\right)$;

(3) $\lambda ZA=\left(1-\left(1-ZT_A\right)^{\lambda},1-\left(1-ZIT_A\right)^{\lambda},\left(ZIF_A\right)^{\lambda},\left(ZF_A\right)^{\lambda}\right),\lambda>0$;

(4) $\left(ZA\right)^{\lambda}=\left(\left(ZT_A\right)^{\lambda},\left(ZIT_A\right)^{\lambda},1-\left(1-ZIF_A\right)^{\lambda},1-\left(1-ZF_A\right)^{\lambda}\right),\lambda>0.$

Definition 6 [17].

Let $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$ and $ZB=\left(ZT_B,ZT_B,ZIF_B,ZF_B\right)$, the DVNN Hamming distance (DVNNHD) between $ZA=\left(ZT_A,ZT_A,ZIF_A,ZF_A\right)$ and $ZB=\left(ZT_B,ZT_B,ZIF_B,ZF_B\right)$ is:

$$DVNNHD\left(ZA,ZB\right)=\frac{1}{4}\left(\begin{matrix}\left|ZT_A-ZT_B\right|+\left|ZIT_A-ZIT_B\right|\\+\left|ZIF_A-ZIF_B\right|+\left|ZF_A-ZF_B\right|\end{matrix}\right) \tag{4}$$

Definition 7.

Let $ZA=\left(ZT_A, ZT_A, ZIF_A, ZF_A\right)$ and $ZB=\left(ZT_B, ZT_B, ZIF_B, ZF_B\right)$, the DVNN Logarithmic distance (DVNNLD) between $ZA=\left(ZT_A, ZT_A, ZIF_A, ZF_A\right)$ and $ZB=\left(ZT_B, ZT_B, ZIF_B, ZF_B\right)$ is administrated:

$$
\begin{aligned}
&DVNNLD\left(ZA, ZB\right)\\
&=\frac{1}{4}\left(\begin{array}{l}
ZT_A \log \frac{2ZT_A}{ZT_A+ZT_B}+ZT_B \log \frac{2ZT_B}{ZT_A+ZT_B}\\
+ZIT_A \log \frac{2ZIT_A}{ZIT_A+ZIT_B}+ZIT_B \log \frac{2ZIT_B}{ZIT_A+ZIT_B}\\
+ZIF_A \log \frac{2ZIF_A}{ZIF_A+ZIF_B}+ZIF_B \log \frac{2ZIF_B}{ZIF_A+ZIF_B}\\
+ZF_A \log \frac{2ZF_A}{ZF_A+ZF_B}+ZF_B \log \frac{2ZF_B}{ZF_A+ZF_B}
\end{array}\right)
\end{aligned}
\tag{5}
$$

3. DVNN-CoCoSo approach for MADM with entropy weight

The DVNN-CoCoSo technique is specifically designed for MADM problems. It involves several key steps:

- Formulating the decision matrix with DVNNs.
- Normalizing the matrix.
- Determining attribute weights using the CRITIC method [20].
- Calculating DVNNWAI and DVNNWGI.
- Implementing three sorting strategies for relative importance.
- Ranking alternatives based on aggregated scores.

This method provides a structured and comprehensive approach to decision-making in complex situations.

Let $ZA=\left\{ZA_1, ZA_2, \cdots, ZA_m\right\}$ be alternatives, $ZG=\left\{ZG_1, ZG_2, \cdots, ZG_n\right\}$ be attributes with weight zw, where $zw_j \in [0,1]$, $\sum_{j=1}^{n} zw_j = 1$. Suppose that assessed information are DVNNs $ZM=\left(ZM_{ij}\right)_{m\times n}=\left(ZT_{ij}, ZIT_{ij}, ZIF_{ij}, ZF_{ij}\right)_{m\times n}$.

Step 1. The DVNN-matrix $ZM=\left(ZM_{ij}\right)_{m\times n}=\left(ZT_{ij}, ZIT_{ij}, ZIF_{ij}, ZF_{ij}\right)_{m\times n}$.

$$ZM = \left[ZM_{ij} \right]_{m\times n} = \begin{bmatrix} ZM_{11} & ZM_{12} & \cdots & ZM_{1n} \\ ZM_{21} & ZM_{22} & \cdots & ZM_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ZM_{m1} & ZM_{m2} & \cdots & ZM_{mn} \end{bmatrix} \tag{6}$$

$$ZM_{ij} = \left(ZT_{ij}, ZIT_{ij}, ZIT_{ij}, ZF_{ij} \right) \tag{7}$$

Step 2. Normalize the $ZR = \left(ZR_{ij} \right)_{m\times n} = \left(ZT_{ij}, ZIT_{ij}, ZIF_{ij}, ZF_{ij} \right)_{m\times n}$ into $NZM = \left(NZM_{ij} \right)_{m\times n} = \left(ZT_{ij}^{N}, ZIT_{ij}^{N}, ZIF_{ij}^{N}, ZF_{ij}^{N} \right)_{m\times n}$.

$$\begin{aligned} NZM_{ij} &= \left(ZT_{ij}^{N}, ZIT_{ij}^{N}, ZIF_{ij}^{N}, ZF_{ij}^{N} \right) \\ &= \begin{cases} \left(ZT_{ij}, ZIT_{ij}, ZIF_{ij}, ZF_{ij} \right), & ZG_j \text{ is benefit attribute} \\ \left(ZF_{ij}, ZIF_{ij}, ZIT_{ij}, ZT_{ij} \right), & ZG_j \text{ is cost attribute} \end{cases} \end{aligned} \tag{8}$$

Step 3. Implement the DVNN negative ideal decision solution (DVNNNIDS) as:

$$DVNNNIDS = \left(DVNNNIDS_1, DVNNNIDS_2, \cdots, DVNNNIDS_n \right) \tag{9}$$

$$DVNNNIDS_j = \left(ZT_j^{N-}, ZIT_j^{N-}, ZIF_j^{N-}, ZF_j^{N-} \right) \tag{10}$$

$$DVNNSV\left(DVNNNIDS_j \right) = \min_i DVNNSV\left(ZT_{ij}^{N}, ZIT_{ij}^{N}, ZIF_{ij}^{N}, ZF_{ij}^{N} \right) \tag{11}$$

Step 4. The CRITIC [18] is utilized to obtain the objective weight.

(1) Apply the DVNNHD and DVNNLD of ZA_i from DVNNNIDS.

$$\begin{aligned} &DVNNHD\left(NZM_{ij}, DVNNNIDS_j \right) \\ &= \frac{1}{4}\left(\begin{matrix} \left| ZT_{ij}^{N} - ZT_j^{N-} \right| + \left| ZIT_{ij}^{N} - ZIT_j^{N-} \right| \\ + \left| ZIF_{ij}^{N} - ZIF_j^{N-} \right| + \left| ZF_{ij}^{N} - ZF_j^{N-} \right| \end{matrix} \right) \end{aligned} \tag{12}$$

$$\begin{aligned} &DVNNLD\left(NZM_{ij}, DVNNNIDS_j \right) \\ &= \frac{1}{4}\left(\begin{matrix} ZT_{ij}^{N} \log \frac{2ZT_{ij}^{N}}{ZT_{ij}^{N} + ZT_j^{N-}} + ZT_j^{N-} \log \frac{2ZT_j^{N-}}{ZT_{ij}^{N} + ZT_j^{N-}} \\ + ZIT_{ij}^{N} \log \frac{2ZIT_{ij}^{N}}{ZIT_{ij}^{N} + ZIT_j^{N-}} + ZIT_j^{N-} \log \frac{2ZIT_j^{N-}}{ZIT_{ij}^{N} + ZIT_j^{N-}} \\ + ZIF_{ij}^{N} \log \frac{2ZIF_{ij}^{N}}{ZIF_{ij}^{N} + ZIF_j^{N-}} + ZIF_j^{N-} \log \frac{2ZIF_j^{N-}}{ZIF_{ij}^{N} + ZIF_j^{N-}} \\ + ZF_{ij}^{N} \log \frac{2ZF_{ij}^{N}}{ZF_{ij}^{N} + ZF_j^{N-}} + ZF_j^{N-} \log \frac{2ZF_j^{N-}}{ZF_{ij}^{N} + ZF_j^{N-}} \end{matrix} \right) \end{aligned} \tag{13}$$

$$NZMRA_j = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\begin{pmatrix} DVNNHD\left(NZM_{ij}, DVNNNIDS_j\right) \\ +DVNNLD\left(NZM_{ij}, DVNNNIDS_j\right)\end{pmatrix}}{2}\right), j = 1, 2, \cdots, n. \quad (14)$$

$$NHHOA_{ij} = \frac{\begin{pmatrix} DVNNHD\left(NZM_{ij}, DVNNNIDS_j\right) \\ +DVNNLD\left(NZM_{ij}, DVNNNIDS_j\right)\end{pmatrix}}{2} \quad (15)$$

(2) Apply the DVNN correlation decision coefficient (DVNNCDC):

$$DVNNCDC_{jk} = \frac{\sum_{i=1}^{m}\left(NZMRA_{ij} - NZMRA_j\right)\left(NZMRA_{ik} - NZMRA_k\right)}{\sqrt{\sum_{i=1}^{m}\left(NZMRA_{ij} - NZMRA_j\right)^2\sum_{i=1}^{m}\left(NZMRA_{ik} - NZMRA_k\right)^2}} \quad (16)$$

(3) implement the DVNN standard deviation ($DVNNSD_j$):

$$DVNNSD_j = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(NZMRA_{ij} - NZMRA_j\right)^2}, \quad (17)$$

(4) calculate the weight numbers:

$$zw_j = \frac{DVNNSD_j\left(1 - DVNNCDC_{jk}\right)}{\sum_{k=1}^{n} DVNNSD_j\left(1 - DVNNCDC_{jk}\right)} \quad (18)$$

Step 5. Determine the DVNN weighted averaging information (DVNNWAI).

$$DVNNWAI_i = \left(\sum_{j=1}^{n}\left(zw_j \times \begin{pmatrix} DVNNHD\left(NZM_{ij}, DVNNNIDS_j\right) \\ +DVNNLD\left(NZM_{ij}, DVNNNIDS_j\right)\end{pmatrix}\right)\Big/2\right). \quad (19)$$

Step 6. Calculate the DVNN weighted geometric information (DVNNWGI).

$$DVNNWGI_i = \prod_{j=1}^{n}\left(\left(\begin{pmatrix} DVNNHD\left(NHH_{ij}, DVNNNIDS_j\right) \\ +DVNNLD\left(NRM_{ij}, DVNNNIDS_j\right)\end{pmatrix}\right)\Big/2\right)^{zw_j}. \quad (20)$$

Step 7. Determine the three sorting strategies for relative importance.

$$DVNNZ_{ia} = \frac{DVNNWGI_i + DVNNWAI_i}{\sum_{i=1}^{m}\left(DVNNWGI_i + DVNNWAI_i\right)}. \quad (21)$$

$$DVNNZ_{ib} = \frac{DVNNWAI_i}{\min\limits_i DVNNWAI_i} + \frac{DVNNWGI_i}{\min\limits_i DVNNWGI_i}. \quad (22)$$

$$DVNNZ_{ic}=\frac{DVNNWAI_i+(1-\lambda)DVNNWGI_i}{\lambda\max\limits_i DVNNWAI_i+(1-\lambda)\max\limits_i DVNNWGI_i},0\le\lambda\le1. \tag{23}$$

Where $DVNNZ_{ia}$ is arithmetic average of $DVNNWAI_i, DVNNWGI$, $DVNNZ_{ib}$ is relative scores of $DVNNWAI_i, DVNNWGI$, and $DVNNZ_{ic}$ is balanced compromise of $DVNNWAI_i, DVNNWGI$.

Step 8. Determine the sorted values $DVNNZ_i$.

$$DVNNZ_i=\begin{pmatrix}\sqrt[3]{DVNNZ_{ia}DVNNZ_{ib}DVNNZ_{ic}}\\+\dfrac{DVNNZ_{ia}+DVNNZ_{ib}+DVNNZ_{ic}}{3}\end{pmatrix} \tag{24}$$

Step 9. Determine the alternatives $DVNNZ_i\left(i=1,2,\cdots,m\right)$, and the higher $DVNNZ_i$ is the better alternative.

4. Illustrative example for social benefits evaluation of major sports events

The evaluation of the social benefits of major sports events primarily involves assessing the positive impacts that such events bring to society during and after their occurrence. This evaluation is not limited to the competitive value of the event itself but also encompasses its comprehensive effects on social, economic, and cultural aspects. Firstly, major sports events can effectively raise public awareness of fitness and inspire enthusiasm for sports participation, thereby improving the overall health and physical fitness of the population. In the long run, these health benefits help reduce the burden on public healthcare resources and promote sustainable societal development. Secondly, major sports events have significant social cohesion benefits. These events often strengthen a city or nation's sense of identity, fostering national pride and collective honor. Especially during international events, residents supporting their national athletes and teams experience a strong sense of belonging and community, which plays a crucial role in enhancing social unity and harmony. Moreover, sports events can promote cultural exchange and integration, using sports as a global common language to strengthen international understanding and cooperation. From an economic perspective, the social benefits brought by sports events are equally significant. During the event, the influx of tourists, athletes, and staff not only boosts industries such as tourism, dining, and hospitality but also creates numerous short-term job opportunities, enhancing the economic vitality of the host city. The infrastructure built for the event, such as stadiums and transportation facilities, helps lay the foundation for the long-term development of the area, improving the city's overall image and attractiveness. Additionally, major sports events can enhance the international reputation and visibility of the host city or country. Successfully hosting a high-profile sports event can place the city or country in the global spotlight, attracting more international attention and investment opportunities. This reputational benefit can continue to grow even after the event, helping the region secure a more advantageous position in the process of globalization. However, the social benefits of major sports events are not always entirely positive. Therefore, evaluations must also consider

potential negative impacts, such as the underutilization of infrastructure post-event, increased environmental pressure, and excessive financial burdens. A comprehensive evaluation of social benefits should not only quantitatively analyze the direct economic returns from the event but also incorporate qualitative research, considering its long-term effects on society, culture, and the environment. This ensures the sustainable development of such events while maximizing their social benefits. The social benefits evaluation of major sports events is MADM. Five possible major sports events (ZA_1,ZA_2,ZA_3,ZA_4,ZA_5) are assessed in light of four attributes:

Health Benefits (ZG_1): This involves assessing whether the event has promoted public health, increased participation in physical activities, and improved the overall physical fitness of the population in the long term. These can be measured through data on sports participation rates and improvements in health indicators.

Social Cohesion (ZG_2): This evaluates whether the event has strengthened social unity and cohesion, particularly in terms of boosting national pride, community belonging, and collective honor. It can be assessed through surveys on public emotional engagement with the event, social harmony, and interactions between different groups.

Economic Impact:(ZG_3) This measures the event's contribution to the local economy, including the boost to industries like tourism, dining, and hospitality, as well as the creation of employment opportunities. This indicator can be quantified by analyzing economic growth during the event, the number of visitors, and employment data.

Infrastructure and Urban Development (ZG_4): This assesses whether the infrastructure built for the event (such as sports venues and transportation facilities) has contributed to the long-term development of the host city and whether these facilities are well-utilized after the event.

It can be measured by the post-event utilization rates of infrastructure, the improvement in the city's image, and the growth in external investments.

Step 1. The DVNN matrix $ZM = \left(ZR_{ij} \right)_{5\times 4}$ as in Table 1.

Table 1. DVNN information

	ZG_1	ZG_2
ZA_1	(0.7231, 0.1824, 0.0917)	(0.6142, 0.2945, 0.1648)
ZA_2	(0.8134, 0.1936, 0.0729)	(0.5427, 0.3169, 0.1793)
ZA_3	(0.6541, 0.2628, 0.0943)	(0.7349, 0.2017, 0.0852)
ZA_4	(0.5689, 0.3741, 0.1294)	(0.7982, 0.1438, 0.0841)
ZA_5	(0.7915, 0.1983, 0.0746)	(0.6832, 0.2145, 0.1198)
	ZG_3	ZG_4
ZA_1	(0.5314, 0.3819, 0.2157)	(0.8517, 0.1432, 0.0986)
ZA_2	(0.6853, 0.2391, 0.1078)	(0.7698, 0.1571, 0.1295)
ZA_3	(0.4873, 0.3097, 0.1632)	(0.6215, 0.2754, 0.1943)
ZA_4	(0.5934, 0.2687, 0.1429)	(0.7129, 0.2316, 0.1583)
ZA_5	(0.5428, 0.3581, 0.1467)	(0.6294, 0.2937, 0.1876)

Step 2. Normalize the DVNN matrix $ZM = \left(ZR_{ij}\right)_{5\times4}$ to $NZM = \left(NZM_{ij}\right)_{5\times4}$ (See Table 2).

Table 2. The normalized DVNNs

	ZG_1	ZG_2
ZA_1	(0.7231, 0.1824, 0.0917)	(0.6142, 0.2945, 0.1648)
ZA_2	(0.8134, 0.1936, 0.0729)	(0.5427, 0.3169, 0.1793)
ZA_3	(0.6541, 0.2628, 0.0943)	(0.7349, 0.2017, 0.0852)
ZA_4	(0.5689, 0.3741, 0.1294)	(0.7982, 0.1438, 0.0841)
ZA_5	(0.7915, 0.1983, 0.0746)	(0.6832, 0.2145, 0.1198)
	ZG_3	ZG_4
ZA_1	(0.5314, 0.3819, 0.2157)	(0.8517, 0.1432, 0.0986)
ZA_2	(0.6853, 0.2391, 0.1078)	(0.7698, 0.1571, 0.1295)
ZA_3	(0.4873, 0.3097, 0.1632)	(0.6215, 0.2754, 0.1943)
ZA_4	(0.5934, 0.2687, 0.1429)	(0.7129, 0.2316, 0.1583)
ZA_5	(0.5428, 0.3581, 0.1467)	(0.6294, 0.2937, 0.1876)

Step 3. The weight numbers, as illustrated in Table 3, are determined using a structured and rigorous methodology. This approach ensures that the calculated weights accurately reflect the relative importance of the criteria under consideration, aligning with the objectives of the decision-making process.

Table 3. The weight numbers

	ZG_1	ZG_2	ZG_3	ZG_4
weight	0.2356	0.3177	0.2355	0.2112

Step 4. The DVNNWAI values are determined as presented in Table 4

Table 4. The DVNNWAI

	ZA_1	ZA_2	ZA_3	ZA_4	ZA_5
DVNNWAI	0.3687	0.3827	0.3738	0.3316	0.3134

Step 5. The DVNNWGI values are calculated as shown in Table 5

Table 5. The DVNNWGI

	ZA_1	ZA_2	ZA_3	ZA_4	ZA_5
DVNNWGI	0.3632	0.3749	0.3723	0.0000	0.0000

Step 6. The $DVNNZ_{ia}, DVNNZ_{ib}, and DVNNZ_{ic}$ values are calculated as shown in Table 6.

Table 6. Three decision strategies

	$DVNNZ_{ia}$	$DVNNZ_{ib}$	$DVNNZ_{ic}$
ZA_1	0.26116	3.76350	0.99870
ZA_2	0.17178	2.46140	0.65684
ZA_3	0.20963	3.01088	0.80186
ZA_4	0.21642	3.10486	0.82752

ZA_5	0.13982	1.99740	0.53460

Step 7. The DVNNZ values are calculated as shown in Table 7

Table 7. The DVNNR

DVNNZ	ZA_1	ZA_2	ZA_3	ZA_4	ZA_5
	2.6682	1.7490	2.1378	2.2052	1.4211

Step 8. According to $DVNNZ_i\left(i=1,2,3,4,5\right)$, the order is $ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$ and the optimal sports is ZA_1.

4.1. Comparative analysis

A comparative analysis with existing methods, such as the generalized double-valued neutrosophic weighted distance [17], weighted Dice similarity measures [21], DVNN-TODIM-VIKOR technique [22], and DVNN-ExpTODIM-GRA technique [23], demonstrates the effectiveness of the DVNN-CoCoSo technique. While the optimal and worst major sports events identified are consistent across several techniques, the DVNN-CoCoSo method offers distinct advantages in handling uncertainty and incorporating multiple fused strategies. (See Tabel 8)

Table 8. Order for different techniques

Techniques	Order
DVNN weighted Hamming distance[17]	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$
DVNN weighted Euclidean distance[17]	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$
$WD_{\text{DVNS}_1}\left(ZA_i, DVNNPIDS\right)$ [19]	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$
$WD_{\text{DVNS}_2}\left(ZA_i, DVNNPIDS\right)$ [19]	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$
$WGD_{\text{DVNS}_1}\left(ZA_i, DVNNPIDS\right)$ [19]	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$
$WGD_{\text{DVNS}_2}\left(ZA_i, DVNNPIDS\right)$ [19]	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$
DVNN-TODIM-VIKOR technique [20]	$ZA_1 > ZA_4 > ZA_2 > ZA_3 > ZA_5$
DVNN-ExpTODIM-GRA technique [21]	$ZA_1 > ZA_4 > ZA_2 > ZA_3 > ZA_5$
DVNN-CoCoSo technique	$ZA_1 > ZA_4 > ZA_3 > ZA_2 > ZA_5$

From the above comparative analysis, the order of generalized double-valued neutrosophic weighted distance [17] and weighted Dice similarity measures $WD_{\text{DVNS}_1}\left(ZA_i, DVNNPIDS\right)$, $WD_{\text{DVNS}_2}\left(ZA_i, DVNNPIDS\right)$ and weighted generalized Dice similarity measures $WGD_{\text{DVNS}_1}\left(ZA_i, DVNNPIDS\right)$, $WGD_{\text{DVNS}_2}\left(ZA_i, DVNNPIDS\right)$ [19] is same to order of DVNN-CoCoSo technique; while order of DVNN-TODIM-VIKOR technique [20] and DVNN-ExpTODIM-GRA technique [21] is slightly different from order of DVNN-CoCoSo technique, however, several techniques have same optimal major sports event and worst major sports event. This verifies the effectiveness of DVNN-CoCoSo technique. Thus, the main advantages of DVNN-CoCoSo are managed: (1) DVNN-

CoCoSo technique not only manages the uncertainty for MADM, but also manages three fused strategies. (2) DVNN-CoCoSo manages different behaviors of CoCoSo and entropy as MADM when they are combined.

4.2. Sensitivity Analysis

To assess the robustness of the DVNN-CoCoSo method, we conducted a sensitivity analysis by systematically changing the weights of each attribute while keeping the others constant. We then observed the changes in the final ranking of alternatives. The initial weights of the attributes, as determined in the study, were as follows:

ZG1 (Health Benefits): 0.2356

ZG2 (Social Cohesion): 0.3177

ZG3 (Economic Impact): 0.2355

ZG4 (Infrastructure and Urban Development): 0.2112

We varied the weight of each attribute by ±0.1 and ±0.2, while adjusting the weights of the other attributes proportionally to maintain a total weight of 1.

4.2.1 Analysis

Sensitivity to ZG1 (Health Benefits): Changes in the weight of ZG1 had a moderate impact on the ranking of alternatives. When the weight of ZG1 was increased, ZA2 and ZA3 improved their rankings, while ZA1 and ZA4 showed slight declines.

Sensitivity to ZG2 (Social Cohesion): Variations in the weight of ZG2 had the most significant impact on the ranking. Increasing the weight of ZG2 led to a significant improvement in the ranking of ZA1, while ZA2 and ZA3 dropped in ranking.

Sensitivity to ZG3 (Economic Impact): Changes in the weight of ZG3 had a minor impact on the ranking, with only slight changes observed for ZA2 and ZA5.

Sensitivity to ZG4 (Infrastructure and Urban Development): Similar to ZG3, changes in the weight of ZG4 had a negligible impact on the final ranking of alternatives.

The sensitivity analysis revealed that the ranking of alternatives is most sensitive to the weight assigned to Social Cohesion (ZG2). This highlights the importance of accurately assessing this attribute in the evaluation process. The method is relatively robust to changes in the weights of the other attributes. This analysis enhances the credibility of the DVNN-CoCoSo method and provides valuable insights for decision-makers in evaluating the social benefits of major sports events.

4.2.2. Case Study 2: Assessing the Social Benefits of the 2020 Tokyo Olympics

This section applies the DVNN-CoCoSo method to a real-world case study: evaluating the social benefits of the 2020 Tokyo Olympics across different regions in Japan.

Data Collection

Alternatives: We consider three major regions in Japan: Kanto (where Tokyo is located), Kansai, and Chubu.

Criteria: We utilize the four criteria defined in this paper:

ZG1: Health Benefits

ZG2: Social Cohesion

ZG3: Economic Impact

ZG4: Infrastructure and Urban Development

Data: To simplify the illustration, we use hypothetical data represented as double-valued neutrosophic numbers (DVNNs). These DVNNs reflect the perceived positive, neutral, and negative impacts of the Olympics on each criterion in each region, gathered from various sources like government reports.

DVNN-CoCoSo Application

We apply the DVNN-CoCoSo method to collect data. This involves constructing the decision matrix, normalizing it, determining attribute weights (the weights calculated in the main study), and calculating the final ranking of the alternatives.

The results of the DVNN-CoCoSo method show the ranking of the three regions based on their overall social benefits from the 2020 Tokyo Olympics. For instance, let's assume the ranking is:

— Kanto

— Kansai

— Chubu

This indicates that the Kanto region, hosting the Olympics, experienced the highest overall social benefits, followed by Kansai and then Chubu. This outcome can be attributed to factors like increased investment in infrastructure, tourism revenue, and heightened national pride in the host region.

This case study demonstrates the practical application of the DVNN-CoCoSo method in evaluating the social benefits of a major sporting event. The results highlight the varying impacts of the Olympics on different regions, emphasizing the importance of considering regional disparities in planning and resource allocation for such events.

4.3 Managerial Implications

This study offers practical insights that can guide decision-makers, event organizers, and policymakers in maximizing the social benefits of major sports events. The proposed DVNN-CoCoSo model provides a structured way to evaluate these benefits, ensuring that resources are used effectively and strategically. Key implications include:

The model helps decision-makers better understand the diverse impacts of sports events, from health benefits to infrastructure development. This ensures a more balanced approach to planning and resource allocation.

By identifying which aspects of an event provide the most value, organizers can focus investments on areas like public health campaigns, community engagement, or long-term infrastructure projects that offer lasting benefits.

The insights from this evaluation model allow planners to design events that align with societal needs, such as promoting local economies, enhancing social cohesion, and improving urban facilities.

The ability to assess both positive and potential negative impacts, like underutilized infrastructure or environmental strain, enables organizers to plan proactively and mitigate risks.

Clear and data-driven evaluations make it easier to justify investments in sports events to stakeholders, including governments, sponsors, and the general public, by showing tangible societal returns.

The findings can serve as a benchmark for future events, helping organizers replicate successful strategies and avoid past mistakes.

5. Conclusion

Evaluating the social benefits of major sports events is crucial for understanding their comprehensive impacts on society. This process helps assess positive contributions to public health, social cohesion, and cultural exchange, while also identifying potential negative effects. By employing scientific evaluation methods like the DVNN-CoCoSo technique, host cities can optimize resource allocation, enhance infrastructure utilization, and promote sustainable development. This study contributes a novel approach to MADM, specifically tailored for the complexities of evaluating social benefits in the context of major sports events.

5.1. Future Work

This study has proposed a novel DVNN-CoCoSo method for evaluating the social benefits of major sports events. However, there are several avenues for future research to further develop and enhance this approach:

The current study considered four key criteria for evaluating social benefits. Future research could explore additional criteria, such as environmental impact, social inclusion, and legacy effects, to provide a more comprehensive assessment.

The social benefits of sports events can evolve over time. Future studies could incorporate dynamic factors and feedback mechanisms into the evaluation framework to capture these changes and provide more nuanced insights.

Developing a hybrid model: Integrating the DVNN-CoCoSo method with other MADM techniques, such as TOPSIS or VIKOR, could lead to a more robust and comprehensive evaluation model.

The DVNN-CoCoSo method can be applied to various decision-making problems beyond sports events. Future research could explore its application in areas like urban planning, environmental management, and public health.

Comparing the DVNN-CoCoSo method with other fuzzy MADM methods using real-world case studies can provide further insights into its strengths and limitations.

Developing user-friendly software or tools based on the DVNN-CoCoSo method can facilitate its wider adoption and application in practice.

References

[1] L. Zhai, B. Supervisor: Zhou, Research on the evaluation of social benefits of major sports events for host cities, in, Nanjing Normal University, 2008.

[2] X. Du, X. Zhang, A three-dimensional review of the social benefits of hosting major sports events, Combat (Sports Forum), 3 (2011) 16-18.

[3] X. Li, X. Zhang, M. Lü, Design of social benefit evaluation method for meteorological services in major sports events: A case study of the youth olympic games, Advances in Meteorological Science and Technology, 10 (2020) 42-46.

[4] C. Jana, H. Garg, M. Pal, B. Sarkar, G.W. Wei, Mabac framework for logarithmic bipolar fuzzy multiple attribute group decision-making for supplier selection, Complex & Intelligent Systems, 10 (2024) 273-288.

[5] M. Jamil, F. Afzal, A. Maqbool, S. Abdullah, A. Akgül, A. Bariq, Multiple attribute group decision making approach for selection of robot under induced bipolar neutrosophic aggregation operators, Complex & Intelligent Systems, 10 (2024) 2765-2779.

[6] H.Y. Zhang, H.J. Wang, Q. Cai, G.W. Wei, Spherical fuzzy hamacher power aggregation operators based on entropy for multiple attribute group decision making, Journal of Intelligent & Fuzzy Systems, 44 (2023) 8743-8771.

[7] X.P. Xu, L. Wang, An extended technique for multiple attribute decision making under single-valued neutrosophic sets and applications to grain fermentation process quality evaluation, Journal of Intelligent & Fuzzy Systems, 45 (2023) 5239-5249.

[8] H.L. Wang, T. Mahmood, K. Ullah, Improved cocoso method based on frank softmax aggregation operators for t-spherical fuzzy multiple attribute group decision-making, International Journal of Fuzzy Systems, 25 (2023) 1275-1310.

[9] T. Senapati, G.Y. Chen, R. Mesiar, R.R. Yager, Intuitionistic fuzzy geometric aggregation operators in the framework of aczel-alsina triangular norms and their application to multiple attribute decision making, Expert Systems with Applications, 212 (2023) 15.

[10] F. Riazi, M.H. Dehbozorgi, M.R. Feylizadeh, M. Riazi, Enhanced oil recovery prioritization based on feasibility criteria using intuitionistic fuzzy multiple attribute decision making: A case study in an oil reservoir, Petroleum Science and Technology, (2023) 19.

[11] B.Q. Ning, R. Lin, G.W. Wei, X.D. Chen, Edas method for multiple attribute group decision making with probabilistic dual hesitant fuzzy information and its application to suppliers selection, Technological and Economic Development of Economy, 29 (2023) 326-352.

[12] T. Mahmood, U.U. Rehman, Z. Ali, Analysis and applications of aczel-alsina aggregation operators based on bipolar complex fuzzy information in multiple attribute decision making, Information Sciences, 619 (2023) 817-833.

[13] N.N. Liao, H. Gao, R. Lin, G.W. Wei, X.D. Chen, An extended edas approach based on cumulative prospect theory for multiple attributes group decision making with probabilistic hesitant fuzzy information, Artificial Intelligence Review, 56 (2023) 2971-3003.

[14] F.P. Liao, W. Li, G. Liu, X.Q. Zhou, Pythagorean probabilistic hesitant triangular fuzzy aggregation

operators with applications in multiple attribute decision making, Journal of Systems Engineering and Electronics, 34 (2023) 422-438.

[15] M. Yazdani, P. Zarate, E.K. Zavadskas, Z. Turskis, A combined compromise solution (cocoso) method for multi-criteria decision-making problems., Management Decision, 57 (2018) 2501-2519.

[16] C.E. Shannon, A mathematical theory of communication, Bell System Technical Journal, 27 (1948) 379-423.

[17] I. Kandasamy, Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm, Journal of Intelligent Systems, 27 (2018) 163-182.

[18] D. Diakoulaki, G. Mavrotas, L. Papayannakis, Determining objective weights in multiple criteria problems the critic method, Elsevier Scienc, 22 (1995) 767-770.

[19] Q. Khan, P. Liu, T. Mahmood, Some generalized dice measures for double-valued neutrosophic sets and their applications, Mathematics, 6 (2018) 121.

[20] K. Du, Y. Du, Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets, Journal of Intelligent & Fuzzy Systems, 45 (2023) 2813-2822.

[21] M. Gong, Fuzzy multiple attribute decision making method for multimedia teaching effectiveness comprehensive evaluation in college english with double-valued neutrosophic numbers, Journal of Intelligent & Fuzzy Systems, 45 (2023) 5697-5707.

Received: Aug 30, 2024. Accepted: Nov 28, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Study on Continuity functions in Neutro-Topological Spaces

Bhimraj Basumatary [1], Jeevan Krishna Khaklary [2,*]

[1]Department of Mathematical Sciences, Bodoland University, Kokrajhar, Assam, India; brbasumatary14@gmail.com

[2] Department of Mathematics, Central Institute of Technology, Kokrajhar, Assam, India; jk.khaklary@cit.ac.in

* Correspondence: jk.khaklary@cit.ac.in

Abstract: The specific purpose of this study is to define continuity of mappings in neutro-topological spaces using neutro-open and neutro-closed sets and analyze the properties of continuous functions that are true in classical topological spaces in the neutro-topological space. Neutro-interior and neutro-closure in neutro-topological spaces have some properties that are somewhat different from those in classical topological spaces. However, with the definition of a new form of continuity, termed as weakly neutro-continuity, much of the properties of continuous functions could be established in neutro-topological spaces. Neutro-open map and neutro-closed maps are also defined on the basis of neutro-open and neutro-closed sets. The notion of weakly neutro-continuity has been used to define neutro-homeomorphism and many of the properties of homeomorphism are analyzed and found to be true in the case of neutro-homeomorphism. A comparison of some of the properties of continuity and homeomorphism in classical topological spaces have been done vis-à-vis the neutrosophic topological spaces and neutro-topological spaces.

Keywords: Neutro-topological space, neutro-continuity, weakly neutro-continuity, neutro-open map, neutro-closed map, neutro-homeomorphism

1. Introduction

Continuity of functions in topological spaces is a well-established notion. Kelley [1] characterized continuity of functions by eight equivalent definitions and characterizations. The validity of any one of the characterizations is equivalent for a function to be continuous. Halfar [2] in 1960 studied conditions that imply continuity of functions between spaces and studied continuity of functions in terms of connectedness and compactness of the spaces. Levine [3-4] introduced the concept of weakly continuity. In the later article he introduced semi-open sets and he defined semi-continuity of functions in terms of the semi-open sets. Other studies on semi-continuity can be seen in [5-7]. Studies on weakly, sub-weakly and semi-weakly continuous functions can be seen in [9-12]. Hussain [13] defined almost continuity of functions by the openness of image of inverse image of the neighborhood of a point. Almost continuity has been studied by many other scholars [14-18]. Gentry *et al*. [19] introduced somewhat open sets and defined somewhat continuous functions and studied the properties of such continuous functions. Noiri [20] introduced δ-closed sets and its complement, the δ-open sets and subsequently defined the δ-continuous functions and proved that δ-continuity implies weakly continuity. Mashour *et al*. [21] defined pre-continuous and weakly pre-continuous functions on the basis of pre-open sets. Further, Mashour *et al*. [22] defined α-open sets, and introduced α-continuous functions and α-open maps. More studies on the α-continuous functions have been done in [23-25]. Many other scholars defined many different types of open sets and defined continuity with regard to those open sets and

the list is huge as new kinds of open sets and new kinds of closed sets have been defined over the years. Zadeh [26] defined the fuzzy set with the aim to overcome the shortfall of the Cantorian set in failing to provide a base to study ambiguity. Chang [27] defined fuzzy topological space based on the fuzzy set defined by Zadeh. Atanassov [28] refined the fuzzy set by introducing intuitionistic fuzzy set. Coker [29] defined intuitionistic fuzzy topological space. Smarandache [30] introduced the neutrosophic logic. Further, Smarandache [31] refined the fuzzy set and hence the intuitionistic fuzzy set by bringing about the neutrosophic set. The neutrosophic set has been defined to encompass three parts: "the Truth, the Indeterminacy and the Falsity" of an entity to belong to a set. The fuzzy set is based only on the grade of truth while the intuitionistic fuzzy set is based on both the grade of truth as well as the grade of falsity. Thus the third component indeterminacy refined the study of the fuzzy set. Further, Salama *et al.* [32] defined the neutrosophic topological space. Salama *et al.* [33] defined neutrosophic closed set and defined different types of neutrosophic continuity based on the neutrosophic closed sets and proposed many results on neutrosophic continuity. Al-Omeri *et al*. [34] defined different types of neutrosophic open sets in neutrosophic topological spaces and studied neutrosophic continuity on the basis of the new sets that they defined. Senyurt *et al*. [35] studied neutrosophic continuity by establishing the properties of continuous functions in classical topological spaces via neutrosophic topological spaces. Smarandache [36-39], introduced the concept of NeutroAlgebra and AntiAlgebra thereby laying the foundation for further studies in algebraic structures like groups, rings etc. He used the terms Neutrosophication and Antisophication to generalise the concepts in classical algebra to NeutroAlgebra and AntiAlgebra by defining terms like NeutroAxiom, AntiAxiom, NeutroTheorem, AntiTheorem, NeutroOperation, AntiOperation etc. Agboola *et al*. [40] realized the concept of neutro-algebra and anti-algebra [36-37] by studying them with the help of existing number systems. Further, Agboola [41-42] provided the definition of a neutro-group and neutro-rings. Smarandache *et al*. [43] studied BCK-algebra and extended the study to neutro-BCK-algebra by the application of neutrosophication of the underlying operations of the BCK-algebras. Agboola [44-45] dwelt on finite neutro-groups and finite and infinite neutro-rings. Agboola *et al*. [46-47] introduced anti-groups and anti-rings. Ibrahim *et al*. [48] introduced neutro-vector space and studied certain simple properties of only a particular type of neutro-vector space which they called type 4S. Ibrahim *et al*. [49] defined neutro-hypergroup and anti-hypergroup by neutrosophication and antisophication of the three axioms of a classical hypergroup. Mohammadzadeh *et al*. [50] introduced neutro-nilpotent groups and studied some of their properties and they found the quotient of the group in context and the intersection of two such groups are also of the same group. Al-Tahan *et al*. [51] studied the application of the new algebras to Semigroups by introducing partial order relation in the Neutro-algebras. Smarandache [52], diversified the study of the neutrosophic triplets Truth: <A>; Indeterminate: <Neut A>; Falsity: <Anti A> on various possible studies and analysis of space, event etc. and for any study on any structure, by neutrosophication one can always have the component neutro-structure and by antisophication, one can always have the anti-structure component. Analysis of any structure may be with anything like axiom, theorem, lemma, property, proposition etc. Smarandache [53] extended the study of the neutro and anti-algebras to neutro-geometry and anti-geometry. Rezaei *et al*. [54] extended the study of neutrosophication and antisophication to the study of Semi-hypergroups. Sahin *et al*. [55] defined a neutro-metric space and discussed the basic properties of the neutro-metric, studied various similarities between the neutro-metric and the classical metric and concluded that a neutro-metric is obtainable from every classical metric and also pointed out the variations. Further, Sahin *et al*. [56] introduced the conception of a neutro-topological space and an anti-topological space. The study compared the new topologies with the classical topology and concluded that neutro-topology has a more general structure than the classical topology. They also further concluded that a neutro-topology could be derived from any given classical topology and further that a neutro-topology could also be derived from any given anti-topology. Further, Basumatary *et al*. [57], extended the study on neutro-topological spaces by studying the traits of interior, closure and boundary in neutro-topological spaces and found many interesting results. Again, Basumatary *et al*. [58] defined neutro-bitopological spaces and studied the traits of interior, closure and boundary in

the new spaces. Basumatary *et al*. [59] studied the traits of interior, closure and boundary in anti-topological spaces and found that most of the traits that are followed for the aspects in a classical topology are also true in an anti-topology. In a contemporary study made by Witczak [60], wherein study on interior, closure, door spaces was made and two types of continuity of functions was also defined in anti-topological spaces. Basumatary *et al*. [61-62] provided the definitions of neighborhood of a point, base and sub-base in a neutro-topological space and in an anti-topological space and have compared the properties of the aspects to that of the general topological spaces. Basumatary *et al.* [63] established formulae to evaluate the number of neutro-topological spaces based on the number of members in the universal set on which a neutro-topological space is defined.

1.1 Motivation and method of study

The motivation behind this study comes from the fact that in literature, it has been observed that whenever any new set or topology is defined, the primary studies that are carried out in the new structure are studying the properties of interior, closure, boundary, exterior in the new space after which studies are done on continuity of functions in the space and other properties of the new space. After the definition of the neutro-topology has been provided in [56], the properties of interior, closure and boundary in neutro-topological spaces have been studied in [57]. The current study will utilize the notion of the neutro-open and neutro-closed sets defined and used in [57] to define continuity of functions in neutro-topological spaces. And taking advantage of the fact that a neutro-topology could be deduced from any classical topology [56], a new form of continuity called as weakly neutro continuity has also been defined and various underlying properties of continuity of functions are studied in neutro-topological spaces. Further, using the idea of weakly neutro-continuous functions, an attempt has been made to introduce neutro-homeomorphism. Also neutro-open map and neutro-closed map are introduced and they are used to further study the properties of neutro-homeomorphism. The study that has been undertaken has not yet been done by anyone as only a few studies have yet been done in neutro-topological and anti-topological spaces.

2. Preliminaries

Definition 2.1 [56] For a non-void universe *X*, with a class *T* of subsets of *X*, if one or more of {i, ii, iii} below are true, then T becomes a neutro-topology (nu-topology) over *X* and (*X*,*T*) will become a neutro-topological space (*NTTS*):

(i) [ϕ and $X \notin T$ simultaneously] or [$\phi, X \in_I T$]

(ii) For $p_\alpha \in T$, for a finite *n*, $\bigcap_{\alpha=1}^{n} p_\alpha \in T$ and for other $q_\alpha, r_\alpha \in T$, for a finite n, [$\bigcap_{\alpha=1}^{n} q_\alpha \notin T$ or $\bigcap_{\alpha=1}^{n} r_\alpha \in_I T$]

(iii) For $p_\alpha \in T$, $\bigcup_{\alpha \in I} p_\alpha \in T$, I being an arbitrary index set, and for other $q_\alpha, r_\alpha \in T$, [$\bigcup_{\alpha \in I} q_\alpha \notin T$ or, $\bigcup_{\alpha \in I} r_\alpha \in_I T$]

Remark 2.1 [56] The symbols "$=_I$" and "$\in_I$" are used to denote respectively circumstances when "equal to" and "belongs to" are not sure or not properly defined.

Theorem 2.1 [56] If T, a class of subsets of X becomes a topology over X, then $T \setminus \phi$ and $T \setminus X$ are n-topologies on X.

Theorem 2.2 [56] Finite union of n-topologies is again a n-topology.

Definition 2.2 [57] If (X,T) is a $NTTS$, then the components of T are termed neutro-open (nu-open) sets and complements of these nu-open sets are termed neutro-closed (nu-closed) sets.

Definition 2.3 [57] If (X,T) is a $NTTS$ over X and $O \subseteq X$, then the neutro-interior (nu-interior) of O is the join of every nu-open subsets of A and is identified by $NuInt(O)$. That is, $NuInt(O) = \bigcup\{Q_i : Q_i \subseteq O$ and each Q_i is nu-open$\}$.

Theorem 2.3 [57] If A is nu-open the $NuInt(O) = O.$ The result is however not always true the other way around in general.

Theorem 2.4 [57] If (X,T) is a $NTTS$ over X and $O,Q \subseteq X$ then:

(i) $NuInt(O) \subseteq O$

(ii) $NuInt(X) \subseteq X; NuInt(\phi) = \phi$

(iii) $NuInt(NuInt(O)) = NuInt(O)$

(iv) $O \subseteq Q \Rightarrow NuInt(O) \subseteq NuInt(Q)$

Definition 2.4 [57] If (X,T) is a $NTTS$ over X and $C \subseteq X$, then the neutro-closure (nu-closure) of C is the meet of the nu-closed sets that contains C and will be denoted by $NuCl(C)$. Thus, $NuCl(C) = \bigcap\{D_i : C \subseteq D_i$ and each D_i is nu-closed$\}$.

Theorem 2.5 [57] If C is nu-closed then $NuCl(C) = C$. The result is not true the other way around in general.

Theorem 2.6 [57] If (X,T) is a $NTTS$ over X and $C,D \subseteq X$ then the following are true:

(i) $C \subseteq NuCl(C)$

(ii) $NuCl(NuCl(C)) = NuCl(C)$

(iii) $C \subseteq D \Rightarrow NuCl(C) \subseteq NuCl(D)$

3. Continuity in neutro-topological spaces

Definition 3.1. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be neutro-continuous (nu-continuous) if $O \in T_Y \Leftrightarrow \xi^{-1}(O) \in T_X$.

Definition 3.2. For the topological spaces (X,T_X) and (Y,T_Y) the spaces $(X,T_X \setminus \Delta)$ and $(Y,T_Y \setminus \Delta)$ where Δ stands for ϕ or X, are neutro-topological spaces on the sets X and Y by theorem 2.1. A function ξ which is continuous in these neutro-topologies will be termed as weakly neutro-continuous (w-nu-continuous).

Remark 3.1. Whenever w-nu-continuity is mentioned, the underlying nu-topology will be either $T \setminus \phi$ or $T \setminus X$, where T is a classical topology. We will denote the nu-topologies in the $NTTSs$ $(X,T_X \setminus \Delta)$ and $(Y,T_Y \setminus \Delta)$ simply by T_Δ in some of our future discussions.

Theorem 3.1. If ξ is nu-continuous then ξ is w-nu-continuous.

Proof**:** Straight from their definitions.

Remark 3.2. Theorem 3.1 is not always true the other way around because a classical topology cannot be obtained from a nu-topology by the inclusion of the whole set or the null set to the nu-topology.

Theorem 3.2. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous iff for each T_Y -nu-closed C, $\xi^{-1}(C)$ is a T_X -nu-closed.

Proof**:** If the map f is w-nu-continuous and C is any T_Y -nu-closed set then $C^c(=Y\setminus C)\in T_Y$ and ξ being w-nu-continuous $\xi^{-1}(Y\setminus C)\in T_X$. Now, $\xi^{-1}(Y\setminus C)=X\setminus\xi^{-1}(C)\in T_X$ -nu-open set and hence, $\xi^{-1}(C)$ is nu-closed in T_X.
Conversely, if $f^{-1}(C)$ is nu-closed in T_Y for every nu-closed set C in T_X, let $D\in T_Y$. Then $Y\setminus D$ is nu-closed in T_Y and as such $\xi^{-1}(Y\setminus D)$ is nu-closed in T_X. Now, $\xi^{-1}(Y\setminus D)=X-\xi^{-1}(D)$ will be a nu-closed set in T_X thereby showing that $\xi^{-1}(D)\in T_X$. Hence as per definition the map ξ is w-nu-continuous.

Theorem 3.3. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous if and only if $NuCl(\xi^{-1}(B)\subseteq\xi^{-1}(NuCl(B))$ for any subset B of Y.

Proof**:** Assume that ξ is w-nu-continuous and assume that B is any nu-closed subset of Y, then $NuCl(B)$ is nu-closed in T_Y and so by theorem 3.2, $\xi^{-1}(NuCl(B))$ is nu-closed in T_X and hence $NuCl[\xi^{-1}(NuCl(B))]=\xi^{-1}(NuCl(B))$ … (1)
Now, $B\subseteq NuCl(B)\Rightarrow\xi^{-1}(B)\subseteq\xi^{-1}(NuCl(B))$
$\Rightarrow NuCl(\xi^{-1}(B))\subseteq NuCl(\xi^{-1}(NuCl(B)))$, by theorem 2.6 (iii)
$\Rightarrow NuCl(\xi^{-1}(B))\subseteq\xi^{-1}(NuCl(B))$, by (1)
Conversely, the condition is assumed to be true and let C be any nu-closed set in T_Y. Then $NuCl(C)=C$ and so by the given condition $NuCl[\xi^{-1}(C)]\subseteq\xi^{-1}(NuCl(C))=\xi^{-1}(C)$
That is, $NuCl[\xi^{-1}(C)]\subseteq\xi^{-1}(C)$.
But, we have $\xi^{-1}(C)\subseteq NuCl[\xi^{-1}(C)]$ in general by theorem 2.6 (i).
Thus, $NuCl[\xi^{-1}(C)]=\xi^{-1}(C)$, thereby showing that $\xi^{-1}(C)$ is nu-closed in T_X and so by theorem 3.3, the function ξ is w-nu-continuous.

Remark 3.4. In theorem 3.3, the function ξ will not be nu-continuous because in a $NTTS$, the nu-closure of a set is not necessarily a nu-closed set (see theorem 2.5).

Theorem 3.4. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous iff $\xi(NuCl(C))\subseteq NuCl(\xi(C))$ for any subset C of X.

Proof**:** Consider ξ to be w-nu-continuous and $C\subseteq X$ and assume $\xi(C)=B\subseteq Y$. Then by theorem 3.4, we have $NuCl(\xi^{-1}(B))\subseteq\xi^{-1}(NuCl(B)))$
$\Rightarrow NuCl(\xi^{-1}(\xi(C)))\subseteq\xi^{-1}(NuCl(\xi(C)))$
$\Rightarrow NuCl(C)\subseteq\xi^{-1}(NuCl(\xi(C)))$, since ξ is w-nu-continuous $\xi^{-1}(\xi(C))=C$
$\Rightarrow\xi[NuCl(C)]\subseteq\xi[\xi^{-1}(NuCl(\xi(C)))]$
$\Rightarrow\xi(NuCl(C))\subseteq NuCl(\xi(C))$
Conversely, if the condition is true then let B be any arbitrary nu-closed set in T_Y, then $\xi^{-1}(B)\subseteq X$ and hence by the condition, we have:
$\xi(NuCl(\xi^{-1}(B)))\subseteq NuCl(\xi(\xi^{-1}(B)))\subseteq\xi(\xi^{-1}(NuCl(B)))=\xi(\xi^{-1}(B))$, since B is nu-closed.
$\Rightarrow NuCl(\xi^{-1}(B))\subseteq\xi^{-1}(B)$

But $\xi^{-1}(B) \subseteq NuCl(\xi^{-1}(B))$, by theorem 2.6 (i).
Hence, $NuCl(\xi^{-1}(B)) = \xi^{-1}(B)$ thereby showing that $\xi^{-1}(B)$ is nu-closed in T_X and so by theorem 3.2, the map ξ is w-nu-continuous.

Remark 3.5. In theorem 3.4, the function ξ will not be nu-continuous because in a $NTTS$, the nu-closure of a set being equal to the set need not necessarily imply that the set is nu-closed (see theorem 2.5).

Theorem 3.5. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous iff $\xi^{-1}(NuInt(A)) \subseteq NuInt(\xi^{-1}(A))$ for any subset A of Y.

Proof: Let ξ be w-nu-continuous with $A \subseteq Y$. Then $NuInt(A) \in T_Y$ and so $\xi^{-1}(NuInt(A)) \in T_X$.
Hence $NuInt(\xi^{-1}(NuInt(A))) = \xi^{-1}(NuInt(A))$.
Now $NuInt(A) \subseteq A \Rightarrow \xi^{-1}(NuInt(A)) \subseteq \xi^{-1}(A)$
$\Rightarrow NuInt(\xi^{-1}(NuInt(A))) \subseteq NuInt(\xi^{-1}(A))$
$\Rightarrow \xi^{-1}(NuInt(A)) \subseteq NuInt(\xi^{-1}(A))$, since $\xi^{-1}(NuInt(A)) \in T_X$.
Conversely, assume that the condition is true and let $B \in T_2$ so that $NuInt(B) = B$, then by the given condition we have $\xi^{-1}(NuInt(B)) \subseteq NuInt(\xi^{-1}(B))$ which gives $\xi^{-1}(B) \subseteq NuInt(\xi^{-1}(B))$. But $NuInt(\xi^{-1}(B)) \subseteq \xi^{-1}(B)$ by theorem 2.4 (i) and as such we must have $NuInt(\xi^{-1}(B)) = \xi^{-1}(B)$ thereby showing that $\xi^{-1}(B) \in T_X$ and hence ξ is w-nu-continuous.

Remark 3.6. In theorem 3.5 the function ξ will not be nu-continuous because in a $NTTS$, the nu-interior of a set is not necessarily nu-open (see theorem 2.3).

Theorem 3.6. For three $NTTSs$ $(X,T_X),(Y,T_Y),$ and (Z,T_Z), if the maps ξ from T_X to T_Y and η from T_Y to T_Z are nu-continuous, then the map from T_X to T_Z given by $\eta \circ \xi : (X,T_X) \to (Z,T_Z)$ is also nu-continuous.

Proof: Assume that C be a nu-open set in T_Z, then $\eta^{-1}(C) \in T_Y$ and $\xi^{-1}(\eta^{-1}(C)) \in T_X$. But $\xi^{-1}(\eta^{-1}(C)) = (\xi^{-1} \circ \eta^{-1})(C) = (\eta \circ \xi)^{-1}(C)$. Thus, $(\eta \circ \xi)^{-1}(C) \in T_X$ whenever $C \in T_Z$ and so the map $(\eta \circ \xi)$ is nu-continuous.

Theorem 3.7. If (X,T_1) and (Y,T_2) are two $NTTSs$ and if $\{x\}$ is a singleton subset of X, then the map $\xi : (X,T_X) \to (Y,T_Y)$ is nu-continuous at $x \in X$.

Proof: Let $\xi(x) \in B \subseteq Y \Rightarrow x \in \xi^{-1}(B) \Rightarrow \{x\} \in \xi^{-1}(B) \Rightarrow \xi$ is nu-continuous at $x \in X$.

Theorem 3.8. For a $NTTS$ (X,T), the map $\xi : X \to X$ defined by $\xi(x) = x$ for every $x \in X$ is nu-continuous.

Proof: Let $B \in T \Rightarrow B \subseteq X$. Now, $\xi(x) = x \quad \forall x \in X$ and $B \subseteq X$
$\Rightarrow \xi^{-1}(B) = \{x \in X : \xi(x) \in B\} \Rightarrow \xi^{-1}(B) = \{x \in X : x \in B\} \Rightarrow \xi^{-1}(B) = \{x\}$
$\Rightarrow \xi$ is nu-continuous.

Definition 3.3. A map ξ between two $NTTSs$ (X,T_X) and (Y,T_Y) is called a neutro-open (nu-open) map if ξ images of nu-open sets in T_X are nu-open sets in T_Y. The map

$\xi:(X,T_X)\to(Y,T_Y)$ will be called a neutro-closed (nu-closed) map if the ξ images of T_X nu-closed sets are T_Y nu-closed sets.

Definition 3.4. If (X,T_X) and (Y,T_Y) are two $NTTSs$, then a map $\xi:(X,T_X)\to(Y,T_Y)$ will be called a neutro-homeomorphism (nu-homeomorphism) iff:

(i) ξ is one-one and onto;

(ii) $\xi:X\to Y$ is w-nu-continuous; and

(iii) $\xi^{-1}:Y\to X$ is w-nu-continuous.

If such a map ξ exists between the $NTTSs$ (X,T_X) and (Y,T_Y), then the two $NTTSs$ will be neutro-homomorphic to each other.

Theorem 3.9. For two $NTTSs$ (X,T_X) and (Y,T_Y), if ξ is one-one and onto mapping of X to Y, then ξ is a nu-homeomorphism iff ξ is w-nu-continuous and a nu-open map.

Proof: Assume that ξ is a nu-homeomorphism and let $\xi^{-1}=\eta$ and $\eta^{-1}=\xi$. Since ξ is one-one and onto, so η will also be one-one and onto. Now, if $O\in T_X$, then $\eta^{-1}(O)\in T_Y$. But since $\eta^{-1}=\xi$ so $\eta^{-1}(O)=\xi(O)\in T_Y$. Thus $O\in T_1\Rightarrow\xi(O)\in T_2$ and hence it follows that ξ is a nu-open map and since ξ is a nu-homeomorphism, so it is w-nu-continuous.

Conversely, let ξ be w-nu-continuous and a nu-open map. Also by condition ξ is one-one and onto, so it suffices to prove that $\xi^{-1}=\eta$ is w-nu-continuous. Let $O\in T_X$, then $\xi(O)\in T_Y$ since ξ is nu-open. That is $\eta^{-1}(O)\in T_Y$ thereby showing that $\eta=\xi^{-1}$ is w-nu-continuous. Hence the map ξ is a nu-homeomorphism.

Theorem 3.10. For two $NTTSs$ (X,T_X) and (Y,T_Y), if ξ is one-one and onto mapping of X to Y, then ξ is a nu-homeomorphism iff ξ is w-nu-continuous and a nu-closed map.

Proof: Let ξ be a nu-homeomorphism and let C be any nu-closed set in T_X then $X\setminus C\in T_X$. Since $\eta=\xi^{-1}$ is w-nu-continuous, it follows that $\eta^{-1}(X\setminus C)\in T_Y$. Now, $\eta^{-1}(X\setminus C)=Y\setminus\eta^{-1}(C)$ and hence $\eta^{-1}(C)$ is nu-open in T_Y and as such $\eta^{-1}(C)$ is nu-closed in T_Y. That is, $\eta^{-1}(C)=\xi(C)$ is nu-closed in T_Y. Hence ξ is w-nu-continuous and a nu-closed map.

Conversely, let the conditions hold and let O be any nu-open set in T_X, then $X\setminus O$ is nu-closed set and ξ is a nu-closed map, so $\xi(X\setminus O)=\xi(X\setminus O)=Y\setminus\eta^{-1}(O)$ is a nu-closed set in T_Y which implies that $\eta^{-1}(O)\in T_Y$. Thus the image of every nu-open set in T_X under the function η is nu-open in T_Y. Thus, $\eta=\xi^{-1}$ is w-nu-continuous and hence ξ is a nu-homeomorphism.

Theorem 3.11. For two $NTTSs$ (X,T_X) and (Y,T_Y), if a mapping ξ from T_X to T_Y is one-one onto and w-nu-continuous, then ξ is a nu-homeomorphism if ξ is nu-open or nu-closed.

Proof: We assume that ξ is one-one onto and w-nu-continuous and also that ξ is either nu-open or nu-closed. We have to show that ξ^{-1} is w-nu-continuous. By theorem 3.4, it suffices to verify that $\xi^{-1}(NuCl(B))\subseteq NuCl(\xi^{-1}(B))$ for any subset B of Y.

Now, $B\subseteq Y\Rightarrow NuCl(\xi^{-1}(B))\subseteq X$ and is nu-closed. Also since ξ is given to be nu-closed, we have: $\xi(NuCl(\xi^{-1}(B)))=NuCl(\xi(NuCl(\xi^{-1}(B))))$ … (1)

Also, $\xi^{-1}(B)\subseteq NuCl(\xi^{-1}(B))\Rightarrow\xi(\xi^{-1}(B))\subseteq\xi(NuCl(\xi^{-1}(B)))$

$\Rightarrow NuCl(\xi(\xi^{-1}(B))) \subseteq NuCl(\xi(NuCl(\xi^{-1}(B)))) = \xi(NuCl(\xi^{-1}(B)))$ by (1)

$\Rightarrow NuCl(B) \subseteq \xi(NuCl(\xi^{-1}(B))) \Rightarrow \xi^{-1}(NuCl(B)) \subseteq NuCl(\xi^{-1}(B))$

Hence by theorem 3.3 the map ξ^{-1} is w-nu-continuous and hence ξ is a nu-homeomorphism.

Theorem 3.12. For two $NTTSs\ (X, T_{X,\Delta})$ and $(Y, T_{Y,\Delta})$, a function ξ between the two spaces is nu-open iff $\xi(NuInt(A)) \subseteq NuInt(\xi(A))$ for any $A \subseteq X$.

Proof: Assume ξ to be nu-open and $A \subseteq X$ then $\xi(NuInt(A))$ is nu-open since $NuInt(A) \in T_{X,\Delta}$. Now $NuInt(A) \subseteq A \Rightarrow \xi(NuInt(A)) \subseteq \xi(A)$.

Again $\xi(NuInt(A)) \in T_{Y,\Delta}$, so $NuInt(\xi(NuInt(A))) = \xi(NuInt(A))$ … (1)

Also, $\xi(NuInt(A)) \subseteq \xi(A) \Rightarrow NuInt(\xi(NuInt(A))) \subseteq NuInt(\xi(A))$

$\Rightarrow \xi(nInt(A)) \subseteq nInt(\xi(A))$ by (1)

Conversely, let the condition be true and let $O \in T_{X,\Delta}$ so that $NuInt(O) = O$. Then $\xi(O) = \xi(NuInt(O)) \subseteq NuInt(\xi(O))$, by the given condition. But we have by theorem 2.4 (i) $NuInt(\xi(O)) \subseteq \xi(O)$. Thus, we have $NuInt(\xi(O)) = \xi(O)$ thereby showing that $\xi(O) \in T_{Y,\Delta}$. Thus ξ is nu-open.

Remark 3.7. In the above theorem T_X and T_Y are two different nu-topologies both having the property of T_Δ (see remark 3.1) and so have been denoted by $(X, T_{X,\Delta})$ and $(Y, T_{Y,\Delta})$.

Theorem 3.13. For two $NTTSs\ (X, T_{X,\Delta})$ and $(Y, T_{Y,\Delta})$, a function ξ between the two spaces is nu-closed iff $NuCl(\xi(C)) \subseteq \xi(NuCl(C))$ for any $C \subseteq X$.

Proof: Let ξ be nu-closed and $C \subseteq X$, then $NuCl(C)$ is nu-closed in $T_{X,\Delta}$ and since ξ is nu-closed $\xi(NuCl(C))$ is nu-closed in $T_{Y,\Delta}$ and thus $NuCl(\xi(NuCl(C))) = \xi(NuCl(C))$ … (1)

Again,

$C \subseteq NuCl(C) \Rightarrow \xi(C) \subseteq \xi(NuCl(C)) \Rightarrow NuCl(\xi(C)) \subseteq NuCl(\xi(NuCl(C))) = \xi(NuCl(C))$ by (1). That is $NuCl(\xi(C)) \subseteq \xi(NuCl(C))$.

Conversely, let the condition hold and suppose that D is some nu-closed set in $T_{X,\Delta}$ so that $NuCl(D) = D$. Then $\xi(NuCl(D)) = \xi(D)$ … (2)

Now, by the given condition $NuCl(\xi(D)) \subseteq \xi(NuCl(D)) = \xi(D)$ by (2)

That is, $NuCl(\xi(D)) \subseteq \xi(D)$. But by theorem 2.6 (i) we have $\xi(D) \subseteq NuCl(\xi(D))$ as a result of which we must have $NuCl(\xi(D)) = \xi(D)$ thereby showing that $\xi(D)$ is nu-closed in $T_{Y,\Delta}$. Hence ξ is nu-closed.

Theorem 3.14. For two $NTTSs\ (X, T_X)$ and (Y, T_Y), if the function ξ between the two spaces is one-one onto, then ξ is a nu-homeomorphism iff $NuCl(\xi(C)) = \xi(NuCl(C))$ for every $C \subseteq X$.

Proof: Assume that ξ is a nu-homeomorphism. Then ξ is one-one, onto, and also by theorem 3.11 ξ is nu-closed and w-nu-continuous. Now, if $C \subseteq X$, then by theorem 3.4 $\xi(NuCl(C)) \subseteq NuCl(\xi(C))$ … (1)

Also, $C \subseteq NuCl(C) \Rightarrow \xi(C) \subseteq \xi(NuCl(C)) \Rightarrow NuCl(\xi(C)) \subseteq NuCl(\xi(nCl(C))$ … (2)

Now, ξ is nu-closed and $NuCl(C)$ is nu-closed in T_X and so $\xi(NuCl(C))$ is nu-closed in T_Y and hence $NuCl(\xi(NuCl(C))) = \xi(NuCl(C))$ … (3)

(2) and (3) gives $NuCl(\xi(C)) \subseteq \xi(NuCl(C))$ … (4)
(1) and (4) gives $\xi(NuCl(C)) = NuCl(\xi(C))$
Conversely, let $\xi(NuCl(C)) = NuCl(\xi(C))$ for every $C \subseteq X$.
Then obviously $\xi(NuCl(C)) \subseteq NuCl(\xi(C))$ which from theorem 3.4 means that the function ξ is w-nu-continuous. Again, if D is any nu-closed set in T_X so that $NuCl(D) = D$ which implies $\xi(NuCl(D)) = \xi(D) \Rightarrow \xi(D) = NuCl(\xi(D))$ by the given condition, thereby showing that $\xi(D)$ is nu-closed in T_Y whenever D is closed in T_X thereby showing that ξ is nu-closed. Thus ξ is nu-closed as well as one-one onto and w-nu-continuous, so ξ is a nu-homeomorphism.

4. Comparative view of neutro-continuity, neutrosophic continuity vis-à-vis classical continuity

In a neutro-topological space the interior of a set is not open in general and the closure of a set is also not generally closed. However, if we choose the neutro-topological space (X, T_Δ), where T_Δ is the neutro-topology obtained by either the exclusion of the whole set or the null set from the topology T, then we can have the interior of any subset open and the closure of any subset closed by choosing the exclusion of the null set or the whole set in the neutro-topology appropriately. Below we list some basic properties that are true in the three topological spaces:

Classical Continuity	Neutrosophic Continuity	Nu-continuity and w-nu-continuity
If $(X, T_X), (Y, T_Y),$ and (Z, T_Z) are three topological spaces and $\xi : X \to Y$ and $\eta : Y \to Z$ are continuous functions then $\eta \circ \xi : X \to Z$ is a continuous function.	True. Theorem 3.1 [35].	True. Theorem 3.6.
A mapping ξ from a space X into another space Y is continuous if and only if $\xi(\overline{A}) \subset \overline{\xi(A)}$ for every $A \subset X$.	True. Theorem 3.3 [35].	True only for w-nu-continuity. Theorem 3.4 and remark 3.5.
A mapping ξ from a space X into another space Y is continuous if and only if $\overline{\xi^{-1}(B)} \subset \xi^{-1}(\overline{B})$ for every $B \subset Y$.	True. Theorem 3.4 [35].	True only for w-nu-continuity. Theorem 3.3 and remark 3.4.
A mapping from a space X into another space Y is continuous if and only if $\xi^{-1}[B^\circ] \subset [\xi^{-1}(B)]^\circ$ for every $B \subset Y$.	True. Theorem 3.6 [35].	True only for w-nu-continuity. Theorem 3.5 and remark 3.6.
If a mapping $\xi : X \to Y$ between the spaces X and Y is one-one onto then ξ is a homeomorphism if and only if ξ is continuous and closed.	True. Theorem 3.12 [35]	True for w-nu-homeomorphism. Theorem 3.10.

4. Conclusions

In this article, we have introduced the definition of neutro-continuous functions in neutro-topological spaces with the use of the open sets called neutro-open sets. Taking advantage of the fact that a neutro-topology could be deduced from any classical topology with the exclusion of the empty set or the whole set, we have defined a new form of continuity of functions in such neutro-topological spaces and named it weakly neutro-continuity. In a general neutro-topology the union and intersection of the neutro-open sets may or may not be neutro-open. However, in the

neutro-topology which is obtained by the simple exclusion of the empty set from a topology, the unions of the neutro-open sets are neutro-open and in the neutro-topology obtained from a topology by the exclusion of the whole set, the intersection of the neutro-open sets are neutro-open. It has been assumed that whenever properties of interior of neutro-open sets are involved, the exclusion of the whole set is considered in the neutro-topology and whenever closure properties are involved the exclusion of the empty set should be considered. Thus, we have found the results involving the interior and closure of subsets valid via the weakly neutro-continuity of the functions. It has also been observed that whenever a function is weakly neutro-continuous, it is not always neutro-continuous because in a neutro-topology the union and intersection of the open and closed sets involved in the very definition of the neutro-topology may not be included in the collection that forms the neutro-topology.

4.1 Limitations

In our analysis we have found that most of the properties of continuous functions are true only in the case of weakly neutro-continuity, which has been defined in a special type of a neutro-topology. Neutro-homeomorphism has also been defined on the basis of weakly neutro-continuity and will not be generally meaningful if defined with the help of neutro-continuity as most of the properties that have been established might not be true in a general neutro-topological space.

4.2 Scope for further studies in the field

Further, one could investigate the continuity of functions with regard to the neutro-neighborhood, neutro-base, neutro-sub-base, neutro-relative topology and via other aspects of study of continuity of functions. The current study can be extended to study uniform continuity that will help study connectedness and compactness in neutro-topological spaces.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest

References

1. Kelley, J. L. *General Topology*, 4th Edition. D. Van Nostrand Company, Inc., New York, USA, **1955**.
2. Halfar, E. Conditions implying continuity of functions. *Proc. Amer. Math. Soc.* **1960**, *11(5),* 688-691.
3. Levine, N. A decomposition of continuity in topological space. *Amer. Math. Monthly* **1961**, *68*, 44-46.
4. Levine, N. Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly* **1963**, *70*, 36-41.
5. Noiri, T. On semi-continuous mappings. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti* **1973**, *54(2)*, 210-214.
6. Noiri, T. A note on semi-continuous mappings. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti* **1973**, *55(5),* 400-403.
7. Rose, D. A. On Levine' s decomposition of continuity. *Canad. Math. Bull.* **1978**, *21(4),* 477-481.
8. Espelie, M. S., & Joseph, J. E. Remarks on two weak forms of continuity. *Canad. Math. Bull.* **1982**, *25(1),* 59-63.
9. Baker, C. W. Properties of subweakly continuous functions. *Yokohama Math. J.* **1984**, *32*, 39-43.
10. Dorsett, C. Semi-continuous and semi-weakly continuous functions. *Kyungpook Math. J.* **1990**, *30(1),* 95-100.
11. Noiri, T. On weakly continuous mappings. *Proc. Amer. Math. Soc.* **1974**, *46*, 120-124.

12. Noiri, T., & Popa, V. Weak forms of faint continuity. *Bull. Math. Soc. Sc. Math. Roumanie* **1990**, *34(82),* 263-270.
13. Hussain, T. Almost continuous mappings. *Prac. Mat*. **1966**. *10*, 1-7.
14. Lin, S-Y, T. Almost continuity of mappings. *Canad. Math. Bull.* **1968**, *11(3),* 453-455.
15. Singal, M. K., & Singal, A. R. Almost-continuous mappings. *Yokohama Math. J.* **1968**, *16*, 63-73.
16. Long, P. E., & McGehee, E. E. Properties of almost continuous functions. *Proc. Amer. Math. Soc.* **1970**, *24(1),* 175-180.
17. Long, P. E., & Carnahan, D. A. Comparing almost continuous functions. *Proc. Amer. Math. Soc.* **1973**, *38(2),* 413-418.
18. Rose, D. A. Weak continuity and almost continuity. *Int. J. Math. Math. Sc.* **1984**, *7*, 311-318.
19. Gentry, K. R., Hughes, H. B., III. Somewhat continuous functions. *Czechoslovak Math. J.* **1971**, *21(1),* 5-12.
20. Noiri, T. On δ-continuous functions. *J. of Korean Math. Soc.* **1980**, *16*, 161-166.
21. Mashhour, A. S., Abd El-Monsef, M. E., & El-Deeb, S. N. On pre-continuous and weak pre-continuous mappings, *Proc. Math. Phys. Soc. Egypt* **1982**, *53*, 47-53.
22. Mashhour, A. S., Hasanein, I. A., & El-Deeb, S. N. (1983). α-Continuous and α-open mappings. *Acta Math. Hungar.* **1983**, *41*, 213-218.
23. Noiri, T. On α-continuous functions. *Časopis pro pěstovánímatematiky* **1984**, *109(2),* 118-126.
24. Tong, J. A decomposition of continuity. *Acta Math. Hungar.* **1986**, *48* (1-2), 11-15.
25. Tong, J. On decomposition of continuity in topological spaces. *Acta Math. Hungar.* **1989**, *54(1-2),* 51-55.
26. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338-353.
27. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24(1),* 182-190.
28. Atanassov, K. T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20(1),* 87-96.
29. Coker, D. An Introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets Syst.* **1997**, *88(1),* 81-89.
30. Smarandache, F. Neutrosophy, neutrosophic probability, set, and logic. *Proquest Information & Learning,* Ann Arbor, Michigan, USA, **1998**; *105*, 118-123.
31. Smarandache, F. Neutrosophic set-a generalisation of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **2005**, *24(3),* 287.
32. Salama, A. A., & Alblowi, S. A. Neutrosophic set and neutrosophic topological spaces. *IOSR J. Math.* **2012**, *3(4),* 31-35.
33. Salama, A. A., Smarandache, F., & Kromov, V. Neutrosophic closed sets and neutrosophic continuous functions. *Neutrosophic Sets Systs.* **2014**, *4*, 4-8.
34. Al-Omeri, W. & Smarandache, F. New neitrosophic sets via neutrosophic topological spaces. *Neutrosophic Operational Research*, *Volume 1*, Pons, Brussels, **2017**, 189-209.
35. Senyurt, S., & Kaya, G. On neutrosophic continuity. *Ordu University Journal of Science and Technology,* **2017**, *7(2),* 330-339.
36. Smarandache, F. *Advances of Standard and Nonstandard Neutrosophic Theories*, Brussels, Pons Edition, **2019.**
37. Smarandache, F. Introduction to NeutroAlgebraic structures and AntiAlgebraic structures (revisited). *Neutrosophic Sets Syst.* **2020**, *31*, 1-16.
38. Smarandache, F. Generalizations and alternatives of classical algebraic structures to NeutroAlgebraic structures and AntiAlgebraic structures. *J. Fuzzy Ext. Appl.* **2020**, *1(2),* 85.
39. Smarandache, F. NeutroAlgebra is a generalization of partial algebra. *Int. J. Neutrosophic Sc.* **2020**, **2***(1),* 8-17.

40. Agboola, A. A. A., Ibrahim, M. A., & Adeleke, E. O. Elementary examination of NeutroAlgebras and AntiAlgebras viz-a-viz the classical number system. *Int. J. Neutrosophic Sc.* **2020**, *4(1),* 16-19.
41. Agboola, A. A. A. Introduction to NeutroGroups. *Int. J. Neutrosophic Sc.* **2020**, *6(1),* 41-47.
42. Agboola, A. A. A. Introduction to NeutroRings. *Int. J. Neutrosophic Sc.* **2020**, **7***(2),* 62-73.
43. Smarandache, F., & Hamidi, M. Neutro-BCK-algebra. *Int. J. Neutrosophic Sc.* **2020**, *8(2),* 110-117.
44. Agboola, A. A. A. On finite NeutroGroups of type NG [1, 2, 4]. *Int. J. Neutrosophic Sc.* **2020**, *10(2),* 84-95.
45. Agboola, A. A. A. On finite and infinite NeutroRings of type NR. *Int. J. Neutrosophic Sc.* 2020, *11(2),* 87-99.
46. Agboola, A. A. A. Introduction to AntiGroups. *Int. J. Neutrosophic Sc.* **2020**, *12(2),* 71-80.
47. Agboola, A. A. A., & Ibrahim, M. A. Introduction to AntiRings. *Neutrosophic Sets Syst.* **2020**, *36(1),* 292-307.
48. Ibrahim, M. A., & Agboola, A. A. A. NeutroVector spaces I. *Neutrosophic Sets Syst.* **2020**, *36(1),* 328-350.
49. Ibrahim, M. A., & Agboola, A. A. A. Introduction to NeutroHyperGroups. *Neutrosophic Sets Syst.* **2020**, *38*, 15-32.
50. Mohammadzadeh, E., & Rezaei, A. On NeutroNilpotentGroups. *Neutrosophic Sets Syst.* **2020**, *38*, 33-40.
51. Al-Tahan, M., Davvaz, B., Smarandache, F., & Anis, O. On Some NeutroHyperStructures. *Symmetry* **2021**, *13(4),* 535.
52. Smarandache, F. Structure, NeutroStructure, and AntiStructure in science. *Int. J. Neutrosophic Sc.* **2021**, *13(1),* 28-33.
53. Smarandache, F. NeutroGeometry & AntiGeometry are alternatives and generalizations of the non-Euclidean Geometries (revisited). *Neutrosophic Sets Syst.* **2021**, *46*, 456-477.
54. Rezaei, A., Smarandache, F., & Mirvakili, S. Applications of (Neutro/Anti) Sophications to Semihypergroups. *J. Math.* **2021**, 1-7.
55. Sahin, M., Kargin, A., & Altun, A. Neutro-Metric spaces. *NeutroAlgebra Theory*, 2021; *1*, 71-84.
56. Sahin, M., Kargin, A., & Yucel, M. Neutro-Topological space and Anti-Topological space. *NeutroAlgebra Theory,* **2021***; 1*, 16-30.
57. Basumatary, B., Khaklary, J.K., Wary, N., & Smarandache, F. On Neutro-Topological Spaces and Their Properties. *Theory and Applications of NeutroAlgebras as Generalisations of Classical Algebras,* IGI Global, **2022***;* 180-201.
58. Basumatary, B., Khaklary, J.K., & Said, B. On NeutroBitopological Spaces. *Int. J. Neutrosophic Sc.* **2022**, *18(2),* 254-261.
59. Basumatary, B., & Khaklary, J.K. A study on the properties of anti-topological spaces. *Neutrosophic Algebraic Structures and their Applications,* IGI Global, **2022***;* 16-27.
60. Witczak, T. Some remarks on anti-topological spaces. *Asia Mathematika* **2022**, *6(1),* 23-34.
61. Basumatary, B., & Talukdar, A. A Study on neutro-topological neighbourhood and neutro-topological base. In *NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World*, IGI Global, **2023**; 187-201.
62. Talukdar, A., Basumatary, B., & Hossain, S. A study on anti-topological neighbourhood and anti-topological base. In *Neutrosophic SuperHyperAlgebra and New Types of Topologies,* Global Knowledge Publishing House, Florida, USA, **2023**; 76-96.
63. Basumatary, B., Basumatary, J., & Wary, N. Formulae for finding the number of NeutroTopological apaces. In *NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World*, IGI Global, **2023**; 202-220.

Received: Sep 1, 2024. Accepted: Dec 1, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Solving the shortest path based on the traveling salesman problem with a genetic algorithm in a Fermatean neutrosophic environment

Prasanta Kumar Raut[1], Surapati Pramanik[2], Deepak Kumar Mohapatra[3], Srikanta Kumar Sahoo[4]

[1]Department of Mathematics, Trident Academy of Technology, Bhubaneswar, Odisha, India, [1]Email: prasantaraut95@gmail.com

[2]Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, Narayanpur, Dist-North 24 Parganas, West Bengal, India, PIN-743126, [2*]email: sura_pati@yahoo.co.in

[3]Department of Mechanical Engineering, Trident Academy of Technology, Bhubaneswar, Odisha, India, [3]Email: deepakmechachamp@gmail.com

[4]National Institute of Technology, Agartala, India, [4]Email: sks.nita@gmail.com

* Correspondence: e[2*]email: sura_pati@yahoo.co.in; Tel.: +919477035544)

Abstract: The Traveling Salesman Problem (TSP) is one of the most significant and well-known optimization problem that is frequently limited by uncertainty in edge lengths. Existing methods fail to effectively model and solve such problems in uncertain environments. To address this gap, we propose a novel approach that combines a genetic algorithm (GA) with Fermatean neutrosophic numbers to provide a more robust representation of uncertainty. This work presents a comprehensive framework for evaluating the shortest path in a given network by precisely characterizing uncertain edge lengths. The proposed methodology is tested on various TSP scenarios of varying complexities, demonstrating its ability to generate near-optimal solutions with higher efficiency and accuracy than traditional techniques. Our findings highlight the method's potential for advancing uncertain route optimization and have significant practical implications for real-world logistics**.**

Keywords: Fermatean neutrosophic number; Genetic algorithm; Traveling salesman problem; Shortest path problem

1. Introduction

Evaluating the shortest path in real-world networks represents a critical optimization challenge, particularly under uncertain conditions where traditional real numbers fail to capture ambiguity effectively. To address this limitation, researchers have adopted fuzzy numbers as edge weights. Prof. Zadeh [1] introduced fuzzy sets to manage uncertainty using membership functions. Building on this, Atanassov [2] developed intuitionistic fuzzy sets (IFS) to incorporate both membership and non-membership degrees, enhancing the capacity to handle ambiguity. After that, Smarandache [3] suggested neutrosophic sets (NS), which include three memberships. Expanding on these concepts, Raut, P. K. [4] introduced Fermatean neutrosophic sets (FNS) by integrating Fermatean fuzzy sets with neutrosophic sets, gaining significant attention in research. Recent studies by Raut, P. K. et al. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and Chakraborty, A. [16, 17] have extended these concepts, particularly in neutrosophic and Fermatean neutrosophic environments.

The TSP is a renowned NP-hard optimization challenge in mathematics and computer science, was initially formulated in the 1930s, gaining prominence post-1950 [18, 19, 20]. The problem involves a salesman traveling through a different city, returning to the initial vertex while minimizing travel cost or distance. Addressing TSP under uncertain conditions, such as variable transportation costs and times influenced by weather or traffic, presents a complex challenge for decision-makers. Fuzzy sets offer a solution by enabling decision-making with imprecise data. Numerous studies [21, 22, 23] have explored on related to fuzzy TSP.

To address these challenges, many researchers have introduced exact, heuristic, and metaheuristic algorithms for TSP. Exact methods offer optimal solutions but become computationally infeasible for larger networks. Heuristic and metaheuristic approaches, such as genetic algorithms, ABC algorithms, and harmony search methods, are more efficient in reducing computational time. Genetic algorithms, which simulate evolutionary processes, stand out for their adaptability. They employ genetic operations like selection, crossover, and mutation to evolve solutions iteratively, demonstrating their efficacy in evaluating TSP and other optimization problems [24, 25, 26, 27, 28, 29, 30].

This study focuses on solving TSP using a genetic algorithm that incorporates Fermatean neutrosophic numbers to represent arc lengths, addressing the limitations of standard evolutionary methods under uncertainty. The proposed approach integrates a novel mathematical model for Fermatean neutrosophic arc lengths and introduces innovative crossover and mutation in the genetic algorithm. To validate the methodology, a numerical example illustrates.

This research article is organized in following sections: In Section-2 explains motivation, In Section-3 research gaps, and in Section-4 presents a flow chart. Section-5 reviews foundational concepts of neutrosophic and Fermatean neutrosophic theories. Section-6 explains the genetic algorithm framework. Section-7 presents the mathematical model for the Fermatean neutrosophic TSP. Section-8 details the proposed evolutionary method. Section-9 evaluates the experimental results, and Section-10 presents the conclusion**.**

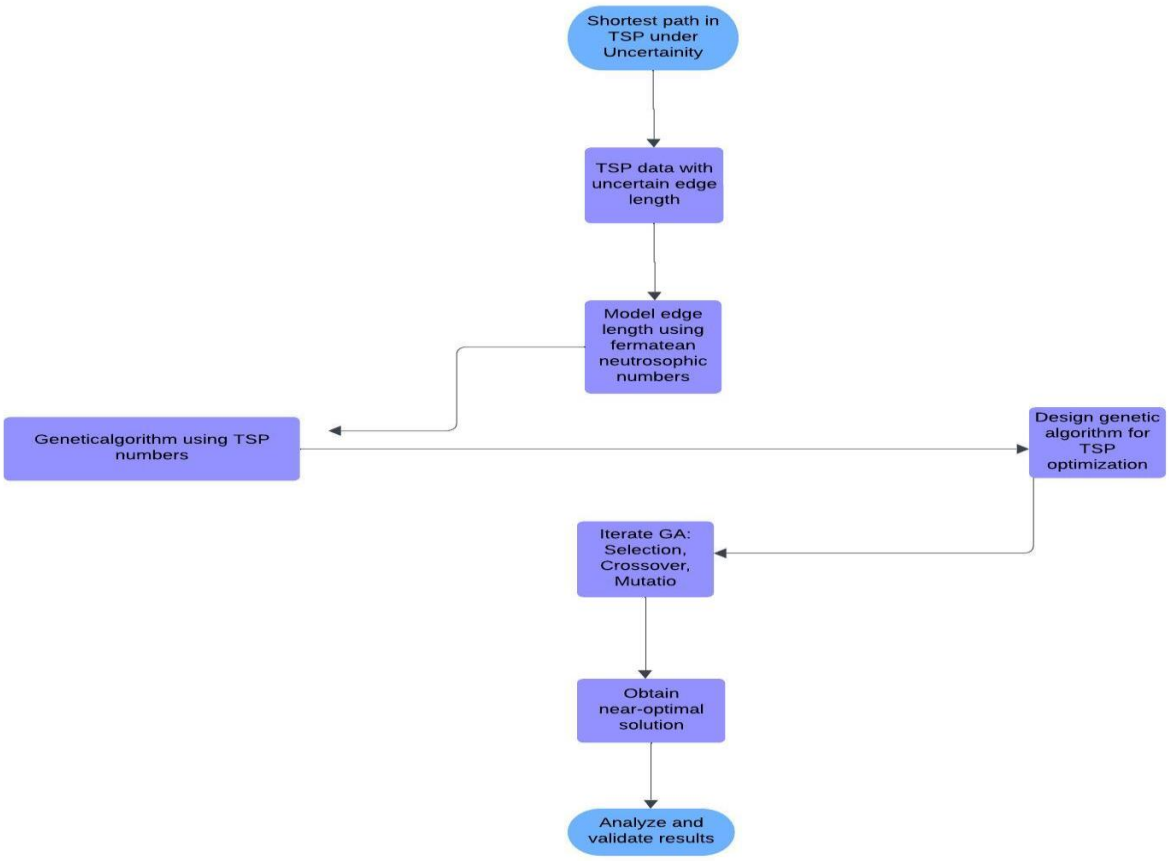
2. Motivation

We emphasized the critical challenges associated with solving the Traveling Salesman Problem (TSP) in uncertain environments, where traditional methods fail to adequately handle the complexities introduced by imprecise and uncertain data. We also discussed the increasing demand for effective and realistic optimization techniques in fields like logistics, transportation, and network design, which are frequently affected by uncertainty in edge lengths.

3. Research Gap

We identified the lack of robust methodologies for addressing the TSP under uncertain conditions as a significant gap in the current literature. While existing approaches use fuzzy or neutrosophic numbers, their ability to express higher degrees of uncertainty and interdependencies is limited. This gap motivated our exploration of Fermatean neutrosophic numbers, which provide a more precise and flexible framework for modelling uncertainty. Our work introduces a novel integration of genetic algorithms with Fermatean neutrosophic numbers to evaluate the shortest path in a network. This approach not only enhances the representation of uncertainty but also improves the efficiency and accuracy of obtaining near-optimal solutions compared to conventional techniques.

4. Flowchart representing the proposed approach


Shortest path in TSP under Uncertainity
TSP data with uncertain edge length
Model edge length using fermatean neutrosophic numbers
Geneticalgorithm using TSP numbers
Design genetic algorithm for TSP optimization
Iterate GA: Selection, Crossover, Mutatio
Obtain near-optimal solution
Analyze and validate results

5. Preliminaries

5.1 Neutrosophic set [5]

Neutrosophic set is an extension of Intuitionistic fuzzy sets, allowing for the representation of uncertainty, vagueness, and indeterminacy.

A Neutrosophic set A is defined as $A = (\breve{x}, \left(\breve{T}_{\breve{A}}(\breve{x}), \breve{I}_{\breve{A}}(\breve{x}), \breve{F}_{\breve{A}}(\breve{x}) \right) : \breve{x} \in \breve{X}.$

$\breve{x} \in U\ ,\ 0 \leq \breve{T}_{\breve{A}}(\breve{x}) + \breve{I}_{\breve{A}}(\breve{x}) + \breve{F}_{\breve{A}}(\breve{x}) \leq 3$

$\breve{T}_{\breve{A}}(\breve{x})\ , \breve{I}_{\breve{A}}(\breve{x}), \breve{F}_{\breve{A}}(\breve{x})$represents degree of truth, indeterminacy and falsity membership respectively.

5.2 Fermatean neutrosophic set [13]

A Fermatean neutrosophic set A in a given universe of discourse U is characterized by three membership functions:$\breve{T}_{\breve{A}}(\breve{x}), \breve{I}_{\breve{A}}(\breve{x}), \breve{F}_{\breve{A}}(\breve{x})$. These functions assign to each element $\breve{x} \in U$ a value within the interval [0,1].

Fermatean neutrosophic set A is defined as:$A = (\breve{x}, \left(\breve{T}_{\breve{A}}(\breve{x}), \breve{I}_{\breve{A}}(\breve{x}), \breve{F}_{\breve{A}}(\breve{x}) \right) : \breve{x} \in \breve{X}.$

$0 \leq (\breve{T}_{\breve{A}}(\breve{x}))^3 + (\breve{F}_{\breve{A}}(\breve{x}))^3 \leq 1\ , 0 \leq (\breve{I}_{\breve{A}}(\breve{x}))^3 \leq 1\ .$

and sum cube of this three-membership degree lies in between [0, 2]

i.e. $0 \leq (\breve{T}_{\breve{A}}(\breve{x}))^3 + (\breve{I}_{\breve{A}}(\breve{x}))^3 + (\breve{F}_{\breve{A}}(\breve{x}))^3 \leq 2$.

5.3 Euclidean Distance between Fermatean Neutrosophic numbers

The Euclidean Distance is a common metric used to measure the distance between two vertices or entities in space. When applied to Fermatean Neutrosophic Numbers (FNNs), which are an extension of neutrosophic numbers, the Euclidean Distance is computed by taking into account the three memberships.

For two Fermatean Neutrosophic Numbers, $\boldsymbol{A} = (\breve{\boldsymbol{T}}_{\boldsymbol{1}}, \breve{\boldsymbol{I}}_{\boldsymbol{1}}, \breve{\boldsymbol{F}}_{\boldsymbol{1}})$, $\boldsymbol{B} = (\breve{\boldsymbol{T}}_{\boldsymbol{2}}, \breve{\boldsymbol{I}}_{\boldsymbol{2}}, \breve{\boldsymbol{F}}_{\boldsymbol{2}})$ the Euclidean Distance is defined as

$$d(A, B) = \sqrt{(\breve{T}_1 - \breve{T}_2)^2 + (\breve{I}_1 - \breve{I}_2)^2 + (\breve{F}_1 - \breve{F}_2)^2}$$

6. Genetic algorithm

The TSP is a most significant optimization problem in the domains of computational science and mathematics. The problem relates to a salesman that begins on a journey to multiple cities before returning to the initial location. This problem has been extensively studied due to its high complexity and practical applications in logistics, network routing, and mapping. While exact algorithms can solve TSP for small problem sizes, they become computationally infeasible as the number of cities increases.

The ideas behind natural genetics are known as genetic algorithms.

This is a powerful metaheuristic approach for solving TSP, offering approximate solutions that can be close to the optimal solution in reasonable computation times. This proposed GA for evaluating the shortest path aims to optimize the TSP by leveraging the principles of evolution.

Objective: The aim of this suggested genetic algorithm (GA) is to identify the most efficient path that starts from a particular city, visits other cities, and then backs to the initial city while minimizing the overall distance traveled.

7. Proposed Methodology

Encoding: Each individual in the population will be encoded as a sequence of integers representing the order of cities to visit, where each integer corresponds to a unique city.

Initialization: A population of potential solutions will be randomly generated, each representing a different permutation of cities. The population size should be large enough to explore a diverse set of solutions.

Fitness Evaluation: The assessment of each individual's fitness in the population is evaluated by the total distance covered following the sequence of cities in their permutation.

Selection: Individuals with greater fitness scores are more likely to be chosen as parents for reproduction.

Crossover: The selected parents will undergo crossover, where their genetic material (permutation of cities) is combined to create offspring.

Mutation: To introduce diversity into the population, a mutation operator is applied to randomly alter the genetic material of some individuals. Mutation can involve swapping or inserting a city in a particular position.

Survival Selection and Termination: The next generation is formed by choosing a specific number of individuals from both the offspring and the existing population, prioritizing higher fitness levels. The cycle of selection, crossover, and mutation continues until a termination criterion is fulfilled.

Result: Once the termination condition is satisfied, the optimal individual in the final population will indicate the shortest path discovered for the TSP.

8. The pseudocode of the genetic algorithm for Fermatean neutrosophic shortest path

Generate an initial population of chromosomes.

Assess the fitness of each chromosome within the population.

Continue executing the given directions until the final condition is satisfied:

Selection:

Choose individuals as parents from the population according to their level of fitness.

Utilize selection methods to select individuals.

Crossover:

Create offspring by merging genetic material from chosen parents. Apply several crossover approaches, such as one-point crossover phenomenon, two-point crossover phenomenon, uniform crossover phenomenon, and others.

Mutation:

Apply mutation to the offspring to introduce genetic diversity

Perform mutation techniques such as bit-flip mutation, swap mutation, inversion mutation, etc.

Evaluate fitness of new individuals

Replacement:

Choose specific individuals to propagate to the subsequent generation, replacing the previous population.

Perform replacement strategies such as generational replacement, elitism, etc.

Termination condition:

Verify whether the termination condition has been satisfied, if satisfied then end.

Return the best most optimal individual found in the final population as the solution

9. Development of a Fermatean Neutrosophic Model for the TSP

The formulation of a Fermatean neutrosophic TSP involves introducing Fermatean neutrosophic or uncertain parameters into the traditional TSP formulation. The goal is to optimize the tour or route of a salesman that begins on a journey to multiple cities before returning to the stating city, while considering the uncertainty or imprecision associated with certain parameters such as distances, costs, times, or demands.

The Fermatean neutrosophic TSP can be defined as follows:

Consider x_{ij} defined a binary decision variable,

where $\begin{cases} x_{ij} = 1, \text{if the salesman visits} \\ x_{ij} = 0, \quad \text{otherwise.} \end{cases}$

Reducing the overall cost, distance, or time of the routes, which can be defined as: minimize $Z = \sum\sum c_{ij} * x_{ij}$

Constraints:

a. Ensure that each city is visited exactly once: $\sum x_{ij} = 1, \forall j \in \text{Cities}$

b. Ensure that the salesman back to the initial city: $\sum x_{ij} = 1$

c. Eliminate subtours (subsets of cities that do not include the starting city): $\sum x_{ij} \leq n - 1$, for all $i \in \text{Cities}, i \neq 1 \sum xij - \sum xji \leq 0$.

Fermatean neutrosophic Parameters: The uncertain parameters, such as distances or costs, can be represented as Fermatean neutrosophic numbers. This involves assigning degrees of membership to different values or intervals of the parameters.

Fermatean neutrosophic Objective Function: The objective function can incorporate Fermatean neutrosophic sets to represent the uncertainty. This can involve maximizing or minimizing a Fermatean neutrosophic value, Fermatean neutrosophic interval, or Fermatean neutrosophic set representing the costs, distances, or time considering their fuzziness.

Fermatean neutrosophic Constraints: The constraints can also employ Fermatean neutrosophic sets to represent the uncertainty associated with parameters. These constraints can include Fermatean neutrosophic equalities or inequalities, where membership degrees are used to model the degree of satisfaction or violation of a constraint.

10. Numerical example

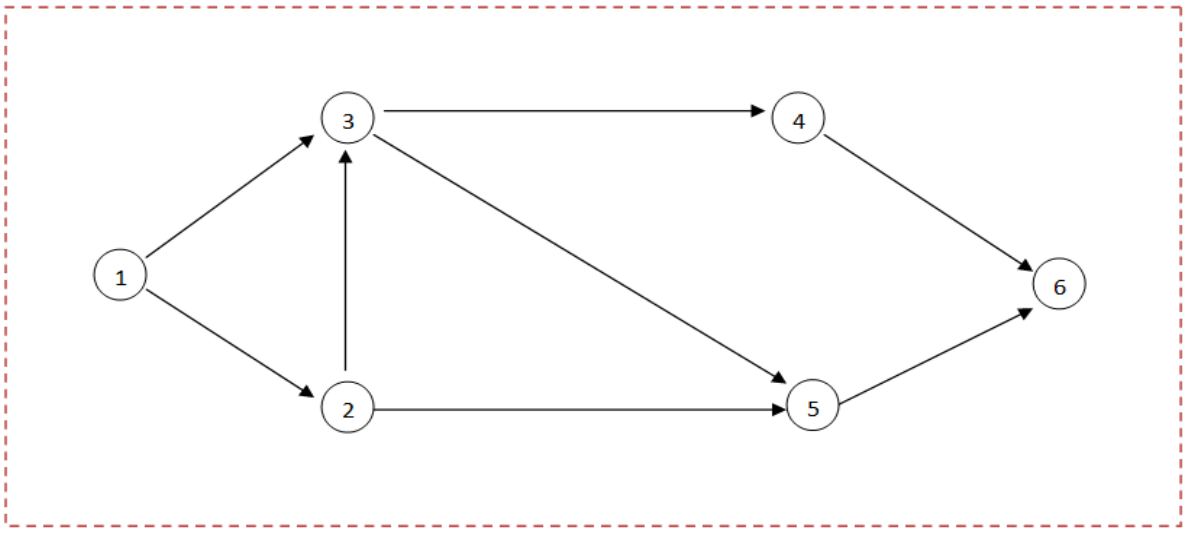


Fig. 1: Routes of TSP with Fermatean neutrosophic number

Step-1: Initialization:

In Table-1, we consider an initial population. We construct a 6-node network consisting of five routes, each of which is arrangement of the 6 cities given in Table 2. The Fermatean neutrosophic number ($\breve{T}$, $\breve{I}$, $\breve{F}$) represents each city.

Table-1. 6 cities edge lengths

City	Fermatean Neutrosophic number
1	<0.6,0.3,0.1>
2	<0.5, 0.3, 0.2>
3	<0.3, 0.6, 0.1>
4	<0.1, 0.7, 0.3>
5	<0.2, 0.1, 0.7>
6	<0.4, 0.4, 0.2>

Table-2. Possible routes

Route no	Possible route
1	$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 6$
2	$1 \rightarrow 3 \rightarrow 4 \rightarrow 6$
3	$1 \rightarrow 3 \rightarrow 5 \rightarrow 6$
4	$1 \rightarrow 2 \rightarrow 5 \rightarrow 6$
5	$1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6$

Step 2: Evaluation (Fitness):

We evaluate the fitness of every route by employing a specific fitness function. In this instance, the function incorporates Fermatean neutrosophic numbers and is designed to minimize the overall distance, taking into consideration any inherent uncertainty.

$$\text{Fitness} = \sum \frac{(T_i * (d_i))}{(I_i + 1)}$$

Where:

T_i denote the node's highest values

I_i denote the node's lowest values

(d_i) denote the is the interval separating two adjacent nodes along the route.

Table-3: Finding Fitness value for first Route by using definition-2.3

Distance between node	Crisp value
1-2	0.14
1-3	0.42
2-3	0.37
2-5	0.61
3-4	0.30
3-5	0.78
4-6	0.42
5-6	0.61

Final Distance =0.14+0.37+0.30+0.42=1.23

$$\text{Fitness} = \sum \frac{(0.6 * 1.23)}{(0.1 + 1)}$$

$$\text{Fitness} = \sum \frac{(0.738)}{(1.1)} = 1.99$$

Similarly, we evaluate 5 Routes

Table-4: Route fitness

Fitness function of Route	Minimize distance
1	1.99
2	0.62
3	0.98
4	0.74
5	1.03

Step 3: Route Selection (Choosing Two Routes):

Utilize a selection method to evaluates the routes that will serve as parents for the next generation. This process increases the likelihood of selecting routes with lower fitness values.

Chosen Routes:

The two routes lowest fitness values are as follows:

- Fitness value of Route 2 is 0.62
- Fitness value of Route 4 is 0.74

Step-4 Crossover (Recombination):

Perform a crossover operation on the selected routes to generate new offspring routes, ensuring the procedure aligns with the framework of neutrosophic numbers.

Genetic Information Combination:

- Parent Route 4: [1, 2, 5, 6]
- Parent Route 2: [1, 3, 4, 6]

Crossover Point after Node 1

- Initial Offspring Route: [1 | 3, 4, 6]

Finalizing the Offspring:

To finalize the offspring, add the missing nodes from the other parent without creating duplicates:

- Final Offspring Route: [1, 3, 4, 6]

Step 5: Mutation (Swapping Two Random Nodes):

Introduce minor random changes to the routes through mutation operators while ensuring compliance with neutrosophic number constraints. For example, swapping nodes 4 and 3 results in the following:

- Mutated Offspring Route: [1, 3, 4, 6]

Step 6: Replacement (Substituting a Parent):

Replace certain existing routes in the population with newly mutated offspring to produce the next generation. For instance, one of the parent routes (e.g., Route 3) can be substituted with the mutated offspring:

- Updated Population: [1, 3, 4, 6]

Step 7: Termination:

This procedure is iteratively performed over multiple generations, progressively improving the solutions. The process ends once the established stopping condition is met. This given example illustrates how a Genetic Algorithm (GA) is used for solving the Traveling Salesman Problem (TSP) in a 6-node network with Fermatean neutrosophic numbers. Advanced genetic operators and fitness evaluation techniques are typically applied in practice to enhance optimization performance.

10. Result: The optimal path determined using Traveling Salesman Problem with Fermatean neutrosophic numbers.

Table 5. The best result of SP

The shortest path problem for FTSP was addressed using linear programming techniques in LINGO software.	Proposed Genetic Algorithm for FTSP Shortest Path:
Minz=0.62	Cost =0.62 and shortest route= $1 \rightarrow 3 \rightarrow 4 \rightarrow 6$

In the given numerical example, the traveling salesman problem of neutrosophic graphs is solved by applying our proposed GAs. We consider six nodes in Figure-1 and the five pathways in Table-2. In this Traveling Salesman problem, we used a genetic algorithm approach to evaluate the SP of given network, and this is the novel idea we applied for fermatean neutrosophic numbers. The first example shows how our suggested

method and language program may discover the best possible shortest path. According to Table-5, the shortest path and cost were same in both instances.

11. A comparative analysis of our approach with alternative methodologies.

Table 4. Comparison of the shortest path and its associated cost.

Methodology	Shortest path	Shortest path cost
Begum, S. G. et al. [30]	3→1→2→4→5→ 6→7	1.86
Dey, A. et al. [31]	3 →1→ 2→ 4→ 5→ 6→7	1.02
Raut, P. K. et al. [32]	3 →1→2→4→5→ 6→7	0.86
Our proposed method in Fermatean Neutrosophic environment	1 → 3 → 4 → 6	0.62

Study	**Methodology**	**Environment**	**Shortest Path Length**	**Strengths**	**Limitations**
Begum, S. G. et al. [30]	Applied an intuitionistic fuzzy set for shortest path evaluation	fuzzy environment	Moderate	Improved handling of hesitation and vagueness in uncertain conditions	Limited scalability for complex real-world networks
Dey, A. et al. [31]	Used a fuzzy-based approach to solve the shortest path problem	Fuzzy environment	Slightly better than [30]	Handles moderate uncertainty well and provides computational efficiency	Lacks precision in modelling higher-order uncertainty
Raut, P. K. et al. [32]	Leveraged a genetic algorithm with neutrosophic numbers	Neutrosophic environment	Better than [31]	Superior uncertainty modeling compared to fuzzy methods and more robust optimization	Computational cost is higher due to the complexity of neutrosophic operations

Our Proposed Method	Combines a genetic algorithm with Fermatean neutrosophic numbers	Fermatean neutrosophic environment	Best result	Achieves the most precise uncertainty modeling and delivers highly efficient solutions	Requires extensive testing in dynamic and real-world applications

We compared our proposed method with three existing methods [30, 31, 32], and the results demonstrate that the new approach is more effective for evaluating the shortest path in a Fermatean neutrosophic environment**.**

12. Conclusions

In this research, we addressed the classic Traveling Salesman Problem (TSP) by integrating a genetic algorithm (GA) with a Fermatean neutrosophic environment, providing a robust framework for solving optimization problems under uncertainty. The proposed approach leverages the exploratory power of GAs to efficiently navigate large search spaces and the flexibility of Fermatean neutrosophic numbers to accurately model uncertainty and imprecision in edge lengths. Through a detailed numerical example, we demonstrated the effectiveness of our method in generating near-optimal solutions. The results highlight the practical applicability of the proposed approach in various domains, such as logistics, transportation networks, and other optimization scenarios that can be modelled as TSP.in future we Explore the integration of genetic algorithms with other metaheuristic methods, such as particle swarm optimization or ant colony optimization, to further enhance solution quality and convergence speed.

References

1. Zadeh, L. A. (1965). Information and control. Fuzzy sets, 8(3), 338-353.

2. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87-96.

3. Smarandache, F. (1999). A unifying field in Logics: Neutrosophic Logic. In Philosophy (pp. 1-141). American Research Press.

4. Raut, P. K., Behera, S. P., Broumi, S., & Mishra, D. (2023). Calculation of shortest path on Fermatean Neutrosophic Networks. Neutrosophic Sets and Systems, 57(1), 22.

5. Panda, N. R., Raut, P. K., Baral, A., Sahoo, S. K., Satapathy, S. S., & Broumi, S. (2025). An overview of neutrosophic graphs. Neutrosophic Sets and Systems, 77, 450-462.

6. Raut, P. K., Satapathy, S. S., Behera, S. P., Broumi, S., & Sahoo, A. K. (2025). Solving the shortest path Problem in an interval-valued Neutrosophic Pythagorean environment using an enhanced A* search algorithm. Neutrosophic Sets and Systems, 76, 360-374.

7. Raut, P. K., Behera, S. P., & Pati, J. K. (2021). Calculation of shortest path in a closed network in fuzzy environment. International Journal of Mathematics Trends and Technology-IJMTT, 67 (11), 31-37.

8. Raut, P. K., Behera, S. P., Broumi, S., & Baral, A. (2024). Evaluation of shortest path on multi stage graph problem using dynamic approach under neutrosophic environment. Neutrosophic Sets and Systems, 64, 113-131.

9. Raut, P. K., Behera, S. P., Broumi, S., & Baral, A. (2024). Evaluation of shortest path by using Breadth-first algorithm under neutrosophic environment. HyperSoft Set Methods in Engineering, 1, 34-45.

10. Bhuvaneshwari, S., Sweety, C., Singh, A., Broumi, S., Talea, M., & Raut, P. K. (2024). A novel and an efficient codas technique to solve real-life magdm problems in fermatean neutrosophic environment. Neutrosophic Sets and Systems, 72, 41-59.

11. Raut, P. K., & Behera, S. P. (2024). Evaluation of shortest path of network by using an intuitionistic pentagonal fuzzy number with interval values. International Journal of Reasoning-based Intelligent Systems, 16(2), 154-159.

12. Raut, P. K., Behera, S. P., Mishra, D., Kumar, V., & Mahanta, K. L. (2023). Fuzzy minimum spanning tree calculation-based approach on acceptability index method. In Intelligent Data Analytics, IoT, and Blockchain (pp. 184-194). Auerbach Publications.

13. Raut, P. K., Behera, S. P., Broumi, S., & Mishra, D. (2023). Calculation of shortest path on Fermatean Neutrosophic Networks. Neutrosophic Sets and Systems, 57, 328-341.

14. Raut, P. K., Behera, S. P., Broumi, S., & Mishra, D. (2023). Calculation of Fuzzy shortest path problem using Multi-valued Neutrosophic number under fuzzy environment. Neutrosophic Sets and Systems, 57, 356-369.

15. Broumi, S., Raut, P. K., & Behera, S. P. (2023). Solving shortest path problems using an ant colony algorithm with triangular neutrosophic arc weights. International Journal of Neutrosophic Science, 20(4), 128-28.

16. Chakraborty, A., Mondal, S., & Broumi, S. (2019). De-neutrosophication technique of pentagonal neutrosophic number and application in minimal spanning tree. Infinite Study.

17. Chakraborty, A. (2020). Minimal spanning tree in cylindrical single-valued neutrosophic arena. In Neutrosophic graph theory and algorithms (pp. 260-278). IGI Global.

18. Bellman, R. (1962). Dynamic programming treatment of the travelling salesman problem. Journal of the ACM (JACM), 9(1), 61-63.

19. Dorigo, M., & Gambardella, L. M. (1997). Ant colonies for the travelling salesman problem. biosystems, 43(2), 73-81.

20. Langevin, A., Soumis, F., & Desrosiers, J. (1990). Classification of travelling salesman problem formulations. Operations Research Letters, 9(2), 127-132.

21. Changdar, C., Mahapatra, G. S., & Pal, R. K. (2014). An efficient genetic algorithm for multi-objective solid travelling salesman problem under fuzziness. Swarm and Evolutionary Computation, 15, 27-37.

22. Maity, S., Roy, A., & Maiti, M. (2015). A modified genetic algorithm for solving uncertain constrained solid travelling salesman problems. Computers & Industrial Engineering, 83, 273-296.

23. Nwobi-Okoye, C. C., Okiy, S., & Obidike, F. I. (2017). Fuzzy based solution to the travelling salesman problem: a case study. In Proceedings of the World Congress on Engineering and Computer Science (Vol. 2, pp. 25-27).

24. Snyder, L. V., & Daskin, M. S. (2006). A random-key genetic algorithm for the generalized traveling salesman problem. European journal of operational research, 174(1), 38-53.

25. Moon, C., Kim, J., Choi, G., & Seo, Y. (2002). An efficient genetic algorithm for the traveling salesman problem with precedence constraints. European Journal of Operational Research, 140(3), 606-617.

26. Choi, I. C., Kim, S. I., & Kim, H. S. (2003). A genetic algorithm with a mixed region searches for the asymmetric traveling salesman problem. Computers & Operations Research, 30(5), 773-786.

27. Ahmed, Z. H. (2010). Genetic algorithm for the traveling salesman problem using sequential constructive crossover operator. International Journal of Biometrics & Bioinformatics (IJBB), 3(6), 96.

28. Majumdar, J., & Bhunia, A. K. (2011). Genetic algorithm for asymmetric traveling salesman problem with imprecise travel times. Journal of Computational and Applied Mathematics, 235(9), 3063-3078.

29. Majumdar, J., & Bhunia, A. K. (2011). Genetic algorithm for asymmetric traveling salesman problem with imprecise travel times. Journal of Computational and Applied Mathematics, 235(9), 3063-3078.

30. Begum, S. G., Praveena, N., & Rajkumar, A. (2022, September). Travelling salesman problem through fuzzy numbers using dynamic programming. In AIP Conference Proceedings (Vol. 2529, No. 1). AIP Publishing.

31. Dey, A., Kumar, R., & Broumi, S. (2022). The Neutrosophic Traveling Salesman problem with Neutrosophic Edge Weight: Formulation and A Genetic Algorithm. International Journal of Neutrosophic Science (IJNS), 19(3).

32.Raut, P. K., Behera, S. P., Broumi, S., Dey, A., Talea, M., & Baral, A. (2024). Evaluation of the shortest path based on the traveling salesman problem with a genetic algorithm in a neutrosophic environment. Neutrosophic Sets and Systems, 68, 187-197.

Received: Sep 2, 2024. Accepted: Dec 2, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Quality Assessment in Higher Education Management using the Modified MARCOS Method with Double-Valued Neutrosophic Numbers: A Case Study

Haihua Ding[1], Haifeng Chen[2*]

[1]Wenzhou University of Technology, 325000, Zhejiang, China (E-mail: 00195015@wzu.edu.cn)

[2]College of Chemistry and Materials Science, Zhejiang Normal University, 321000, Zhejiang, China

*Corresponding author, E-mail: shchf2023@163.com; shchf@zjnu.cn

https://orcid.org/0009-0004-2590-1240

Abstract

Quality is the core and soul of educational management, and educational management is the carrier of quality. Only by improving the efficiency and quality of educational management can we promote the healthy and long-term development of higher education. In the new era, universities should actively introduce education quality assurance theory and advanced management concepts, reform the education management quality assurance system, and thus play multiple educational functions such as enrollment management, student training, degree awarding, and quality evaluation. Based on the theoretical and event exploration of the quality assurance path of higher education management, China's higher education will inevitably embark on a unique path of innovative development. The quality assessment of higher education management is considered a multiple-attribute decision-making (MADM) problem. Recently, the MARCOS approach has been utilized to advance MADM approaches. Double-valued neutrosophic sets (DVNSs) serve as optimal decision-making approaches to represent uncertainty in data during the evaluation of higher education management in academic institutions. In this research, the MARCOS approach is developed for MADM using DVNSs. Subsequently, the double-valued neutrosophic number MARCOS (DVNN-MARCOS) approach is proposed for MADM. Finally, a numerical example is provided to validate the DVNN-MARCOS model in the context of quality evaluation for higher education management.

Keywords: MADM; DVNSs; MARCOS model; higher education management

1. Introduction

Education management is a guaranteed service rooted in the education industry of universities, laying a solid foundation for improving the quality of education and achieving stable development of school teaching [1]. With the development and progress of society, the management mode of higher education has also undergone profound changes. The quality assurance system is a comprehensive evaluation instituted by the goals, forms, and contents of educational management, and ensures the smooth implementation of educational management work through reasonable means, thereby playing the necessary educational auxiliary role[2]. With the rapid popularization of the concept of education popularization, it is imperative to improve the quality of higher education management and reform the higher education management system[3]. Based on this, the author has analyzed in detail the basic connotation and characteristics of quality assurance in higher education management and proposed a path concept for quality assurance in higher education management, to assist in the construction of a new ecosystem of higher education management[4]. At present, there is no unified definition for the concept of quality assurance in education management, which can only be analyzed from the perspective of basic definition and theoretical connotation. Quality of education management refers to the judgment or measurement of the achievement of established goals by the subject, while the guarantee is to protect it from infringement or destruction and maintain the stability of its supporting structure[5]. Based on studying a large amount of literature on education management quality assurance in the past, the author believes that although there are still significant differences in the subjects and measurement standards of education management quality assurance, research and measures for education management quality assurance should ultimately be implemented on the fundamental aspects of education management quality assurance, such as setting evaluation procedures and methods, establishing quality standards, and establishing quality evaluation institutions, to guide government departments Social organizations, parents, and other educational stakeholders provide "high-quality evidence" to enhance public trust in higher education, enabling universities to gain more development space, opportunities, and resources, and truly achieve high-quality development of educational management[6]. The quality assurance of educational

management usually has two significant characteristics[7]. One is the characteristic of the times, which reflects that universities must adopt different education security systems at different stages (development stage of universal higher education, development stage of elite higher education, and development stage of mass higher education). The quality assurance mechanism of higher education management has left a distinct imprint of the times on higher education and is a product of universities adapting to the needs of the times[8]. The second is the system characteristics, specifically the need for an integrated operation to improve the quality of education management [9, 10]. At the level of mechanism, it aims to ensure the smooth implementation of various educational reform projects in universities, and integrate the student-centered concept into teaching practice, in order to improve the efficiency of university management, the quality of talent cultivation, and the level of education. At the level of content assurance, it usually includes factors such as assurance activities, assurance systems, assurance measures, and assurance culture. At the level of mechanism construction, there is a development system and evaluation system that includes quality assurance. Finally, at the content level, it usually includes main elements such as government agencies, social organizations, and universities.

In 1965, Zadeh [11] introduced his groundbreaking theory of fuzzy sets (FSs), aimed at addressing various types of uncertainties. This innovative concept has been successfully applied to model uncertainty across numerous real-world domains. More recently, a new theory known as neutrosophic logic and sets has emerged. The term "neutrosophic" refers to the knowledge of neutral thought, which represents a key distinction between fuzzy and intuitionistic fuzzy logic and sets. Neutrosophic logic, introduced by Florentin [12], posits that each proposition is evaluated with three components: a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F). In the neutrosophic sets (NSs), each element of the universe is characterized by these three degrees, all of which fall within the non-standard unit interval [0, 1]. NSs have been successfully applied in various fields, including decision-making [13-15], and have broad applications in natural sciences, operations research, economics, management science, military strategy, and urban planning. They are particularly useful in decision-making scenarios where the ambiguity and complexity of attributes make it difficult to express or evaluate problems using real numbers. However, because NSs are challenging to apply directly in real-life situations, Wang, Smarandache, Zhang,

and Sunderraman [16] proposed the concept of SVNSs in 2005. In this framework, the degrees of truth, indeterminacy, and falsity for any element of an NS are restricted to the standard unit interval [0, 1]. The SVNSs generalize several concepts, including classical sets, fuzzy sets, intuitionistic fuzzy sets, and paraconsistent sets, offering a broad range of possibilities for solving real-world problems.

The quality evaluation of higher education management in academic institutions is considered a MADM problem. Recently, the MARCOS model [17] has been utilized to address MADM challenges. While several techniques have been used to process both qualitative and quantitative information simultaneously, they still face limitations, particularly in dealing with DVNSs [18,19], which are well-suited for representing uncertain data during the quality evaluation of higher education management. The primary goal of this research is to overcome the shortcomings of conventional classroom teaching quality evaluation methods, fully account for objective attribute weight, and manage uncertain data. To achieve this, the DVNSs and MARCOS approach were utilized to address the quality evaluation of higher education management. The DVNN-MARCOS approach. Finally, a comparative analysis was instituted between the DVNN-MARCOS approach and existing techniques to validate the effectiveness of the DVNN-MARCOS approach.

2. Preliminary Concepts and Definitions

The concept of Double-Valued Neutrosophic Sets (DVNSs) has been recognized as a valuable tool in managing uncertainty and indeterminacy within complex systems. This framework extends traditional neutrosophic sets by introducing dual membership functions for truth, indeterminacy, and falsity, enabling a more nuanced representation of ambiguous data. In the study presented by Kandasamy [18], DVNSs were employed to explore their applications in constructing minimum-spanning trees and developing clustering algorithms. This usage highlights the practical relevance and adaptability of DVNSs in addressing challenges across various domains in uncertainty analysis.

Definition 1. DVNS is an extension of the classical neutrosophic set, characterized by a pair of membership functions for each of the truth, indeterminacy, and falsity values. Formally, a DVNS as:

$$ZA=\left\{\left(\theta,ZT_A\left(\theta\right),ZIT_A\left(\theta\right),ZIF_A\left(\theta\right),ZF_A\left(\theta\right)\right)\middle|\theta\in\Theta\right\}, \quad (1)$$

with $ZT_A\left(\theta\right)$ be truth-membership, $ZIT_A\left(\theta\right)$ be indeterminacy leaning towards $ZT_A\left(\theta\right)$, $ZIF_A\left(\theta\right)$ be indeterminacy leaning towards $ZF_A\left(\theta\right)$, $ZF_A\left(\theta\right)$ is falsity-membership, $ZT_A\left(\theta\right),ZIT_A\left(\theta\right),ZIF_A\left(\theta\right),ZF_A\left(\theta\right)\in\left[0,1\right]$, $0\le ZT_A\left(\theta\right)+ZIT_A\left(\theta\right)+ZIF_A\left(\theta\right)+ZF_A\left(\theta\right)\le 4$.

The DVNN is expressed as $ZA=\left(ZT_A,ZIT_A,ZIF_A,ZF_A\right)$, where $ZT_A,ZIT_A,ZIF_A,ZF_A\in\left[0,1\right]$, $0\le ZT_A+ZIT_A+ZIF_A+ZF_A\le 4$.

Definition 2. The Score Value Number (SVN) for Double-Valued Neutrosophic Numbers (DVNNs) is defined as:

Let $ZA=\left(ZT_A,ZIT_A,ZIF_A,ZF_A\right)$ and $ZB=\left(ZT_B,ZIT_B,ZIF_B,ZF_B\right)$

$$SVN\left(ZA\right)=\frac{\left(2+ZT_A+ZIT_A-ZIF_A-ZF_A\right)}{4},\ SVN\left(ZA\right)\in\left[0,1\right].$$

$$SVN\left(ZB\right)=\frac{\left(2+ZT_B+ZIT_B-ZIF_B-ZF_B\right)}{4},\ SVN\left(ZB\right)\in\left[0,1\right]. \quad (2)$$

Definition 3. Let $ZA=\left(ZT_A,ZIT_A,ZIF_A,ZF_A\right)$ and $ZB=\left(ZT_B,ZIT_B,ZIF_B,ZF_B\right)$ be DVNNs, The Accuracy Value Number (AVN) for a DVNN is defined as:

$$AVN\left(ZA\right)=\frac{\left(ZT_A+ZIT_A+ZIF_A+ZF_A\right)}{4},\ AVN\left(ZA\right)\in\left[0,1\right]\ . \quad (3)$$

$$AVN\left(ZB\right)=\frac{\left(ZT_B+ZIT_B+ZIF_B+ZF_B\right)}{4},\ AVN\left(ZB\right)\in\left[0,1\right]\ . \quad (3)$$

The AVN provides an evaluation of the accuracy or reliability of a DVNN by considering the difference between its truth and falsity components. This measure plays a crucial role in comparing and ranking DVNNs, especially in scenarios where indeterminacy is not a primary concern.

Definition 4

Let $ZA=\left(ZT_A, ZIT_A, ZIF_A, ZF_A\right)$ and $ZB=\left(ZT_B, ZIT_B, ZIF_B, ZF_B\right)$,

$SVN\left(ZA\right)=\frac{\left(2+ZT_A+ZIT_A-ZIF_A-ZF_A\right)}{4}$, $SVN\left(ZB\right)=\frac{\left(2+ZT_B+ZIT_B-ZIF_B-ZF_B\right)}{4}$,

$AVN\left(ZA\right)=\frac{\left(ZT_A+ZIT_A+ZIF_A+ZF_A\right)}{4}$, $AVN\left(ZB\right)=\frac{\left(ZT_B+ZIT_B+ZIF_B+ZF_B\right)}{4}$, if

$SVN\left(ZA\right)<SVN\left(ZB\right)$, $ZA<ZB$; if $SVN\left(ZA\right)=SVN\left(ZB\right)$, (1)if $AVN\left(ZA\right)=AVN\left(ZB\right)$, $ZA=ZB$; (2) if $AVN\left(ZA\right)>AVN\left(ZB\right)$, $ZA<ZB$.

Definition 5 Let $ZA=\left(ZT_A, ZIT_A, ZIF_A, ZF_A\right)$ and $ZB=\left(ZT_B, ZIT_B, ZIF_B, ZF_B\right)$ be two DVNNs, the operations are defined as:

(1) $ZA\oplus ZB=\left(ZT_A+ZT_B-ZT_AZT_B, ZIT_A+ZIT_B-ZIT_AZIT_B, ZIF_AZIF_B, ZF_AZF_B\right)$;
(2) $ZA\otimes ZB=\left(ZT_AZT_B, ZIT_AZIT_B, ZIF_A+ZIF_B-ZIF_AZIF_B, ZF_A+ZF_B-ZF_AZF_B\right)$;
(3) $\lambda ZA=\left(1-\left(1-ZT_A\right)^{\lambda}, 1-\left(1-ZIT_A\right)^{\lambda}, \left(ZIF_A\right)^{\lambda}, \left(ZF_A\right)^{\lambda}\right), \lambda>0$;
(4) $\left(ZA\right)^{\lambda}=\left(\left(ZT_A\right)^{\lambda}, \left(ZIT_A\right)^{\lambda}, 1-\left(1-ZIF_A\right)^{\lambda}, 1-\left(1-ZF_A\right)^{\lambda}\right), \lambda>0.$

Definition 6. Let $ZA=\left(ZT_A, ZIT_A, ZIF_A, ZF_A\right)$ and $ZB=\left(ZT_B, ZIT_B, ZIF_B, ZF_B\right)$, the distance measure between $ZA=\left(ZT_A, ZIT_A, ZIF_A, ZF_A\right)$ and $ZB=\left(ZT_B, ZIT_B, ZIF_B, ZF_B\right)$ is defined as[20]:

$$DVNNDM\left(ZA, ZB\right)=\frac{1}{4}\left(\begin{array}{l} ZT_A\log\frac{2ZT_A}{ZT_A+ZT_B}+ZT_B\log\frac{2ZT_B}{ZT_A+ZT_B} \\ +ZT_A\log\frac{2ZIT_A}{ZIT_A+ZIT_B}+ZIT_B\log\frac{2ZIT_B}{ZIT_A+ZIT_B} \\ +ZT_A\log\frac{2ZIF_A}{ZIF_A+ZIF_B}+ZIF_B\log\frac{2ZIF_B}{ZIF_A+ZIF_B} \\ +ZF_A\log\frac{2ZF_A}{ZF_A+ZF_B}+ZF_B\log\frac{2ZF_B}{ZF_A+ZF_B} \end{array}\right) \tag{4}$$

3. DVNN-MARCOS approach for MADM

This section outlines the MARCOS (Measurement Alternatives and Ranking according to the Compromise Solution) method within the framework of Double-Valued Neutrosophic Numbers (DVNN) for Multi-Attribute Decision Making (MADM). The steps of the proposed method are detailed as follows:

Step 1: Create the Initial Decision Matrix

- Construct the decision matrix using DVNN values, where each alternative is evaluated against the criteria using DVNNs.
- Apply the Score Function to convert DVNNs into crisp values, providing a standardized decision matrix.
- Combine individual decision matrices (if applicable) into a single aggregated decision matrix.

Step 2. Creating the extended initial matrix. Expand the initial decision matrix by including the ideal (best) and anti-ideal (worst) solutions based on the criteria. This ensures a comprehensive comparison framework for all alternatives.

$$R = \begin{matrix} AAI \\ .. \\ AI \end{matrix} \begin{bmatrix} r_{aa1} & \cdots & r_{aan} \\ \vdots & \ddots & \vdots \\ r_{ai1} & \cdots & r_{ain} \end{bmatrix} \tag{5}$$

Where AAI refers to the cost criteria and AI refers to the positive criteria.

$$AAI = \min r_{ij} \; for \; positive \text{ criteria } and \max r_{ij} for \; cost \text{ criteria} \tag{6}$$

$$AI = \max r_{ij} \; for \; positive \text{ criteria } and \min r_{ij} for \; cost \text{ criteria} \tag{7}$$

Step 3. Normalize the extended matrix.

$$P_{ij} = \frac{r_{ai}}{r_{ij}} \; for \; cost \text{ criteria} \tag{8}$$

$$P_{ij} = \frac{r_{ij}}{r_{ai}} \; for \; positive \text{ criteria} \tag{9}$$

Step 4. Determine the weighted matrix. Multiply the normalized matrix by the weights of the criteria to form the weighted normalized matrix:

$$q_{ij} = w_j * n_{ij} \tag{10}$$

Step 5. Computing the utility degree. Calculate the utility degree for each alternative relative to the ideal and anti-ideal solutions:

$$K_i^- = \frac{S_i}{S_{aai}} \tag{11}$$

$$K_i^+ = \frac{S_i}{S_{ai}} \tag{12}$$

$$S_i = \sum_{i=1}^{n} q_{ij} \tag{13}$$

Step 6. Compute the utility function. Derive the utility function for each alternative using the utility degrees, which represent its relative closeness to the ideal solution.

$$f(K_i) = \frac{K_i^- + K_i^+}{1 + \frac{1-f(K_i^+)}{f(K_i^+)} + \frac{1-f(K_i^-)}{f(K_i^-)}} \tag{14}$$

$$f(K_i^-) = \frac{K_i^+}{K_i^+ + K_i^-} \tag{15}$$

$$f(K_i^+) = \frac{K_i^-}{K_i^+ + K_i^-} \tag{16}$$

Step 7. Order the alternatives.

- Rank the alternatives in descending order based on their utility function values.
- The alternative with the highest utility function value is considered the most optimal solution.

4. A Case study

The rise and fall of higher education are closely related to national development, social order, and individual well-being. Education management usually includes many important links such as enrollment management, student training, degree awarding, quality evaluation, etc. Therefore, universities need to not only do a good job in administrative management but also accelerate the construction of management mechanisms. Firstly, universities must continue to improve the enrollment management system for bachelor's and master's degrees to ensure the quality of students; Secondly, universities must deeply understand the significance of talent cultivation and establish a comprehensive talent cultivation mechanism to ensure the quality and efficiency of talent cultivation, such as implementing professional title evaluation mechanism, pre-employment appointment mechanism, employee assessment mechanism, service system, teaching mechanism, scientific research system, etc. In addition, universities must establish a guarantee system for credit allocation, degree awarding, course learning, and project research to help students complete their studies; Finally, universities must establish a quality assurance system for degree awarding, taking academic constraints, incentives, and management as entry points to effectively

improve the academic quality of universities. In addition, universities must establish a quality evaluation mechanism that is compatible with the above system to supervise the implementation of educational policies and the implementation of educational systems, to compensate for the shortcomings and deficiencies of university education management. The key to improving the quality of education is talent cultivation, which requires cooperation from multiple parties such as the government, society, and universities to establish a multi-level internal quality assurance system (such as enrollment quality assurance system, training quality assurance system, degree quality assurance system, etc.), comprehensively promoting the rapid development of higher education from the perspectives of ideology, organization, and activities. Firstly, university managers must clearly guide themselves with internal guarantees and promote the organic integration of administrative management and professional teaching, such as establishing and improving constraint, assessment, and incentive systems, as well as improving quality management, control, and evaluation in areas such as scientific research and development, cultural services, and education, in order to promote the long-term development of universities; Secondly, it is necessary to establish an external evaluation system with the participation and governance of multiple subjects, strictly control the quality of education management, pay attention to the value and effectiveness of subject participation, take internal university evaluation as the core, supplemented by external government evaluation and social evaluation, and form a stable pattern of internal and external coordinated development; Finally, through administrative means, promote the effective integration of internal and external evaluation mechanisms and embed them into educational management work, thereby deepening the management function of internal and external guarantee mechanisms. The quality evaluation of higher education management is MADM. Six provincial comprehensive universities are evaluated with 15 attributes.

In the first step, three experts evaluate the decision matrix, which is presented in Table 1. Their input serves as the foundation for further calculations and analysis.

Next, using Equation (5), an extended version of the decision matrix is constructed. This step ensures that all relevant data is prepared for the subsequent normalization process.

Following this, Equation (8) is applied to normalize the decision matrix. The normalized results are shown in Table 2. After normalization, the criteria weights are calculated, and these weights are illustrated in Figures (1a) and (1b) which illustrate the following points:

The weights for the criteria show significant variability, indicating that different criteria hold varying levels of importance.

Some criteria, such as "Governance and Leadership" (0.0711) and "Strategic Planning and Policy Development" (0.0707), have relatively higher weights, suggesting their strong influence or importance in the context.

The lowest weights are observed for "Use of Technology in Management and Learning" (0.0630) and "Sustainability and Environmental Practices" (0.0622), indicating these criteria are less influential compared to others.

Several criteria, such as "Quality of Academic Programs" (0.0699) and "Financial Management and Resource Allocation" (0.0668), are in the mid-range, reflecting moderate importance.

There are sharp rises and drops in the weights across the criteria:

For instance, after the peak at "Governance and Leadership" (0.0711), the weight for "Use of Technology in Management and Learning" (0.0630) drops sharply.

Similarly, the weight increases again for "Strategic Planning and Policy Development" (0.0707), indicating a strong emphasis on these areas.

Criteria such as "Governance and Leadership", "Strategic Planning and Policy Development", and "Institutional Reputation and Accreditation" have consistently higher weights, emphasizing their critical roles in the evaluation.

Weights for criteria like "Sustainability and Environmental Practices" and "Monitoring and Evaluation Systems" are relatively consistent but low, showing they are not as highly prioritized.

The variability in the weights highlights that some aspects, like leadership, policy development, and academic quality, are deemed more critical, possibly because they directly impact the overall outcomes or goals.

Lower weights for criteria like "Sustainability" or "Technology Usage" could indicate these are either less prioritized in this context or less impactful in achieving the objectives.

The steep changes between certain criteria suggest a strong delineation of priorities.

Subsequently, the weighted decision matrix is created using Equation (9). This process adjusts the normalized values by their respective weights, and the results are displayed in Table 3.

In the fifth step, the utility degree is calculated using Equations (11) through (13). These calculations help in determining how well each alternative aligns with the defined criteria.

The next step involves computing the utility function using Equations (14) through (16). This function provides a comprehensive measure of the effectiveness of each alternative.

Finally, based on the computed values, the alternatives are ranked. The final order of the alternatives is shown in Figure 2, providing a clear comparison of their relative performance.

Table 1. Opinions of the first expert.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.16, 0.47,0.28, 0.59)	(0.79, 0.23, 0.80, 0.34)
C_2	(0.12, 0.65, 0.41, 0.69)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.65, 0.71, 0.20, 0.35)
C_3	(0.65, 0.71, 0.20, 0.35)	(0.79, 0.23, 0.80, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)
C_4	(0.79, 0.23, 0.80, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.32, 0.66,0.91, 0.34)
C_5	(0.16, 0.47,0.28, 0.59)	(0.44, 0.45,0.26, 0.53)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)
C_6	(0.44, 0.45,0.26, 0.53)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.16, 0.47,0.28, 0.59)
C_7	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.32, 0.66,0.91, 0.34)	(0.79, 0.23, 0.80, 0.34)
C_8	(0.12, 0.65, 0.41, 0.69)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.65, 0.71, 0.20, 0.35)
C_9	(0.65, 0.71, 0.20, 0.35)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)
C_{10}	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.79, 0.23, 0.80, 0.34)	(0.32, 0.66,0.91, 0.34)
C_{11}	(0.16, 0.47,0.28, 0.59)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.44, 0.45,0.26, 0.53)
C_{12}	(0.44, 0.45,0.26, 0.53)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)
C_{13}	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.79, 0.23, 0.80, 0.34)
C_{14}	(0.12, 0.65, 0.41, 0.69)	(0.65, 0.71, 0.20, 0.35)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)
C_{15}	(0.12, 0.65, 0.41, 0.69)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)	(0.32, 0.66,0.91, 0.34)	(0.12, 0.65, 0.41, 0.69)

Table 2. The normalized data.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	1	0.850242	0.830918	0.954911	0.850242	0.908213
C_2	0.594306	0.864769	0.626335	0.6121	0.70344	1
C_3	1	0.669039	0.864769	0.677343	0.615658	0.652432

C_4	0.903846	1	0.879808	0.831731	0.915064	0.831731
C_5	0.868313	0.834019	0.711934	0.783265	1	0.813443
C_6	1	0.75321	0.947218	0.907275	0.736091	0.801712
C_7	0.826087	1	0.850242	0.830918	0.89533	0.908213
C_8	0.594306	0.864769	0.626335	0.6121	0.70344	1
C_9	1	0.644128	0.871886	0.670225	0.622776	0.645314
C_{10}	1	0.84434	0.834906	0.878931	0.886792	0.816038
C_{11}	0.795181	0.990964	0.808735	0.799699	1	0.948795
C_{12}	0.957447	0.788754	1	0.975684	0.775076	0.849544
C_{13}	1	0.832536	1	0.832536	0.832536	0.934609
C_{14}	0.594306	1	0.644128	0.622776	0.696323	0.615658
C_{15}	1	0.744856	0.864198	0.775034	0.716049	0.746228

Table 3. The weighted normalized data.

	A_1	**A_2**	**A_3**	**A_4**	**A_5**	**A_6**
C_1	0.063119	0.053667	0.052447	0.060273	0.053667	0.057326
C_2	0.041543	0.060449	0.043782	0.042787	0.049172	0.069902
C_3	0.071146	0.047599	0.061525	0.04819	0.043802	0.046418
C_4	0.056982	0.063044	0.055467	0.052436	0.057689	0.052436
C_5	0.059765	0.057404	0.049001	0.053911	0.068828	0.055988
C_6	0.067962	0.051189	0.064375	0.06166	0.050026	0.054486
C_7	0.051333	0.06214	0.052834	0.051633	0.055636	0.056436
C_8	0.041543	0.060449	0.043782	0.042787	0.049172	0.069902
C_9	0.07075	0.045572	0.061686	0.047419	0.044062	0.045656
C_{10}	0.063044	0.053231	0.052636	0.055411	0.055907	0.051446
C_{11}	0.053158	0.066246	0.054064	0.05346	0.06685	0.063427
C_{12}	0.063464	0.052282	0.066285	0.064673	0.051376	0.056312
C_{13}	0.064175	0.053428	0.064175	0.053428	0.053428	0.059978
C_{14}	0.039393	0.066285	0.042696	0.041281	0.046156	0.040809
C_{15}	0.066567	0.049583	0.057527	0.051592	0.047666	0.049674

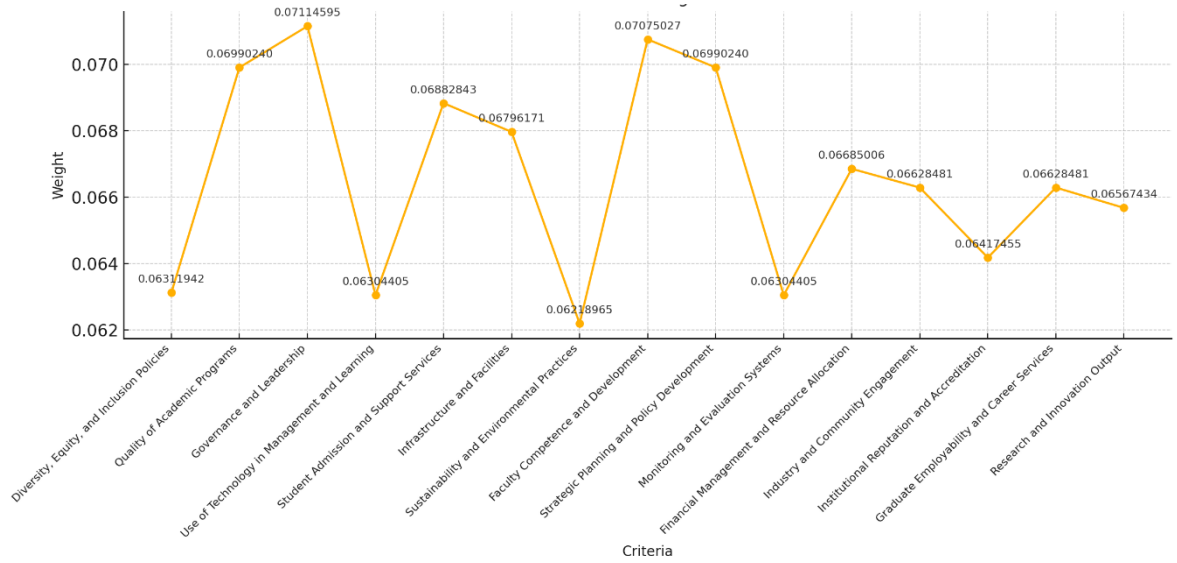


Figure 1a. The criteria weights.

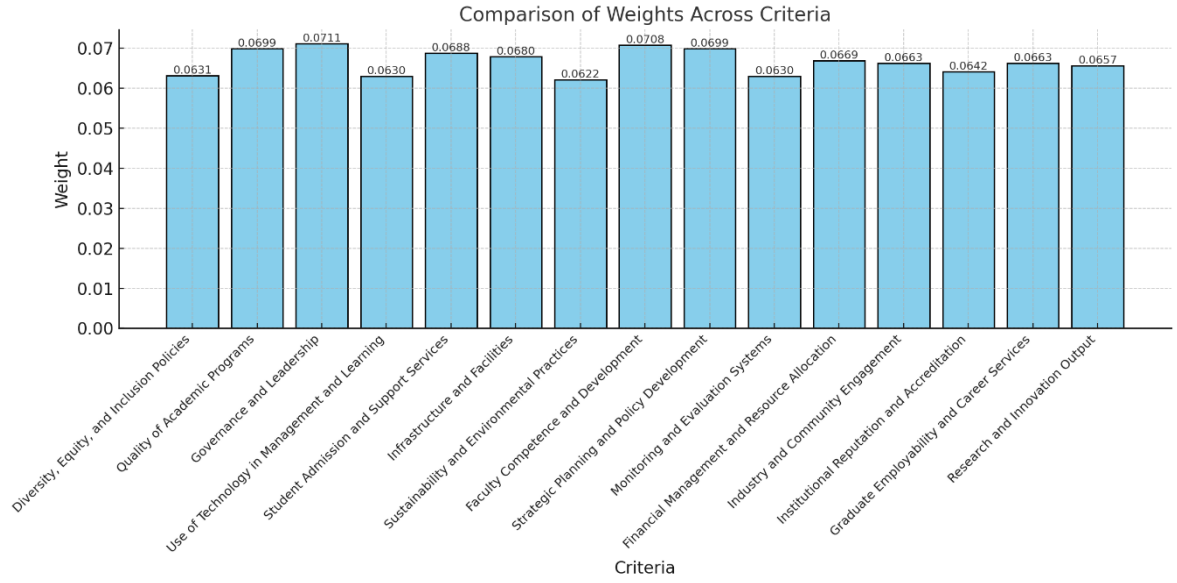


Figure 1b. Comparison of weights across criteria

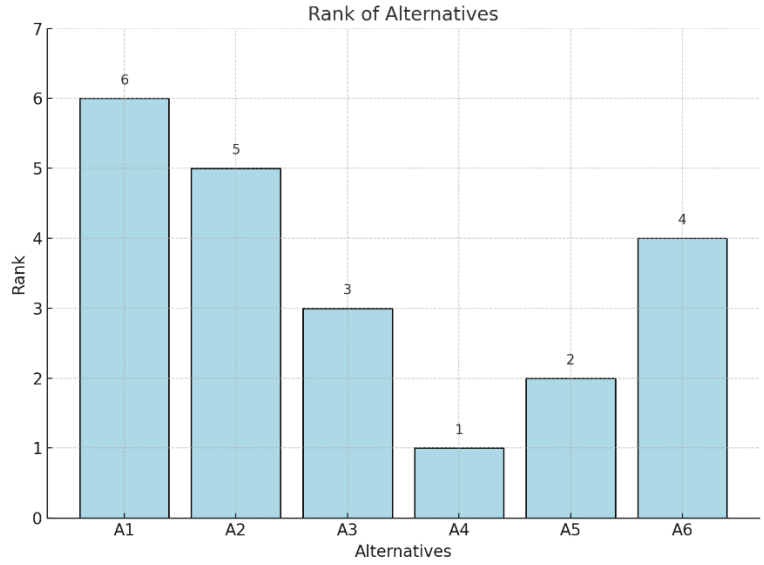


Figure 2. The rank of alternatives.

5. Analysis and Discussion

We conducted a sensitivity analysis by varying the criteria weights across multiple scenarios to examine the ranking of alternatives. Specifically, the criteria weights were adjusted in eight different cases, as illustrated in Figure 3. Using these modified weights, the proposed method was applied to evaluate the alternatives under each case.

The resulting rankings for the alternatives under these scenarios are presented in Figure 4. The analysis demonstrates that the rankings of the alternatives remain consistent and stable, even when the criteria weights are altered across different cases. This stability highlights the robustness of the proposed method.

Figure 3. The different criteria weights.

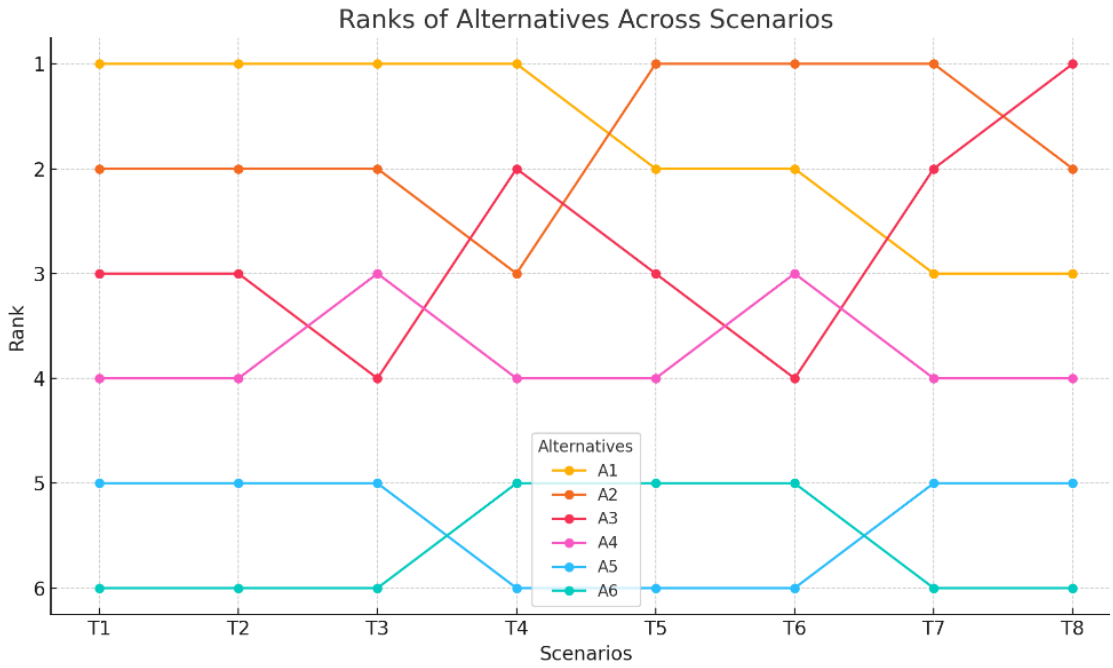


Figure 4. The different ranks of alternatives.

This part shows the comparative analysis between the proposed method and other methods to show the strength of the proposed method. We used the criteria weights of this study and ranked the other methods as shown in Figure 5, this figure compares the ranks of alternatives (A1 to A6) using five different methods: Proposed, CoCoSo method, VIKOR method, Taxonomy method, and EDAS method. Each bar represents the rank assigned to an alternative by a specific method, with the y-axis inverted to display the best rank (1) at the top.

Observations:

For A1, all methods rank it at 6, showing consensus that it is the least favorable alternative.

Similarly, A2 has a consistent rank of 5 across all methods.

For A3, the ranks vary significantly:

Proposed and CoCoSo methods rank it at 4.

VIKOR and EDAS methods rank it at 3.

Best-Performing Alternative:

A4 consistently ranks as 1 across all methods, indicating unanimous agreement that it is the most favorable choice.

A5 is ranked 2 by most methods, except for EDAS, which places it at 4.

A6 has ranks ranging from 3 to 4, showing slight variations in its evaluation.

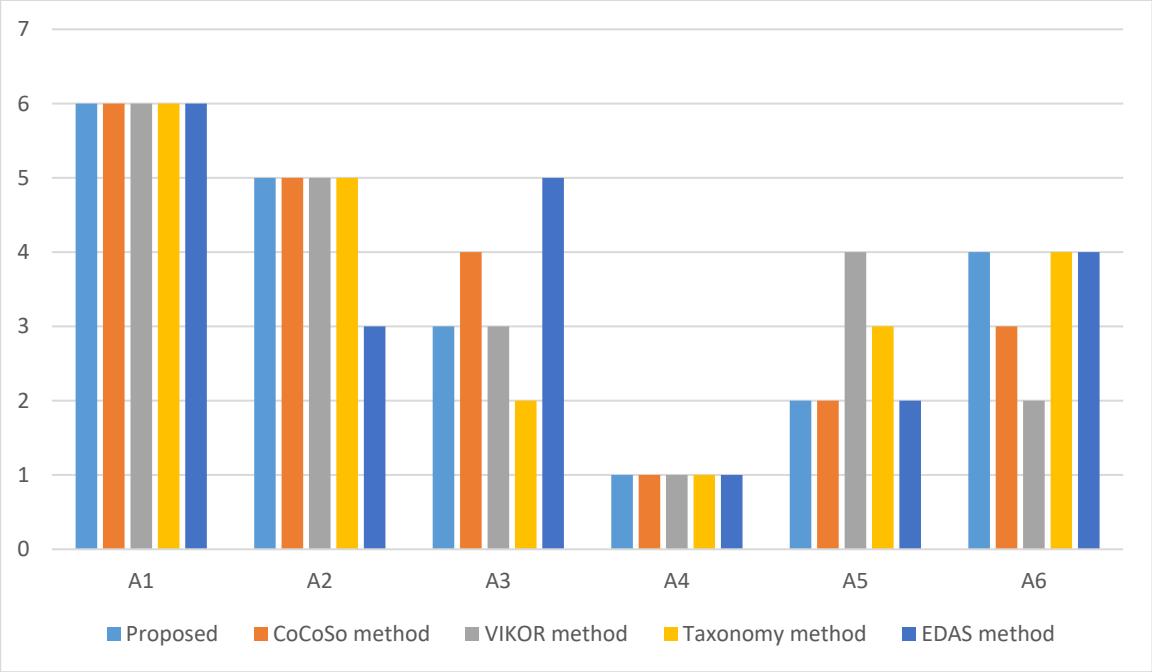


Figure 5. The comparative analysis.

6. Conclusion

The guaranteed environment for educational management must be based on talent cultivation and educational management, integrating individual, government, society, and other aspects to achieve high-quality development of higher education management. Firstly, create a legal environment. The legal system provides strong theoretical guidance and practical constraints for the construction of a quality assurance system for higher education management. Secondly, create an atmosphere of "quality culture". Creating a "quality culture" atmosphere is a prerequisite and main content for improving the quality of education management. It requires highly coordinated and consistent understanding and practice among diverse social entities, in order to form a culture of integration and progress, good quality construction, quality supervision, and quality evaluation. The quality evaluation of higher education management in academic institutions is considered a MADM problem. DVNSs are employed as an effective tool to represent uncertainty during the evaluation process. In this research, the MARCOS method is adapted for MADM for DVNSs. Subsequently, the DVNN- MARCOS approach is developed for MADM. The key contributions of this research are outlined: (1) a novel MADM approach is introduced based on MARCOS

approach under DVNSs, and (2) a numerical example for evaluating higher education management in academic institutions, utilizing the DVNN- MARCOS model, is provided.

7. Research Limitations

First, the paper has a limitation in terms of the number of case studies presented. It validates the DVNN- MARCOS model using only one numerical example, which may not be sufficient to demonstrate the model's broad applicability. The lack of diverse case studies, particularly in different types of universities or across various countries, limits the generalizability of the model and its application to a wider range of educational management contexts. Second, the study does not adequately account for dynamic factors that can influence the quality of higher education management. In practice, educational management is affected by constant changes such as policy shifts, technological developments, and variations in faculty quality. The paper's analysis is based on static data, which overlooks the impact of these dynamic and evolving factors on the quality evaluation process. The paper does not sufficiently address how these subjective factors can be combined with objective methods to create a more balanced and rational decision-making model.

8. Three directions for further research

One potential direction for further research is to expand the scope of case studies. Future work could apply the DVNN- MARCOS model to a wider variety of real-world scenarios, including different types of universities, such as research institutions, application-oriented universities, or vocational colleges. Additionally, testing the model in different countries with varying educational management systems could help verify its robustness and adaptability in diverse contexts. Another direction is to incorporate dynamic decision-making models into the DVNN- MARCOS framework. Higher education management operates in a constantly changing environment, influenced by factors such as policy reforms, advancements in technology, and fluctuations in faculty quality. Future research could explore how dynamic factors can be incorporated into the model, perhaps through time-series analysis or dynamic fuzzy decision-making techniques, to enhance the model's ability to cope with these changes. A third direction for further research is to investigate how subjective and objective weights can be more effectively combined. Future studies could explore methods for integrating these subjective and objective factors, potentially through the use

of multi-criteria decision-making techniques, expert systems, or even machine learning, to improve the rationality and accuracy of the evaluation process.

References

[1] K.J. Cassidy, M.N. Sullivan, Z.J. Radnor, Using insights from (public) services management to improve student engagement in higher education, Studies in Higher Education, 46 (2021) 1190-1206.

[2] L.G. Bravo, N. Nistor, B.C. Ramírez, I.G. Soto, M.V. Contreras, M.N. Vives, P.M. Robles, Higher education managers' perspectives on quality management and technology acceptance: A tale of elders, mediators, and working bees in times of covid-19, Computers in Human Behavior, 131 (2022) 11.

[3] J. Mampaey, W. Lambrechts, Reactive brand management in higher education: The case of a teacher training programme in the netherlands, Journal of Marketing for Higher Education, (2022) 15.

[4] G.V. Olariu, S. Brad, Preventive risk management of resource allocation in romanian higher education by assessing relative performance of study programs with dea method, Sustainability, 14 (2022) 17.

[5] D. Tahirbegi, Higher music education students' experiences and management of performance anxiety: A qualitative study, Psychology of Music, 50 (2022) 1184-1196.

[6] F. Yahiaoui, K. Chergui, N. Aissaoui, S.K.M. Brika, I.A. Lamari, A.A. Musa, M. Almezher, The impacts of total quality management practices in algerian higher education institutions, Frontiers in Psychology, 13 (2022) 14.

[7] K. Yildirim, S. Yenipinar, Compliance with international quality standards: Evaluation of turkish higher education institutions in terms of management dimensions, Egitim Ve Bilim-Education and Science, 47 (2022) 223-247.

[8] L.F. Cheah, M.Y. Cheng, K.W. Hen, The effect of quality management practices on academics' innovative performance in malaysian higher education institutions, Studies in Higher Education, 48 (2023) 643-656.

[9] X.N. Geng, X.N. Geng, Combining grey relation analysis with mcgdm to evaluate the higher education management quality with interval-valued neutrosophic sets, Journal of Intelligent & Fuzzy Systems, 45 (2023) 3791-3805.

[10] A.S. Ibidunni, B.Y. Nwaodu, L.E. Mdaka, Bringing quality management to perspective in higher education institutions' research output: A focus on selected private universities in nigeria, Heliyon, 9 (2023) 11.

[11] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[12] F.A. Smarandache, Unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[13] M. Karak, A. Mahata, M. Rong, S. Mukherjee, S.P. Mondal, S. Broumi, B. Roy, A solution technique of transportation problem in neutrosophic environment, Neutrosophic Systems with Applications, 3 (2023) 17-34.

[14] N. Martin, S. Broumi, S. Sudha, R. Priya, Neutrosophic marcos in decision making on smart manufacturing system, Neutrosophic Systems with Applications, 4 (2023) 12-32.

[15] S.I. Abdel Aal, Neutrosophic framework for assessment challenges in smart sustainable cities based on iot to better manage energy resources and decrease the urban environment's ecological impact, Neutrosophic Systems with Applications, 6 (2023) 9-16.

[16] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single valued neutrosophic sets, Infinite Study, 12 (2010) 20110.

[17] K. Karunanithi, C. Han, C.J. Lee, W.C. Shi, L. Duan, Y. Qian, Identification of a hemodynamic parameter for assessing treatment outcome of edas in moyamoya disease, Journal of Biomechanics, 48 (2015) 304-309.

[18] I. Kandasamy, Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm, Journal of Intelligent Systems, 27 (2018) 163-182.

[19] D. Diakoulaki, G. Mavrotas, L. Papayannakis, Determining objective weight in multple criteria problems:The critic method, Computers & Operations Research, 22 (1995) 763-770.

[20] F.Y. Xiao, A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems, Ieee Transactions on Systems Man Cybernetics-Systems, 51 (2021) 3980-3992.

Received: Sep 2, 2024. Accepted: Dec 3, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

Elevating Service Quality in Mass Sports Events Through Trapezoidal Neutrosophic Weighted Product Models

Yifang Zhao *

Inner Mongolia Medical University, Hohhot, 010000, Inner Mongolia, China

*Corresponding author Email: 20150023@immu.edu.cn

Abstract

The evaluation of service quality in mass sports events is a structured process assessing aspects such as event organization, venue facilities, safety measures, staff services, and information dissemination. This evaluation assists organizers in identifying strengths and weaknesses, enabling optimization of services to enhance participant and spectator satisfaction. This study applies to the Weighted Product Method (WPM) integrated with Trapezoidal Neutrosophic Numbers (TNN) to address uncertainties inherent in multi-attribute decision-making. By presenting numerical examples and comparative analyses, the study demonstrates the effectiveness of the TNN-WPM technique. The findings provide a reliable framework for systematically evaluating service quality in mass sports events, promoting their development and broader societal impact.

Keywords: Multiple-attribute group decision-making (MAGDM); TNN; TNN-WPM model; service quality evaluation

1. Introduction

As society progresses, the material living standards of people in China have greatly improved. Along with the high-pressure and fast-paced lifestyle, more and more people have begun to focus on their physical and mental well-being. Sports have become an integral part of many people's lives. An increasing number of individuals are engaging in sports, leading to a growing sports population, while various sports clubs and associations have sprung up like mushrooms after the rain, and the sports industry has developed rapidly. Local governments and enterprises have seized this opportunity, actively organizing various mass sports events, which not only promote local economic development but also enhance local visibility and contribute to the talent pool for competitive sports. The flourishing development of mass sports events holds great significance for achieving nationwide fitness and building a strong sporting nation. However, mass sports events are derivatives of competitive sports and differ from competitive sports in terms of both entertainment value and intrinsic worth. As such, their organization and management also differ. The development of mass sports events in China

started relatively late, and there are still many issues in terms of organization and management. Analyzing strategies to optimize the organization and management of large-scale mass sports events in China is of great importance. It can help promote the institutionalization, standardization, and scientific management of these events, encourage more people to participate in sports activities, foster the development of the national fitness movement, and support the healthy growth of the sports industry. Wang, Dai and Cheng [1] examined the role of government in managing mass sports events in Shanghai. They used expert interviews and questionnaires to analyze the current state of government involvement, identifying gaps in government functions and proposing a governance model to better coordinate between stakeholders. Following this, Tan [2] explored the management models of large-scale mass sports events in China post-Beijing Olympics. The study highlighted the growing demand for mass sports events and identified weaknesses in the current organizational models, offering suggestions for optimizing management structures to support the expanding market. Moving into 2016, Li [3] explored the use of WeChat for promoting mass sports events. The study emphasized the importance of strategic content delivery, timing, and technical features, such as voting activities, to improve event marketing. Li's research suggested leveraging WeChat's unique features like service accounts and micro-communities to enhance ongoing event operations. The following year, Miao and Li [4] analyzed the role of new media in promoting mass sports events. They used case studies and expert interviews to investigate the reasons behind media path dependency and proposed that combining online and offline activities could be an innovative solution for future event promotion. In 2018, Zhou and Li [5] focused on large-scale mass sports events, identifying challenges such as limited event types, outdated operational models, and a lack of brand awareness. They proposed expanding event variety, enhancing administrative functions, and building stronger event brands to address these issues. The following year, Zhou and Li [6] examined the governance of mass sports events within the context of China's national fitness strategy. They advocated for a governance approach centered on precision, collaboration, legal protection, and sustainability to ensure the efficient and long-term development of mass sports events. In 2020, Hu and Jing [7] investigated the challenges faced by mass sports events in Guangdong Province, suggesting that the region should leverage its unique cultural and natural resources to create brand-specific events. They also emphasized the need for legislative support and a more coordinated management structure within the Guangdong-Hong Kong-Macao Greater Bay Area. Around the same time, Wu [8] examined the importance of linking mass sports events across the Greater Bay Area, suggesting that such integration could promote cultural exchange and provide a platform for the internationalization of Chinese mass sports. Wang [9] studied mass sports event governance from a social governance perspective. He proposed innovative governance models that would stimulate the active participation of multiple stakeholders, clarify roles, and create a co-governance framework for event management. In 2022, Zhang, Zhang, Zheng and Tian [10] explored the digital governance of mass sports events. They

argued for the necessity of integrating digital tools to enhance management efficiency and suggested building information platforms to support precise supervision and service. More recently, Ma, Jiang and Wang [11] focused on the legal aspects of managing emergencies in mass sports events. They recommended clarifying the responsibilities of safety regulators, establishing qualification systems for event organizers, and enhancing legal frameworks to cover civil liability. Finally, in 2024, Zhang and Zhang [12] conducted a comprehensive analysis of the governance mechanisms of mass sports events from a multi-stakeholder perspective. Their study identified challenges such as unclear governance responsibilities and insufficient resource support, and they proposed further refining collaboration and risk assessment mechanisms to improve event governance.

The service quality evaluation of mass sports events presents a MAGDM. To address this, traditional WPM technique [13] has been utilized as a means to manage MAGDM. In the context of service quality evaluation of mass sports events, TNN [14] are administrated as valuable decision tool to handle uncertain information in light with neutrosophic sets [15-18]. While various techniques have been developed to optimize the traditional WPM technique [13], there is a scarcity of methods that apply the WPM technique to manage MAGDM using TNN. In this study, we propose the TNN-WPM approach, which integrates the traditional WPM method with TNN. This novel approach is applied to MAGDM for evaluating the service quality of mass sports events. To demonstrate the effectiveness and key advantages of the proposed technique, a numerical example focusing on the service quality evaluation of mass sports events is presented, followed by a comparative analysis with existing methods.

The key contributions of this study are outlined as follows:

- **Extension of the WPM Technique**: We extend the traditional WPM approach by incorporating TNN, which enhances its applicability in scenarios involving uncertainty and linguistic information.
- **Development of the TNN-WPM Technique for MAGDM**: A new TNN-WPM technique is developed specifically for MAGDM, utilizing TNN to better handle uncertainty and improve decision-making accuracy in complex environments.
- **Numerical Demonstration**: We illustrate the application of the TNN-WPM technique through a detailed numerical study on the service quality evaluation of mass sports events, providing practical insights into its implementation.
- **Comparative Analysis**: A comparative analysis is conducted to evaluate the performance of the TNN-WPM technique against existing methods. The results confirm the effectiveness and reliability of the proposed approach for service quality evaluation in mass sports events.

This study demonstrates that the TNN-WPM technique is a valuable tool for MAGDM, especially in scenarios where uncertainty and linguistic assessments play a significant role. The method offers a more comprehensive and accurate evaluation of service quality, making it a useful approach for decision-makers in the context of mass sports events.

To achieve this, the study follows a structured approach: Section 2 introduces the concept of TNN. Section 3 focuses on the application of the TNN-WPM technique for MAGDM. In Section 4, a numerical example is provided, specifically analyzing the service quality evaluation of mass sports events. Additionally, a comparative analysis is conducted to further validate and examine the results obtained from the TNN-WPM technique. This structure allows for a clear and systematic exploration of the method's effectiveness in MAGDM scenarios.

2. Preliminaries

In order to provide a trapezoidal neutrosophic set based on the combination of trapezoidal fuzzy numbers and a single-valued neutrosophic set, together with its score and accuracy functions, this section extends a trapezoidal intuitionistic fuzzy set—which is preferred in practice—to a single-valued neutrosophic set.

We provide the following concept of a trapezoidal neutrosophic set (TNS), which is an extension of a trapezoidal intuitionistic fuzzy set [23,24].

Definition 1

Let x be a universe of discourse, TNS can be defined as:

$$\boldsymbol{N} = \{(\boldsymbol{x}, \boldsymbol{T_N}(\boldsymbol{x}), \boldsymbol{N_N}(\boldsymbol{x}), \boldsymbol{F_N}(\boldsymbol{x}))|\boldsymbol{x} \in \boldsymbol{X}\} \tag{1}$$

Where $\boldsymbol{T_N}(\boldsymbol{x}) = \left(\boldsymbol{t_N^1}(\boldsymbol{x}), \boldsymbol{t_N^2}(\boldsymbol{x}), \boldsymbol{t_N^3}(\boldsymbol{x}), \boldsymbol{t_N^4}(\boldsymbol{x})\right) : \boldsymbol{X} \to [\boldsymbol{0}, \boldsymbol{1}]$

$$\boldsymbol{I_N}(\boldsymbol{x}) = \left(\boldsymbol{i_N^1}(\boldsymbol{x}), \boldsymbol{i_N^2}(\boldsymbol{x}), \boldsymbol{i_N^3}(\boldsymbol{x}), \boldsymbol{i_N^4}(\boldsymbol{x})\right) : \boldsymbol{X} \to [\boldsymbol{0}, \boldsymbol{1}] \tag{2}$$

$$\boldsymbol{T_N}(\boldsymbol{x}) = \left(\boldsymbol{f_N^1}(\boldsymbol{x}), \boldsymbol{f_N^2}(\boldsymbol{x}), \boldsymbol{f_N^3}(\boldsymbol{x}), \boldsymbol{f_N^4}(\boldsymbol{x})\right) : \boldsymbol{X} \to [\boldsymbol{0}, \boldsymbol{1}] \tag{3}$$

$$\boldsymbol{0} \leq \boldsymbol{t_n^4}(\boldsymbol{x}) + \boldsymbol{i_n^4}(\boldsymbol{x}) + \boldsymbol{f_n^4}(\boldsymbol{x}) \leq \boldsymbol{3}, \boldsymbol{x} \in \boldsymbol{X} \tag{4}$$

Definition 2

Let $\boldsymbol{n_1} = \left((\boldsymbol{a_1}, \boldsymbol{b_1}, \boldsymbol{c_1}, \boldsymbol{d_1}), (\boldsymbol{e_1}, \boldsymbol{f_1}, \boldsymbol{g_1}, \boldsymbol{h_1}), (\boldsymbol{l_1}, \boldsymbol{m_1}, \boldsymbol{n_1}, \boldsymbol{p_1})\right)$ and

$$\boldsymbol{n_2} = \left((\boldsymbol{a_2}, \boldsymbol{b_2}, \boldsymbol{c_2}, \boldsymbol{d_2}), (\boldsymbol{e_2}, \boldsymbol{f_2}, \boldsymbol{g_2}, \boldsymbol{h_2}), (\boldsymbol{l_2}, \boldsymbol{m_2}, \boldsymbol{n_2}, \boldsymbol{p_2})\right)$$

$$\boldsymbol{n_1} \oplus \boldsymbol{n_2} = \begin{Bmatrix} (\boldsymbol{a_1} + \boldsymbol{a_2} - \boldsymbol{a_1}\boldsymbol{a_2}, \boldsymbol{b_1} + \boldsymbol{b_2} - \boldsymbol{b_1}\boldsymbol{b_2}, \boldsymbol{c_1} + \boldsymbol{c_2} - \boldsymbol{c_1}\boldsymbol{c_2}, \boldsymbol{d_1} + \boldsymbol{d_2} - \boldsymbol{d_1}\boldsymbol{d_2}), \\ (\boldsymbol{e_1}\boldsymbol{e_2}, \boldsymbol{f_1}\boldsymbol{f_2}, \boldsymbol{g_1}\boldsymbol{g_2}, \boldsymbol{h_1}\boldsymbol{h_2}), (\boldsymbol{l_1}\boldsymbol{l_2}, \boldsymbol{m_1}\boldsymbol{m_2}, \boldsymbol{n_1}\boldsymbol{n_2}, \boldsymbol{p_1}\boldsymbol{p_2}) \end{Bmatrix} \tag{5}$$

$$\boldsymbol{n_1} \otimes \boldsymbol{n_2} = \begin{Bmatrix} (\boldsymbol{a_1}\boldsymbol{a_2}, \boldsymbol{b_1}\boldsymbol{b_2}, \boldsymbol{c_1}\boldsymbol{c_2}, \boldsymbol{d_1}\boldsymbol{d_2}), \\ (\boldsymbol{e_1} + \boldsymbol{e_2} - \boldsymbol{e_1}\boldsymbol{e_2}, \boldsymbol{f_1} + \boldsymbol{f_2} - \boldsymbol{f_1}\boldsymbol{f_2}, \boldsymbol{g_1} + \boldsymbol{g_2} - \boldsymbol{g_1}\boldsymbol{g_2}, \boldsymbol{h_1} + \boldsymbol{h_2} - \boldsymbol{h_1}\boldsymbol{h_2}), \\ (\boldsymbol{l_1} + \boldsymbol{l_2} - \boldsymbol{l_1}\boldsymbol{l_2}, \boldsymbol{m_1} + \boldsymbol{m_2} - \boldsymbol{m_1}\boldsymbol{m_2}, \boldsymbol{n_1} + \boldsymbol{n_2} - \boldsymbol{n_1}\boldsymbol{n_2}, \boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p_1}\boldsymbol{p_2}) \end{Bmatrix} \tag{6}$$

$$\gamma\, \boldsymbol{n_1} = \begin{Bmatrix} (\boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{a_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{b_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{c_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{d_1})^{\gamma}), \\ (\boldsymbol{e_1^{\gamma}}, \boldsymbol{f_1^{\gamma}}, \boldsymbol{g_1^{\gamma}}, \boldsymbol{h_1^{\gamma}}), (\boldsymbol{l_1^{\gamma}}, \boldsymbol{m_1^{\gamma}}, \boldsymbol{n_1^{\gamma}}, \boldsymbol{p_1^{\gamma}}) \end{Bmatrix} \tag{7}$$

$$\boldsymbol{n_1^{\gamma}} = \begin{Bmatrix} (\boldsymbol{a_1^{\gamma}}, \boldsymbol{b_1^{\gamma}}, \boldsymbol{c_1^{\gamma}}, \boldsymbol{d_1^{\gamma}}), \\ (\boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{e_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{f_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{g_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{h_1})^{\gamma}), \\ (\boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{l_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{m_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{n_1})^{\gamma}, \boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{o_1})^{\gamma}) \end{Bmatrix} \tag{8}$$

Definition 3

The score function of trapezoidal neutrosophic number (TNN) can be computed as:

$$\boldsymbol{S(n)} = \frac{1}{3}\left(2 + \frac{a+b+c+d}{4}\right) - \left(\frac{e+f+g+h}{4}\right) - \left(\frac{l+m+n+p}{4}\right) \quad (9)$$

Definition 4

The accuracy function of TNN can be computed as:

$$\boldsymbol{A(n)} = \left(\frac{a+b+c+d}{4}\right) - \left(\frac{l+m+n+p}{4}\right) \quad (10)$$

3. Weighted Product Method (WPM) with Trapezoidal Neutrosophic Numbers (TNN)

The TNN-WPM (Trapezoidal Neutrosophic Number Weighted Product Method) framework combines the traditional Weighted Product Method (WPM) with Trapezoidal Neutrosophic Numbers (TNN) to address uncertainty and linguistic assessments in multi-attribute decision-making.

3.1 WPM

WPM evaluates alternatives based on a set of criteria by computing the product of the normalized performance scores raised to the power of their respective weights. It is particularly effective in decision-making where multiplicative relationships are meaningful, such as evaluating proportional contributions.

3.2 TNN

TNN extends traditional fuzzy and intuitionistic fuzzy sets by incorporating truth (T), indeterminacy (I), and falsity (F) memberships, each represented by trapezoidal numbers.

Example of TNN: (a, b, c, d), (e, f, g, h), (i, j, k, l) (a,b,c,d),(e,f,g,h),(i,j,k,l), where:

a, b, c, d a,b,c,d: Membership to truth

e, f, g, h e,f,g,h: Membership to indeterminacy

i, j, k, l i,j,k,l: Membership to falsity.

This representation handles vagueness and uncertainty in data more effectively than crisp or fuzzy numbers**.**

3.3 Practical Applications

The practical implications of the TNN-WPM technique extend across multiple domains due to its capacity to process uncertain and linguistic information effectively. Key applications include:

- Cultural Event Management: Ensuring high service quality in cultural festivals by evaluating logistics, attendee satisfaction, and staff performance.
- Healthcare Services: Assessing patient satisfaction and service efficiency to improve overall healthcare delivery under uncertain feedback.
- Public Transport Systems: Enhancing service quality through the evaluation of scheduling, safety, and passenger satisfaction.

The adoption of TNN-WPM in these sectors facilitates precise, unbiased decision-making, fostering enhanced operational outcomes.

3.4 Comparative Analysis of Other Techniques

The TNN-WPM technique was benchmarked against similar MAGDM methods to assess its unique strengths:

Technique	Strengths	Weaknesses
TNN-WPM	Manages overall and individual uncertainty; Simplicity of application	Does not account for psychological factors
TNN-TODIM	Incorporates psychological behaviors	Complex structure and calculations
TNN-VIKOR	Balances compromise solutions	Limited capability for handling uncertainty

3.5 Advantages of TNN-WPM

- TNN allows modeling of truth, indeterminacy, and falsity, capturing nuanced uncertainties in decision-making.
- Effective in real-world situations with multiple conflicting criteria and vague inputs.
- Despite its robust modeling, the method remains computationally straightforward compared to other multi-attribute decision-making methods.
- TNN-WPM is adaptable to various fields, including sports management, healthcare, education, and urban planning.

3.6 Steps in the TNN-WPM Framework

The steps of TNN-WPM are:

A. Create a decision matrix.

The decision matrix is created between the criteria and alternatives by the opinions of decision makers.

$$K = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mn} \end{bmatrix} \tag{11}$$

B. apply the score function to obtain one value.

C. Combined the decision matrix.

D. Compute the criteria weights.

The criteria weights are computed using the average method.

E. Normalize the decision matrix.

$$D_{ij} = \frac{k_{ij}}{\max k_{ij}} \text{ for positive criteria} \tag{12}$$

$$D_{ij} = \frac{\min k_{ij}}{k_{ij}} \text{ for cost criteria} \quad (13)$$

F. Compute the weighted normalized decision matrix.

$$y_{ij} = w_j D_{ij} \quad (14)$$

G. Rank the alternatives.

$$R_i = \prod_{j=1}^{n} y_{ij} \quad (15)$$

4. Numerical examples and comparative analysis

The evaluation of service quality in mass sports events refers to a systematic assessment of various aspects such as event organization, venue facilities, service personnel, information dissemination, and event safety, to comprehensively measure the overall quality of services and the performance of each component. The core goal of this evaluation is to ensure that participants and spectators have a positive experience, increase their satisfaction, and continuously improve and optimize the event. First, event organization is one of the key aspects of evaluation. The preparation of the event, scheduling, and contingency plans are directly related to the smooth operation of the entire event. Whether the event is organized rigorously and reasonably, and whether it can effectively handle unexpected situations, serves as a fundamental standard for evaluating service quality. Second, the completeness and safety of venue facilities are also important evaluation factors. Good venue conditions, clear signage systems, and reasonable seating arrangements can significantly enhance the experience of participants. At the same time, safety measures at the venue, such as first aid facilities, emergency exits, and security personnel, directly impact the service quality of the event. The professionalism and attitude of service personnel are crucial factors that directly affect service quality. Whether volunteers, referees, or the event's operational management team, their service attitude, professional competence, and communication skills have a direct impact on the satisfaction of participants and spectators. In addition, the timeliness and accuracy of information dissemination are also key aspects of service quality. By using effective communication channels to clearly convey event schedules, rules, transportation information, and other details, any confusion caused by information asymmetry can be avoided, thereby improving the efficiency of event organization. Finally, the service quality evaluation system for the event must include feedback from participants and spectators. By collecting their opinions and suggestions, organizers can deeply understand the shortcomings in the event services and make adjustments and optimizations based on the actual situation. Overall, the evaluation of service quality in mass sports events is not only a tool for assessing the success of the event but also provides valuable references and

improvement directions for future event organizations. It effectively enhances the brand image of the event, promotes the continued development of mass fitness activities, and facilitates the widespread dissemination of sports culture. The service quality evaluation of mass sports events is MAGDM. Seven possible mass sports events are assessed with 18 attributes as shown in Figure 1.

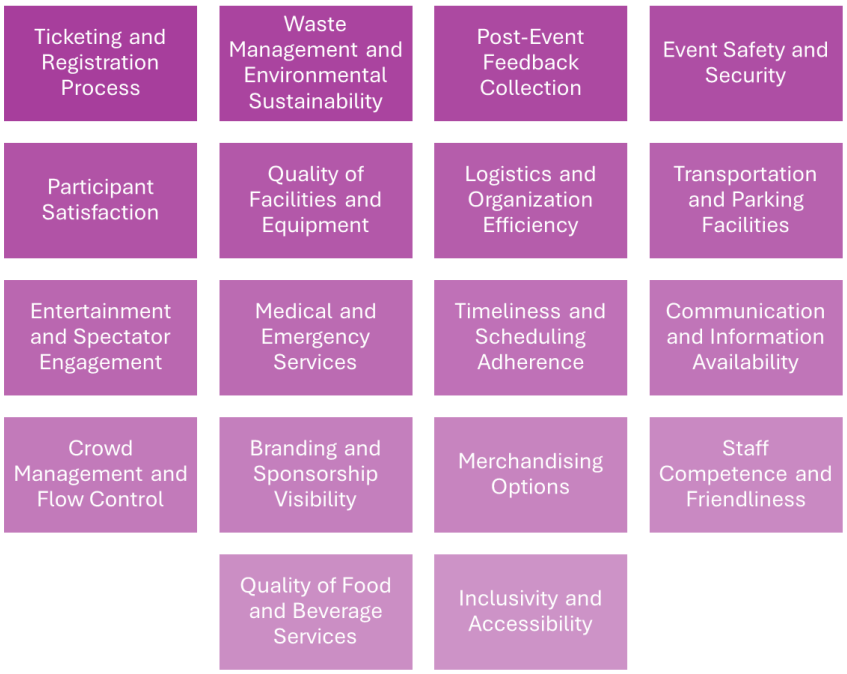


Figure 1. Detailed List of Selection Criteria

4.1 Illustrative Example

1. Problem:

A sports event organizer wants to evaluate three venues (A1, A2, A3) based on:

C1: Venue capacity.

C2: Accessibility.

C3: Cost.

2. Decision Matrix (TNN):

	C1 (Beneficial)	C2 (Beneficial)	C3 (Cost)
A1	(0.5,0.6,0.7,0.8) (0.5, 0.6, 0.7, 0.8) (0.5,0.6,0.7,0.8)	(0.4,0.5,0.6,0.7) (0.4, 0.5, 0.6, 0.7) (0.4,0.5,0.6,0.7)	(0.3,0.4,0.5,0.6) (0.3, 0.4, 0.5, 0.6) (0.3,0.4,0.5,0.6)
A2	(0.6,0.7,0.8,0.9) (0.6, 0.7, 0.8, 0.9) (0.6,0.7,0.8,0.9)	(0.5,0.6,0.7,0.8) (0.5, 0.6, 0.7, 0.8) (0.5,0.6,0.7,0.8)	(0.4,0.5,0.6,0.7) (0.4, 0.5, 0.6, 0.7) (0.4,0.5,0.6,0.7)
A3	(0.4,0.5,0.6,0.7) (0.4, 0.5, 0.6, 0.7) (0.4,0.5,0.6,0.7)	(0.6,0.7,0.8,0.9) (0.6, 0.7, 0.8, 0.9) (0.6,0.7,0.8,0.9)	(0.2,0.3,0.4,0.5) (0.2, 0.3, 0.4, 0.5) (0.2,0.3,0.4,0.5)

3. Steps:

Calculated Scores:

S(A1, C1)=0.65 , S(A1,C2)=0.55S(A1, C2) = 0.55S , S(A1,C3)=0.45 , etc.

Normalize Scores:

Normalize using beneficial or cost criteria formulas.

Apply Weights:

Assign weights to criteria (W_C1=0.4, W_C2=0.3, W_C3=0.3

Rank Alternatives:

Computing the final scores: VA1, V_{A2}, VA_3

Rank A2>A3>A1

4.2 Detailed Explanation of the Process

Step A: Creating a Decision Matrix Using TNN

The process begins by constructing a decision matrix, as shown in Table 1, where:

Rows represent alternatives such as different venues for a sports event.

Columns represent the evaluation criteria, such as safety, accessibility, and cost.

Each entry in the matrix is represented by a TNN, which captures uncertainty in the evaluation. The decision matrix in Table 1 provides a comprehensive overview of the alternatives and their performance under each criterion.

Step B: Applying the Score Function to Simplify Evaluations

The next step is to simplify the TNN values in Table 1 into single, representative scores using the score function. This function converts the complex TNN into a single numeric value, making it easier to compare alternatives.

For example: A venue evaluated as "mostly good with slight uncertainty" might be simplified to a score of 7.5 out of 10. The result is a revised decision matrix where each entry is now represented by a single score instead of a TNN.

Step C: Consolidating the Decision Matrix

After simplifying the evaluations in Step B, the decision matrix is updated into a consolidated format. This new matrix ensures that all the evaluations are now consistent, simplifying further analysis. The updated decision matrix retains the structure of Table 1, but now all entries are clear and comparable numerical scores.

Step D: Assigning Weights to Criteria

Not all criteria have the same level of importance. For instance, safety might be more critical than cost when evaluating venues. To account for this, weights are assigned to each criterion based on their significance. The weights are typically derived from expert opinions or stakeholder inputs. Figure 2 illustrates the assigned weights for each criterion, clearly showing how much influence each one has in the evaluation process.

For example: Safety might have a weight of 0.4, while cost might have a weight of 0.2.

These weights are crucial for ensuring that the final evaluation aligns with the priorities of the decision-makers.

Step E: Normalizing the Scores (Equation 12)

The scores from the consolidated decision matrix are normalized to ensure all criteria are measured on the same scale. This normalization process adjusts the scores so that they fall within a consistent range, typically between 0 and 1. For beneficial criteria (e.g., higher values are better), normalization ensures that the highest score is assigned a value of 1.

For cost criteria (e.g., lower values are better), normalization ensures that the lowest score is assigned a value of 1. The results of this normalization are shown in Table 2, where all scores have been adjusted for comparability.

Step F: Computing the Weighted Normalized Decision Matrix (Equation 14)

Using the normalized scores from Table 2, the next step is to calculate the weighted normalized scores by applying the weights from Figure 2. This involves multiplying each normalized score by the corresponding weight of the criterion.

The resulting weighted normalized decision matrix reflects both the performance of each alternative and the importance of the criteria.

Step G: Ranking the Alternatives

Finally, the scores for each alternative are aggregated to calculate an overall performance score. These overall scores are used to rank the alternatives.

The rankings, as shown in Figure 3, clearly identify the best-performing alternative based on the weighted normalized scores.

Table 1. Expert 1's Decision Matrix Analysis

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))
C_2	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))
C_3	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))
C_4	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))
C_5	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))
C_6	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))
C_7	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))
C_8	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))
C_9	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))
C_{10}	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))
C_{11}	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))
C_{12}	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))
C_{13}	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))
C_{14}	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))
C_{15}	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))
C_{16}	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))
C_{17}	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))
C_{18}	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))	((0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1))	((0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9))	((0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))	((0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3))	((0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2))

Figure 2. Weighted Analysis of Criteria

Table 2. Data Normalization Results

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.943262	1	0.879433	0.978723	0.950355	0.77305	0.929078
C_2	0.851351	0.831081	1	0.756757	0.851351	0.810811	1
C_3	0.5	1	0.853333	0.886667	0.66	0.993333	0.5
C_4	0.98	0.946667	1	0.853333	0.84	0.826667	0.94
C_5	0.88	0.86	0.893333	0.826667	0.966667	1	0.92
C_6	0.94	0.78	0.94	0.92	0.833333	0.873333	1
C_7	0.938776	1	0.904762	0.884354	0.959184	0.843537	0.891156
C_8	0.857143	0.897959	0.884354	0.789116	0.789116	0.673469	1
C_9	0.627586	0.97931	0.855172	1	0.965517	0.806897	0.682759
C_{10}	1	0.898649	0.736486	0.777027	0.837838	0.932432	0.851351
C_{11}	0.825503	0.899329	0.966443	0.95302	0.731544	1	0.926174
C_{12}	0.946309	0.61745	0.993289	0.95302	0.899329	0.718121	1
C_{13}	0.932432	1	0.864865	0.851351	0.837838	1	0.831081
C_{14}	0.904762	0.877551	0.959184	0.680272	0.891156	0.510204	1
C_{15}	0.5	1	0.913333	0.886667	0.82	0.94	0.5
C_{16}	1	0.904762	0.938776	0.931973	0.904762	0.795918	0.904762

C_{17}	0.825503	0.899329	0.751678	0.892617	0.771812	1	0.926174
C_{18}	0.959184	0.911565	0.904762	0.680272	1	0.836735	0.965986

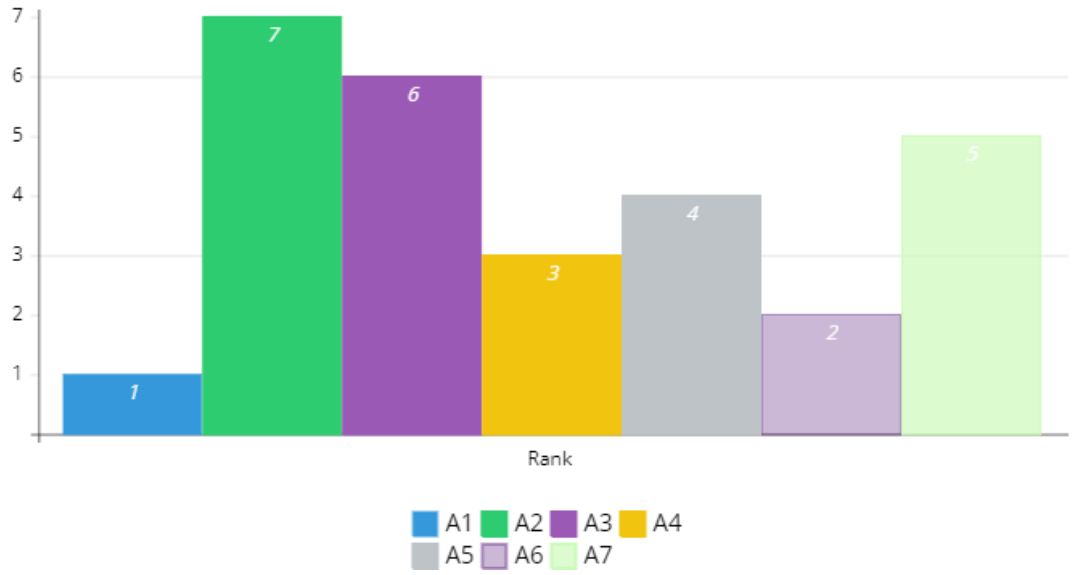


Figure 3. Evaluation Results: Alternative Rankings

Figure 4 shows comparative analysis. When comparing the results of the TNN-WPM model with those of existing models, slight differences in the calculated outcomes are observed. Despite these minor variations, the identification of both the best and the worst mass sports events remains consistent across all models. This consistency demonstrates that the TNN-WPM technique is an effective and reliable approach for MAGDM. The subtle differences in results can be attributed to distinct factors. For instance, the TNN-VIKOR technique addresses the influence of individual decision-makers on the final outcome, while the TNN-TODIM model accounts for the psychological behaviors of DMs. By contrast, the TNN-WPM technique incorporates both individual and overall uncertainty within the framework of TNN.

However, despite its strengths, the TNN-WPM technique does have a limitation. It does not account for the psychological behavior of decision-makers, a factor that can play a significant role in real-world decision-making scenarios, especially in MAGDM contexts. Techniques like the TNN-WPM model, which explicitly consider decision-makers' psychological tendencies, may be better suited for situations where such behavior is a critical factor. Nonetheless, the TNN-WPM technique remains a highly effective tool for addressing uncertainty, offering a reliable and straightforward approach to MAGDM, even if it does not fully capture the complexity of human decision-making behaviors.

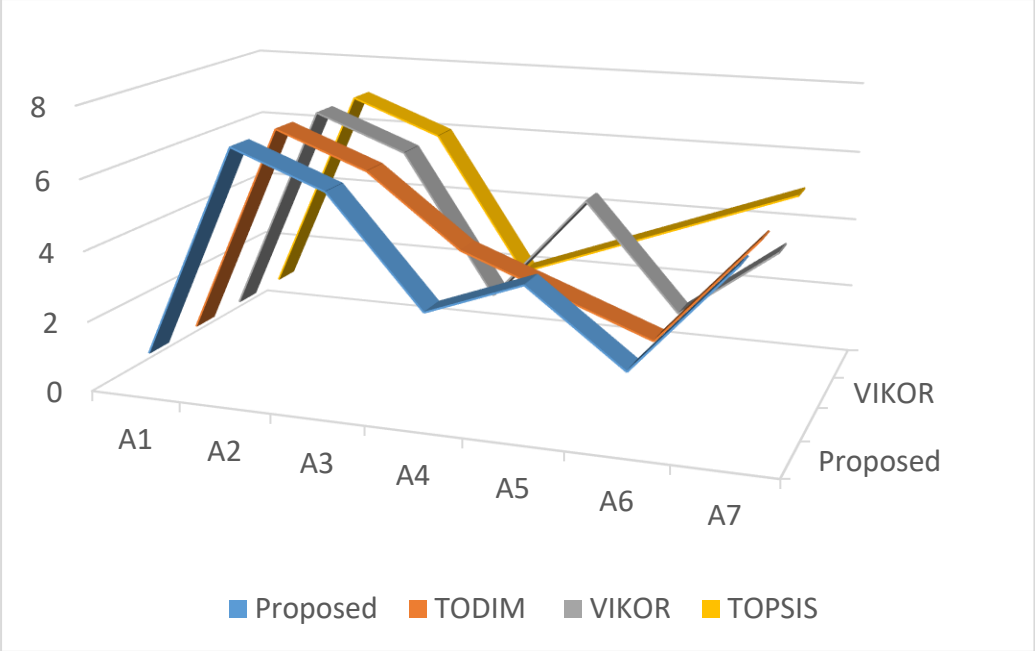


Figure 4. Rankings of Alternatives: Comparative Analysis

6. Conclusion

The evaluation of service quality in mass sports events involves a comprehensive and systematic assessment of various aspects to ensure a positive experience for participants and spectators. It specifically examines event organization, venue facilities, safety measures, the service provided by volunteers and staff, as well as event promotion and information dissemination. By gathering feedback from participants, spectators, and other stakeholders, the strengths and weaknesses of the event services are analyzed. This evaluation not only helps event organizers optimize service processes and improve the overall quality of the event but also enhances public engagement and satisfaction, promoting word-of-mouth and brand building. Additionally, it contributes to the development of mass fitness initiatives and increases the influence and popularity of sports culture in society. A high-quality event service evaluation system encourages more people to participate in sports activities and elevates the public perception of the event. The problem of service quality evaluation of mass sports events is MAGDM. This study presents the application of TNN-WPM technique, which is based on the traditional WPM technique with TNN. By incorporating WPM technique along with TNN, TNN-WPM technique is implemented for MAGDM with TNN. Additionally, numerical example focusing on the service quality evaluation of mass sports events is provided to validate the advantages of TNN-WPM. Comparative analysis is administrated to highlight the strengths of this technique. The key contributions of this study are summarized: (1) Extension of WPM technique to encompass TNN, expanding its applicability; (2)

Implementation of TNN-WPM technique for MAGDM, utilizing the advantages of TNN; (3) Demonstration of TNN-WPM technique through numerical example involving the service quality evaluation of mass sports events. Comparative analysis is administrated to further illustrate the advantages of TNN-WPM technique.

Acknowledgment

The work was supported by the Social Science Project of Ordos City in 2023, Project No. 2023p392, 2023k.

References

[1] Y. Wang, J. Dai, H. Cheng, Lun zhengfu zai shanghai qunzhongxing tiyu saishi zhong de juese dingwei, Nanjing Tiyu Xueyuan Xuebao (Shehui Kexue Ban), 24 (2010) 85-89.

[2] S. Tan, Goujian daxing qunzhongxing tiyu saishi de guanli moshi, Shang, (2014) 266.

[3] S. Li, Weixin zai qunzhongxing tiyu saishi yingxiao tuiguang de yingyong yanjiu, Dangdai Tiyu Keji, 6 (2016) 98-99.

[4] Y. Miao, Y. Li, Qunzhongxing tiyu saishi zhong de lujing yilai yu chuanbo chuangxin—yi tiaotiao wu app weili, Dangdai Tiyu Keji, 7 (2017) 151-154.

[5] Y. Zhou, M. Li, Xinshidai woguo daxing qunzhongxing tiyu saishi de fazhan kunjing yu gaige celue, Zhongguo Xuexiao Tiyu (Gaodeng Jiaoyu), 5 (2018) 18-22.

[6] B. Zhou, Y. Li, Quanmin jiankang guojia zhanlue xia de qunzhongxing tiyu saishi shehui zhili jinlu yanjiu, Shenyang Tiyu Xueyuan Xuebao, 38 (2019) 56-61.

[7] B. Hu, H. Jing, Guangdongsheng qunzhongxing tiyu saishi fazhan kunjing yu lujing yanjiu, Guangzhou Tiyu Xueyuan Xuebao, 40 (2020) 35-38.

[8] S. Wu, Xinshidai yuegangao dawanqu qunzhongxing tiyu saishi huodong de liandong fazhan yanjiu, Jilin Tiyu Xueyuan Xuebao, 36 (2020) 16-20.

[9] Z. Wang, Jiyu shehui zhili lilun xia qunzhongxing tiyu saishi zhili de wenti yu chulu, Jilin Gongcheng Jishu Shifan Xueyuan Xuebao, 37 (2021) 62-65.

[10] L. Zhang, R. Zhang, Y. Zheng, L. Tian, Qunzhongxing tiyu saishi huodong de shuzihua zhili fenxi, Tiyu Keji Wenxian Tongbao, 30 (2022) 82-86.

[11] L. Ma, J. Jiang, B. Wang, Woguo qunzhongxing tiyu saishi tufa shijian yingdui de falü sikao, Lanzhou Wenli Xueyuan Xuebao (Shehui Kexue Ban), 39 (2023) 101-107.

[12] J. Zhang, E. Zhang, Duoyuan zhuti zhili shijiao xia woguo qunzhongxing tiyu saishi zhili jizhi de kunjing yu tupo, Shandong Tiyu Keji, 46 (2024) 1-6.

[13] X. Shang, W. Jiang, A note on fuzzy information measures, Pattern Recognition Letters, 18 (1997) 425-432.

[14] J. Wang, G.W. Wei, Y. Wei, Models for green supplier selection with some 2-tuple linguistic neutrosophic number bonferroni mean operators, Symmetry-Basel, 10 (2018) 36.

[15] V. Christianto, F. Smarandache, The convergence of ikigai and design thinking: Crafting a purposeful framework, Sustainable Machine Intelligence Journal, 7 (2024) (1):1-8.

[16] M. Mohamed, A. Elsayed, M. Voskoglou, Collaboration of vague theory and mathematical techniques

for optimizing healthcare by recommending optimal blockchain supplier, Sustainable Machine Intelligence Journal, 6 (2024) (5):1-15.

[17] F.A. Smarandache, Unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[18] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single valued neutrosophic sets, Infinite Study, 12 (2010) 20110.

Received: Sep 3, 2024. Accepted: Dec 4, 2024

Neutrosophic Sets and Systems, Vol. 77, 2025
University of New Mexico

Cloud Computing and Neutrosophic Logic: Navigating Uncertainty in Educational Administration

Ibrahim Yasser[1], Aya A. Abd El-Khalek[2] ,Huda E. Khalid[3*], A. A. Salama[4] ,Ahmed K. Essa[3]

[1] Engineering Faculty, University of Mansoura, Mansoura, 35516, Egypt,
Ibrahim_yasser@mans.edu.eg

[2] Communications and Electronics Engineering Department, Nile Higher Institute for Engineering and Technology, Mansoura, 35524, Egypt, ayayakot94@gmail.com

[3]Telafer University, The Administration Assistant for the President of the Telafer University, Telafer, Iraq; https://orcid.org/0000-0002-0968-5611 , dr.huda-ismael@uotelafer.edu.iq
ahmed.k.essa@uotelafer.edu.iq; https://orcid.org/0000-0001-6153-9964

[4] Department of Mathematics and Computer Science, Faculty of Sciences, Port Said University, Port Said, 42526, Egypt.
drsalama44@gmail.com, ahmed_salama_2000@sci.psu.edu.eg

*Correspondence: dr.huda-ismael@uotelafer.edu.iq

Abstract: Educational institutions face inherent uncertainties in student performance, stakeholder priorities, and data analysis. This paper explores how cloud computing, with its data storage, analytics, and collaboration tools, can address these uncertainties in educational administration. We propose a neutrosophic framework utilizing "truth," "falsity," and "indeterminacy" values to represent these complexities within educational data. Real-world case studies analyze how cloud computing can reduce uncertainties and improve decision-making. Our findings reveal that cloud computing offers significant potential for navigating the "fog" of educational administration, but its effectiveness is enhanced by considering the "indeterminacy" inherent in educational data through the neutrosophic framework. This research contributes a unique neutrosophic perspective to the use of cloud computing for improved and more informed decision-making in educational administration.

Keywords: Cloud Computing, Educational Administration, Neutrosophic Logic, Uncertainty, Decision-Making, Educational Data, Case Studies.

1. Introduction

The educational landscape is undergoing a dramatic transformation fueled by technological advancements. These advancements are reshaping teaching methods, learning materials, and access to knowledge [1, 2]. To navigate this dynamic environment and ensure optimal learning outcomes, educational institutions require innovative solutions that address the inherent complexities of the educational experience.

One significant challenge faced by educational administrators is the presence of uncertainties. These uncertainties can manifest in various forms, such as:

- Predicting student potential due to diverse learning styles and external influences [14].
- Aligning educational goals and resource allocation with the differing priorities of parents, teachers, and administrators [17].
- Making informed decisions based on incomplete datasets that may lack comprehensive data on student progress, teacher effectiveness, and resource utilization [14].

These uncertainties create a "fog" that can obscure the path towards achieving optimal learning outcomes. Fortunately, emerging technologies like cloud computing offer-promising tools to navigate this fog [6, 7].

Cloud computing provides educational institutions with a suite of powerful tools, including:

- Scalable data storage and analytics platforms that allow for the collection and analysis of vast amounts of data, leading to a more holistic understanding of student performance and educational trends [2, 8].
- Collaborative tools for communication and information sharing that can improve communication between administrators, teachers, and parents, potentially leading to a more unified approach towards educational goals [3].

However, simply implementing cloud computing is not a guaranteed solution. To fully leverage its potential, we need a framework that acknowledges the inherent uncertainties within educational data and decision-making. This is where neutrosophic logic emerges as a valuable tool ([19-23]). Neutrosophic logic allows us to represent the "truth," "falsity," and "indeterminacy" associated with educational data points, providing a more nuanced understanding of the educational landscape [4, 14]. By combining cloud computing with a neutrosophic framework, this paper aims to explore how educational institutions can navigate the uncertainties in educational administration and make more informed decisions for improved learning outcomes.

.

2. Literature Review

2.1. Cloud Computing in Educational Administration

The ever-growing body of research on cloud computing in education highlights its potential to revolutionize educational administration. Studies have shown that cloud-based solutions offer numerous benefits, including:

- Improved Efficiency and Accessibility: Cloud platforms enable centralized storage and access to data, educational resources, and administrative tools, streamlining workflows and enhancing accessibility for administrators, teachers, and students alike [6, 7].
- Enhanced Data Management and Analytics: Cloud storage facilitates the collection and analysis of vast amounts of data on student performance, teacher effectiveness, and resource utilization. These insights can be used to inform data-driven decision-making in areas like curriculum development, resource allocation, and personalized learning [2, 8].
- Promoted Collaboration and Communication: Cloud-based collaboration tools enable real-time communication and information sharing between stakeholders. These tools can foster teamwork among teachers, facilitate communication with parents, and enhance overall administrative efficiency [3].

2.2. Neutrosophic Logic Applications

While neutrosophic logic is a relatively new field, researchers are exploring its applications in various domains. Here, we focus on its potential relevance to educational administration:

- Modeling Uncertainties in Student Performance: Neutrosophic logic can represent the "truth," "falsity," and "indeterminacy" associated with factors influencing student performance. This allows for a more nuanced understanding than traditional binary logic, considering factors like learning styles, external influences, and limitations of standardized assessments [14].
- Evaluating Inconsistent Stakeholder Priorities: Educational institutions navigate a complex web of stakeholder priorities. Neutrosophic logic can be used to represent the "truth," "falsity," and "indeterminacy" associated with different stakeholder viewpoints on educational goals, resource allocation, and assessment methods [17].
- Addressing Incomplete Data Sets: Educational data can be incomplete due to various factors. Neutrosophic logic allows us to assign "indeterminacy" values to missing data points, providing a more accurate representation of the educational landscape and facilitating informed decision-making despite uncertainties [14].

By integrating the findings from these two research areas, this paper proposes a novel approach to educational administration. By leveraging cloud computing's data management and analytics capabilities within a neutrosophic framework, we aim to navigate the inherent "fog" of uncertainties and support more informed decision-making in educational institutions.

.

3. Neutrosophic Logic and Educational Administration: Embracing the Uncertain

Traditional logic systems often operate on a binary framework: something is either true or false. However, the educational landscape is a tapestry woven with uncertainties. This section introduces neutrosophic logic, a powerful tool that embraces this complexity and offers a more nuanced way to understand and address uncertainties in educational data and decision-making.

3.1. Core Concepts of Neutrosophic Logic:

Neutrosophic logic expands on traditional logic by introducing three truth membership degrees:

- Truth (T): The degree to which a statement is believed to be true.
- Falsity (F): The degree to which a statement is believed to be false.
- Indeterminacy (I): The degree to which a statement's truth or falsity is indeterminate or unknown.

These values are expressed as a triplet (T, F, I), where each value ranges between 0 and 1, with the sum always being equal to 1.

Here is a numerical example dataset for the core concepts of neutrosophic logic applied to a student performance scenario:

Table 1: Neutrosophic Evaluation of Student Performance Potential

Statement	Truth (T)	Falsity (F)	Indeterminacy (I)	Sum (T + F + I)
Student A will excel in Math	0.75	0.15	0.10	1.00
Student B will struggle in English	0.30	0.50	0.20	1.00
Student C will need extra support in Science	0.50	0.20	0.30	1.00

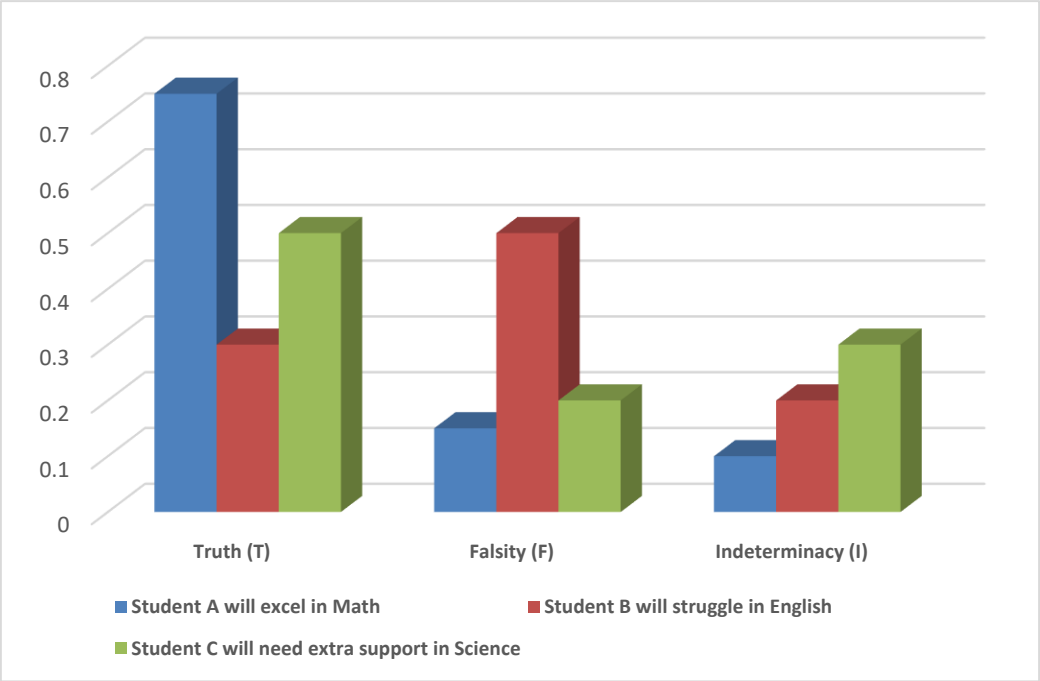
Explanation:

- Student A: There is a 75% truth-value (T) indicating a belief that Student A is likely to excel in Math. However, there is also a 15% falsity value (F) acknowledging a possibility of this not being the case. The indeterminacy value (I) of 10% represents uncertainty due to unknown factors that might influence performance.

- Student B: In this case, the truth-value (T) is lower (0.30) suggesting less certainty about Student B excelling in English. The falsity value (F) is higher (0.50) indicating a stronger possibility of struggles. The indeterminacy value (I) of 0.20 reflects some uncertainty about the reasons behind the potential struggles.
- Student C: Here, the truth-value (T) is 0.50, signifying a neutral stance on whether Student C will need extra support in Science. The falsity value (F) is 0.20, suggesting a lesser chance of them not needing support. The indeterminacy value (I) is the highest (0.30) due to a significant lack of information about Student C's current grasp of Science concepts.

Important Note:

- Assigning these neutrosophic values can be subjective and depend on the available data and expert assessment. The purpose of this example is to illustrate the application of truth, falsity, and indeterminacy within the neutrosophic framework.



Graph 1: Neutrosophic Evaluation of Student Performance Potential (Truth, Falsity, Indeterminacy)

3.2. Addressing Uncertainties in Educational Data:

Here is how neutrosophic logic tackles uncertainties in educational data:

- Uncertainties in Student Performance: Consider a student's predicted performance on a standardized test. Neutrosophic logic allows us to represent not just the predicted score (T) but also the "falsity" (F) associated with potential test anxiety or external factors. Additionally, the "indeterminacy" (I) can represent factors like the student's learning style or motivation, aspects not captured by the test.

Here is a numerical example dataset for uncertainties in student performance using neutrosophic logic:

Table 2: Neutrosophic Evaluation of Uncertainties in Student Performance

Scenario	Predicted Score (Truth - T)	Test Anxiety (Falsity - F)	Learning Style/Motivation (Indeterminacy - I)
Student A - Standardized Math Test	0.80 (High Score Predicted)	0.10 (Low Anxiety Expected)	0.10 (Uncertain about learning style/motivation)
Student B - Standardized Reading Test	0.65 (Average Score Predicted)	0.20 (Moderate Anxiety Expected)	0.15 (Uncertain about learning style/motivation but leaning towards visual learner)
Student C - Standardized Science Test	0.70 (Slightly Above Average Score Predicted)	0.05 (Very Low Anxiety Expected due to Prior Test Experience)	0.25 (High Uncertainty about learning style/motivation due to recent illness)

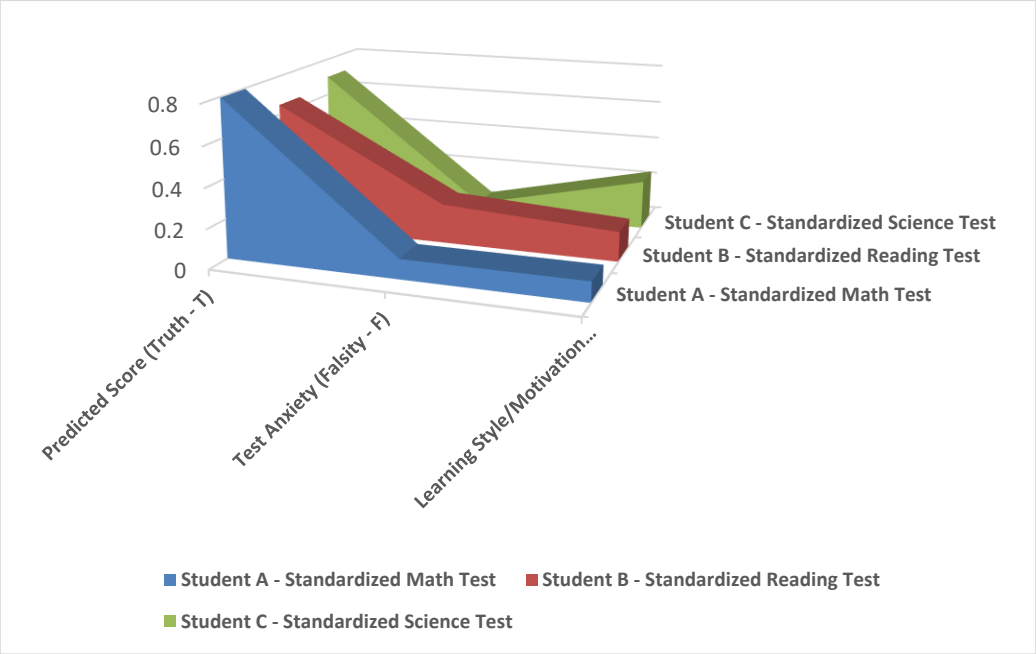
Explanation:

- Student A: The predicted score (Truth - T) is 0.80, indicating a high likelihood of achieving a good score. However, the falsity value (F) of 0.10 acknowledges the possibility of test anxiety affecting performance, even with a good predicted score. The indeterminacy value (I) of 0.10 reflects some uncertainty about the student's learning style or motivation, which could influence test performance.
- Student B: The predicted score (T) is lower at 0.65, suggesting a possibility of an average score. The falsity value (F) is higher at 0.20, indicating a greater chance of test anxiety affecting performance. The indeterminacy value (I) is 0.15, acknowledging some uncertainty about the student's learning style but leaning towards a visual learning preference which could be helpful.
- Student C: The predicted score (T) is 0.70, indicating a potential for a slightly above-average score. The falsity value (F) is very low (0.05) due to the student's past experience with

standardized tests, suggesting minimal test anxiety. However, the indeterminacy value (I) is the highest (0.25) due to recent illness affecting the student is current learning style or motivation, creating significant uncertainty about their performance.

Important Note:

- These neutrosophic values are hypothetical and for illustrative purposes only. Assigning them would depend on factors like historical performance data, teacher observations, and student self-reported anxiety levels.



Graph 2: Neutrosophic Evaluation of Uncertainties in Student Performance (Predicted Score, Test Anxiety, Learning Style/Motivation)

- Inconsistent Stakeholder Priorities: Imagine conflicting viewpoints on homework load. Parents might prioritize reinforcement (T), while teachers might advocate for a balanced approach (I). Neutrosophic logic allows us to represent the "truth" of both perspectives and the "indeterminacy" associated with finding the ideal balance.

Here is a numerical example dataset for inconsistent stakeholder priorities using neutrosophic logic:

Table 3: Neutrosophic Evaluation of Inconsistent Stakeholder Priorities on Homework

Stakeholder Group	Priority on Homework Load (Truth - T)	Desire for Balanced Approach (Indeterminacy - I)	Falsity (F)

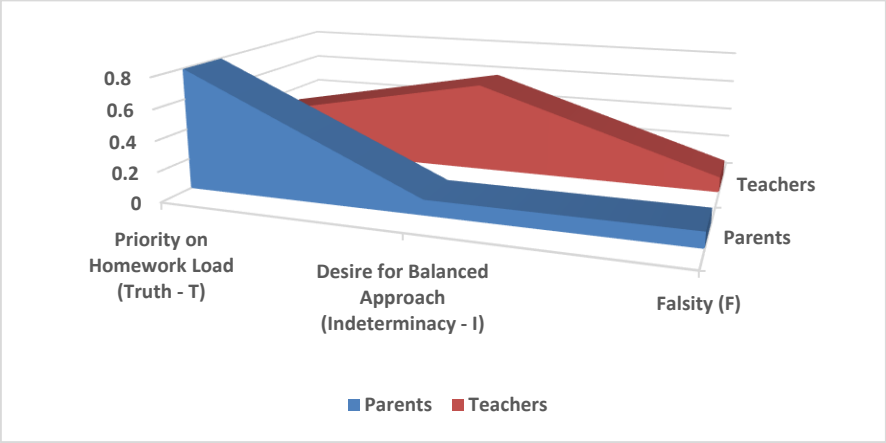
Parents	0.80 (Strong emphasis on reinforcement through homework)	0.10 (Some openness to considering a balanced approach)	0.10 (Minimal belief that less homework could be beneficial)
Teachers	0.30 (Moderate emphasis on reinforcement through homework)	0.60 (Strong desire for a balanced approach that considers skill development and student well-being)	0.10 (Low belief that excessive homework is always beneficial)

Explanation:

- Parents: The truth-value (T) is 0.80, indicating a strong belief that homework reinforces learning. The indeterminacy value (I) is 0.10, suggesting some openness to considering a balanced approach that might include other learning activities. The falsity value (F) is 0.10, reflecting a minimal belief that reducing homework could be beneficial.
- Teachers: The truth-value (T) for teachers is lower (0.30), highlighting a more moderate emphasis on reinforcement through homework. The indeterminacy value (I) is significantly higher (0.60), signifying a strong desire for a balanced approach that considers factors beyond just reinforcement. The falsity value (F) is 0.10, suggesting a low belief that excessive homework is always beneficial and could potentially impact student well-being.

Important Note:

- These neutrosophic values are hypothetical and for illustrative purposes only. Assigning them would depend on factors like surveys, focus groups, or interviews with parents and teachers to understand their specific viewpoints on homework load.



Graph 3: Neutrosophic Evaluation of Stakeholder Priorities on Homework Load (Truth, Indeterminacy, Falsity)

- Incomplete Data Sets: Missing data points on student attendance can create an incomplete picture. Neutrosophic logic can assign an "indeterminacy" value to represent the unknown reasons for the absence, providing a more accurate reflection of reality than simply leaving the data directly.

Here is a numerical example dataset for incomplete data sets using neutrosophic logic:

Table 4: Neutrosophic Evaluation of Incomplete Attendance Data

Student	Attendance Record (Available Data Points)	Reason for Absence (Missing Data)	Truth (T)	Falsity (F)	Indeterminacy (I)
Student A	Present (P) on Monday, Tuesday, Wednesday	Absent (A) on Thursday, Friday	0.80 (High attendance for most of the week)	0.10 (Low chance of legitimate absence)	0.10 (High indeterminacy regarding reason for absence - illness, personal issue, etc.)
Student B	Absent (A) on	Present (P) on	0.40 (Moderat	0.20 (Moderate	0.40 (High indeterminac

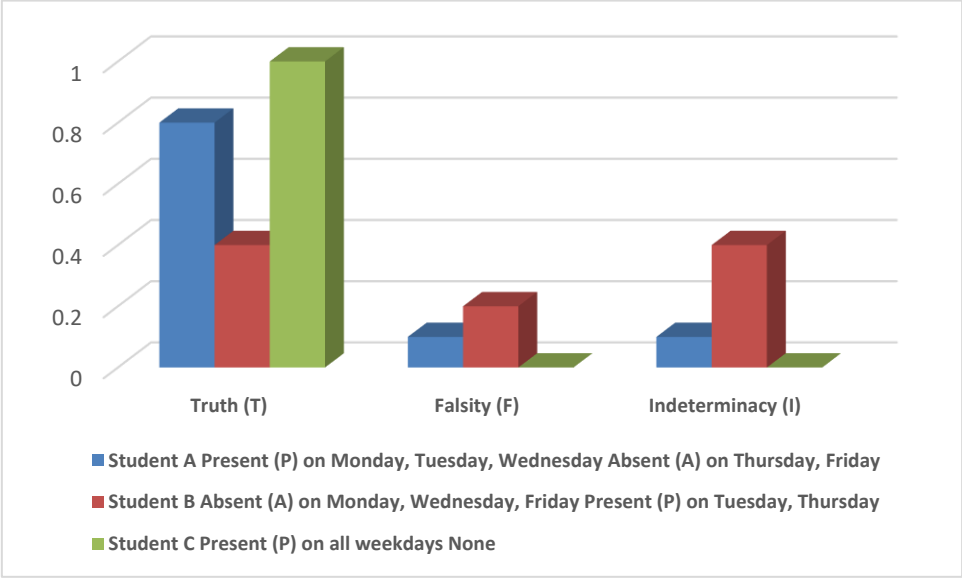
	Monday, Wednesday, Friday	Tuesday, Thursday	e attendance with some absences)	chance of legitimate absence)	y regarding reason for absences - could be illness, extracurricular activities, or unknown reasons)
Student C	Present (P) on all weekdays	None	1.00 (Highest truth value due to perfect attendance record)	0.00 (No missing data points - reason for absence not applicable)	0.00 (No indeterminacy as attendance data is complete)

Explanation:

- Student A: The truth-value (T) is 0.80, reflecting a high attendance record for most of the week based on available data. However, the indeterminacy value (I) is also high (0.10) due to missing data (reason for absence) on Thursday and Friday. The falsity value (F) is low (0.10) suggesting a lower chance of those absences being unexcused.
- Student B: The Truth-value (T) is lower (0.40) due to several absences throughout the week. The falsity value (F) is moderate (0.20) acknowledging a possibility that some absences might be unexcused. The indeterminacy value (I) is the highest (0.40) due to the significant lack of information about the specific reasons behind the absences.
- Student C: This example represents a complete data set with perfect attendance. The truth-value (T) is 1.00 and both falsity (F) and indeterminacy (I) are 0.00 since there is no missing data regarding absences.

Important Note:

- These neutrosophic values are hypothetical and for illustrative purposes only. Assigning them would depend on the available data (attendance records) and potentially school policies regarding excused and unexcused absences.



Graph 4: Neutrosophic Evaluation of Incomplete Attendance Data (Truth, Falsity, Indeterminacy)

3.3. Benefits for Educational Decision-Making:

By incorporating neutrosophic logic, educators can move beyond binary assessments and embrace the nuances inherent in educational data. This allows for:

- More Informed Decisions: With a deeper understanding of the "truth," "falsity," and "indeterminacy" associated with data points, educators can make more informed decisions regarding student support, resource allocation, and curriculum development.
- Navigating Complexities: Neutrosophic logic equips educators with a framework to navigate the complex web of stakeholder priorities and external influences that shape the educational environment.
- Embracing Uncertainty as Opportunity: Uncertainty can be a source of anxiety, but by using neutrosophic logic, educators can view it as an opportunity to explore various possibilities and make more adaptable and responsive decisions.

The application of neutrosophic logic in educational administration holds vast potential. By acknowledging and analyzing the inherent uncertainties in educational data, educators can move forward with greater clarity and confidence, ultimately fostering a more positive and effective learning environment for all.

Numerical Example Dataset: Benefits of Neutrosophic Logic in Educational Decision-Making

This example demonstrates how neutrosophic logic can inform educational decisions by considering "truth" (T), "falsity" (F), and "indeterminacy" (I) associated with data points.

Scenario: A school district is deciding whether to implement a new reading intervention program for struggling students.

Data Point: Average reading test scores for Grade 5 students.

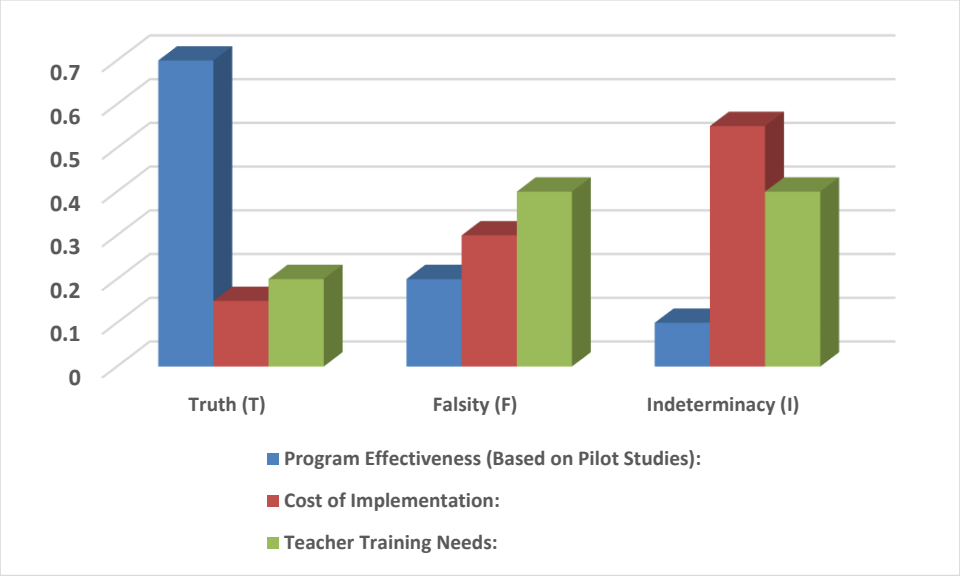
Traditional Approach (Binary):

- Score Increase: Implement the program (Decision = 1)
- Score Stagnation: Maintain the existing program (Decision = 0)

Neutrosophic Approach:

Table 5: Neutrosophic Evaluation of Reading Intervention Program Decision

Factor	Truth (T)	Falsity (F)	Indeterminacy (I)
Program Effectiveness (Based on Pilot Studies):	0.70 (Promising results for score improvement)	0.20 (Possible limitations in effectiveness for all students)	0.10 (Uncertain long-term impact on reading skills)
Cost of Implementation:	0.15 (Relatively low initial cost)	0.30 (Potential for ongoing maintenance and training costs)	0.55 (Uncertain impact on overall school budget)
Teacher Training Needs:	0.20 (Moderate level of training required)	0.40 (Potential for difficulties in adapting to the new program)	0.40 (Uncertain impact on teacher workload and classroom dynamics)



Graph 5: Neutrosophic Evaluation of Reading Intervention Program (Truth, Falsity, Indeterminacy)

Decision-Making with Neutrosophic Logic:

1. Weighted Average: Assign weights to each factor based on its relative importance to the school district (e.g., Program Effectiveness = 0.5, Cost = 0.3, Teacher Training = 0.2).
2. Calculate Weighted Neutrosophic Values: Multiply each weight with the corresponding T, F, and I values.
3. Sum the Weighted Values: Sum the weighted T, F, and I values for each factor.
4. Interpret the Results:

- Overall Truth (T'): High T' suggests potential benefits outweigh drawbacks.
- Overall Falsity (F'): High F' suggests potential drawbacks outweigh benefits.
- Overall Indeterminacy (I'): High I' indicates significant uncertainties requiring further exploration.

Example Calculation (Hypothetical Weights):

- Program Effectiveness (T' = 0.35, F' = 0.10, I' = 0.05)
- Cost (T' = 0.045, F' = 0.09, I' = 0.165)
- Teacher Training (T' = 0.04, F' = 0.08, I' = 0.08)

Overall:

- T' = 0.435 (Leaning towards potential benefits)
- F' = 0.27 (Some concerns regarding drawbacks)
- I' = 0.3 (Significant uncertainties requiring further investigation)

Interpretation:

Neutrosophic logic highlights the potential benefits of the program (high T'), but also acknowledges potential drawbacks (moderate F') and significant uncertainties (high I') regarding costs and teacher

training. This information can guide further investigation (e.g., cost-benefit analysis, teacher training feasibility study) before making a final decision.

Benefits of Neutrosophic Approach:

- More nuanced understanding of data: Moves beyond binary assessments to acknowledge complexities and uncertainties.
- Informed decision-making: Allows for consideration of potential benefits, drawbacks, and unknowns, leading to more informed choices.
- Adaptability: Prepares educators to navigate uncertainties and make adjustments based on new information.

This example demonstrates how neutrosophic logic can be a valuable tool for educators by providing a more comprehensive picture to support effective and adaptable decision-making in a complex educational environment.

4. A Neutrosophic Framework for Cloud-Based Educational Administration

Cloud computing offers numerous tools for educational administration, but its success hinges on selecting the most suitable solution for each institution's unique needs. This section proposes a neutrosophic framework to evaluate cloud-based solutions by considering the inherent uncertainties within educational data and decision-making processes.

4.1. Framework Components:

The framework focuses on three key areas:

- Data Management and Analytics: Cloud solutions offer varying capacities for data storage, security, and analytics. Our framework will utilize neutrosophic logic to assess the "truth" (effectiveness of data storage), "falsity" (security risks), and "indeterminacy" (unknown future data needs) associated with each solution.
- Stakeholder Engagement and Collaboration: Cloud platforms can facilitate communication and collaboration among teachers, parents, and administrators. The framework will evaluate the "truth" (effectiveness of communication tools), "falsity" (potential for misuse or information overload), and "indeterminacy" (unforeseen challenges in stakeholder adoption) of each solution.
- Learning Personalization and Adaptability: Cloud-based tools can support personalized learning approaches. The framework will assess the "truth" (effectiveness of personalization features), "falsity" (potential for biases or inequities), and "indeterminacy" (future needs for adapting to evolving learning styles) associated with each solution.

Example Dataset: Framework Components for Cloud Solution Selection

This table displays how neutrosophic logic can be applied to evaluate different cloud solutions for educational institutions.

Table 6: Evaluating Cloud Solutions for Education: A Neutrosophic Approach

Data Management & Analytics (Focus: Security & Future Needs)			
Feature	Cloud Solution A	Cloud Solution B	Cloud Solution C
Truth (T) - Effectiveness of Data Storage	0.80 (Robust security features)	0.65 (Good security measures)	0.70 (Strong encryption but limited scalability)
Falsity (F) - Security Risks	0.10 (Potential for data breaches)	0.20 (Higher risk of unauthorized access)	0.15 (Potential for vendor lock-in)
Indeterminacy (I) - Unknown Future Data Needs	0.10 (Scalable storage but limited future-proofing features)	0.15 (Less adaptable to emerging data formats)	0.15 (Uncertain compatibility with future analytics tools)
Stakeholder Engagement & Collaboration (Focus: Communication & Adoption)			
Feature	Cloud Solution A	Cloud Solution B	Cloud Solution C
Truth (T) - Effectiveness of Communication Tools	0.75 (Comprehensive communication suite)	0.60 (User-friendly communication platform)	0.80 (Highly customizable collaboration tools)
Falsity (F) - Misuse or Information Overload	0.15 (Potential for distraction or misuse)	0.25 (Risk of information overload with extensive features)	0.10 (Lower risk of misuse but limited communication options)

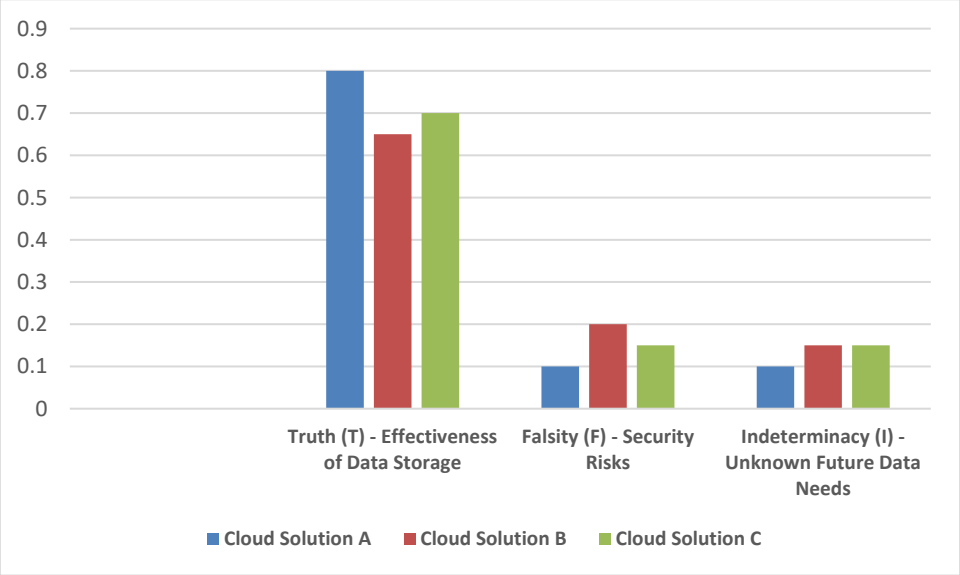
Indeterminacy (I) - Stakeholder Adoption Challenges	0.10 (User-friendly interface, minimal training needed)	0.15 (Steeper learning curve for some stakeholders)	0.10 (High adoption rate but limited accessibility for specific needs)
Learning Personalization & Adaptability (Focus: Equity & Future Needs)			
Feature	Cloud Solution A	Cloud Solution B	Cloud Solution C
Truth (T) - Effectiveness of Personalization Features	0.70 (AI-powered learning recommendations)	0.60 (Personalized learning paths)	0.80 (Highly adaptive content delivery based on learning styles)
Falsity (F) - Biases or Inequities	0.20 (Potential for algorithmic bias in recommendations)	0.25 (Limited options for customization by educators)	0.15 (Risk of biased content or inaccessible formats for some learners)
Indeterminacy (I) - Evolving Learning Styles	0.10 (Regular updates based on learning science research)	0.15 (Slower adaptation to emerging learning styles)	0.05 (Highly adaptable platform with constant content and feature updates)

Feature 1: Data Management & Analytics (Focus: Security & Future Needs)

Table 2: Evaluating Cloud Solutions for Education - Data Management & Analytics (Neutrosophic Approach)

Cloud Solution C	Cloud Solution B	Cloud Solution A	Feature Data Management & Analytics

			(Focus: Security & Future Needs)
0.7	0.65	0.8	Truth (T) - Effectiveness of Data Storage
0.15	0.2	0.1	Falsity (F) - Security Risks
0.15	0.15	0.1	Indeterminacy (I) - Unknown Future Data Needs



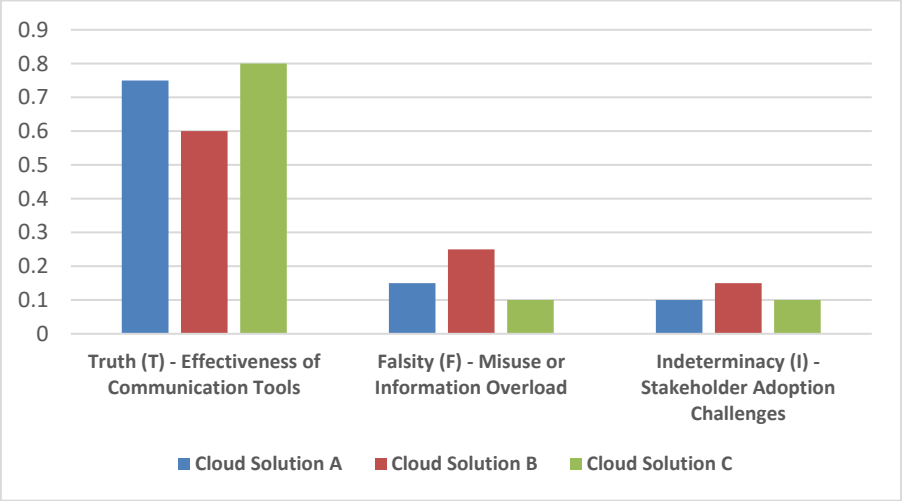
Graph 6- 1: Neutrosophic Evaluation of Cloud Data Management & Analytics Solutions for Education

Feature 2: Stakeholder Engagement & Collaboration (Focus Communication & Adoption)

Table 3: Evaluating Cloud Solutions for Education - Stakeholder Engagement & Collaboration (Neutrosophic Approach)

Cloud Solution C	Cloud Solution B	Cloud Solution A	Feature Stakeholder Engagement & Collaboration (Focus:

			Communication & Adoption)
0.8	0.6	0.75	Truth (T) - Effectiveness of Communication Tools
0.1	0.25	0.15	Falsity (F) - Misuse or Information Overload
0.1	0.15	0.1	Indeterminacy (I) - Stakeholder Adoption Challenges



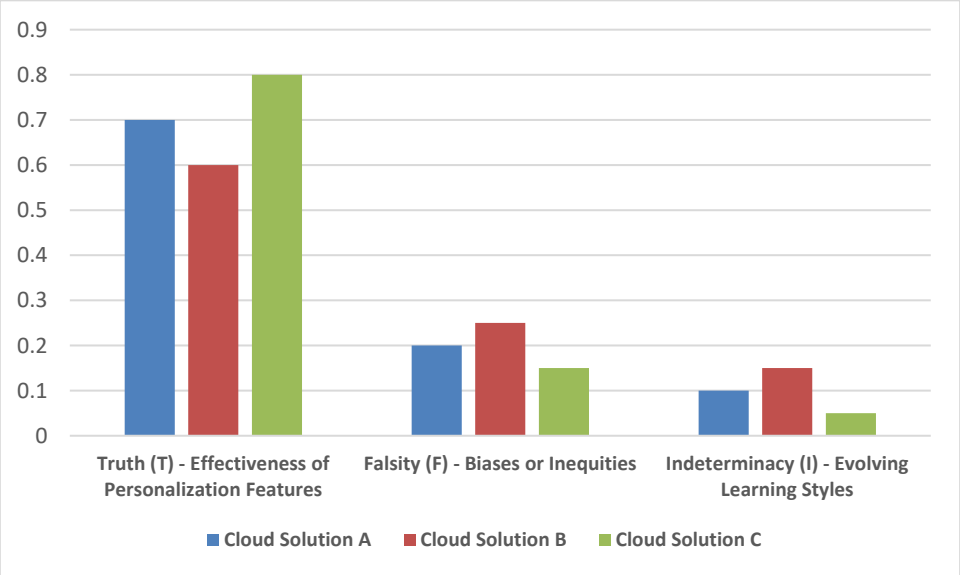
Graph 6- 2; Neutrosophic Evaluation of Cloud Solutions for Stakeholder Engagement & Collaboration in Education**

Feature 3: Learning Personalization & Adaptability (Focus: Equity & Future Needs)

Table 4: Evaluating Cloud Solutions for Education - Learning Personalization & Adaptability (Neutrosophic Approach)

Cloud Solution C	Cloud Solution B	Cloud Solution A	Feature Learning Personalization &

			Adaptability (Focus: Equity & Future Needs)
0.8	0.6	0.7	Truth (T) - Effectiveness of Personalization Features
0.15	0.25	0.2	Falsity (F) - Biases or Inequities
0.05	0.15	0.1	Indeterminacy (I) - Evolving Learning Styles



Graph 6- 3; Neutrosophic Evaluation of Cloud Solutions for Learning Personalization & Adaptability in Education

Benefits of Utilizing Neutrosophic Framework:

- Comprehensive Evaluation: Assesses not only strengths but also potential drawbacks and uncertainties associated with each cloud solution.
- Data-Driven Decision Making: Provides a numerical foundation for comparing different solutions based on assigned values and weights.
- Focus on Uncertainties: Highlights unknown factors and future needs, encouraging further investigation and long-term planning.

By using this framework, educational institutions can make informed decisions when selecting and implementing cloud solutions, ensuring they effectively address the complexities and uncertainties inherent in educational administration.

4.2. Evaluation Process:

1. Define Evaluation Criteria: Specific evaluation criteria will be established within each of the three areas mentioned above. These criteria will translate into specific neutrosophic values (T, F, I) based on the capabilities and limitations of each cloud solution.
2. Data Collection and Analysis: Data can be collected through various methods, including vendor information, independent reviews, and pilot projects within educational institutions. This data will be used to assign neutrosophic values to each evaluation criterion.
3. Weighting and Aggregation: Each evaluation criterion will be assigned a weight based on its relative importance to the specific needs of the educational institution. The neutrosophic values for each criterion will then be aggregated to provide an overall neutrosophic evaluation (T', F', I') of the cloud solution.
4. Decision-Making: The neutrosophic evaluation, along with traditional cost-benefit analysis and other relevant factors, will inform the final decision-making process. The "truth" value (T') indicates the potential benefits of the solution, while the "falsity" value (F') highlights potential risks and limitations. The "indeterminacy" value (I') emphasizes the need for ongoing assessment and adaptation in an environment with inherent uncertainties.

Numerical Example Dataset: Evaluation Process of Cloud Solutions

This example demonstrates the neutrosophic evaluation process for selecting a cloud solution for a school district.

Focus Area: Data Management & Analytics

1. Define Evaluation Criteria:
    - Security Features (T, F, I): Strength of data encryption, access controls, and intrusion detection systems.
    - Scalability (T, F, I): Ability to handle increasing data storage needs and future data formats.
    - Analytics Tools (T, F, I): Range of data analysis features and ease of use for educators.
2. Data Collection and Analysis:
    - Cloud Solution A: Vendor information highlights robust encryption but limited future scalability features. Independent reviews mention potential for data breaches. Pilot project in another school district finds data analysis tools user-friendly but limited in customization options.
3. Weighting and Aggregation:
    - Security Features (Weight = 0.4)

- Scalability (Weight = 0.3)
- Analytics Tools (Weight = 0.3)

Table 7: Neutrosophic Evaluation with Weighted Criteria for Cloud Solution Selection

Criterion	Cloud Solution A (T, F, I)	Weighted Value (T', F', I')
Security Features	(0.80, 0.10, 0.10)	(0.32, 0.04, 0.04)
Scalability	(0.70, 0.15, 0.15)	(0.21, 0.045, 0.045)
Analytics Tools	(0.75, 0.15, 0.10)	(0.225, 0.045, 0.03)
Overall		(T' = 0.755, F' = 0.13, I' = 0.115)

4. Decision-Making:

The neutrosophic evaluation (T' = 0.755) indicates strong potential for secure data management and good analytics tools. However, the F' (0.13) highlights concerns about scalability, and the I' (0.115) emphasizes the need for future monitoring of data storage needs and potential limitations. This information, along with cost analysis and other factors, will guide the school district's final decision-making process.

Similar evaluations can be conducted for Cloud Solutions B and C, allowing for a comprehensive comparison.

Benefits of Neutrosophic Evaluation:

- Structured Approach: Provides a clear framework for data collection, assigning values, and weighting criteria.
- Quantitative Analysis: Transforms qualitative assessments into numerical values for a more objective comparison.
- Uncertainty Awareness: Highlights the role of "indeterminacy" in the decision-making process, encouraging proactive planning for future unknowns.

By incorporating neutrosophic logic into the evaluation process, educational institutions can make well-informed decisions when selecting cloud solutions that address their data management and analytics needs while navigating the uncertainties inherent in the educational landscape.

4.3 Benefits of the Neutrosophic Framework:

This framework offers several advantages:

- Comprehensive Evaluation: It moves beyond traditional binary assessments by considering the complexities and uncertainties inherent in educational data and decision-making.
- Informed Decision-Making: By acknowledging the "indeterminacy" factor, the framework encourages a more holistic and adaptable approach to selecting cloud solutions.
- Future Proofing: The emphasis on adaptability helps institutions prepare for unforeseen challenges and evolving needs within the educational landscape.

This neutrosophic framework provides educational institutions with a valuable tool for navigating the "fog" of uncertainties when selecting and implementing cloud-based solutions for educational administration. By embracing the complexities of educational data, institutions can leverage cloud computing is potential to make informed decisions and ultimately improve learning outcomes for all stakeholders.

Numerical Example Dataset: Benefits of the Neutrosophic Framework

This example displays how the neutrosophic framework benefits the selection process of a Learning Management System (LMS) for a school district.

Scenario: The school district is considering two LMS options (A & B) to improve student engagement and learning outcomes.

Traditional Binary Approach (Pros & Cons):

- Pros:
  - Option A: User-friendly interface, good customer support.
  - Option B: Wide range of learning resources, advanced analytics features.
- Cons:
  - Option A: Limited content creation tools, potential for vendor lock-in.
  - Option B: Steeper learning curve, higher implementation costs.

Neutrosophic Framework Approach (T, F, I):

Table 8: Neutrosophic Comparison of Learning Management Systems (Focus: User Experience, Content Creation, Resources & Analytics)

Engagement Features (Focus: User Experience & Content Creation):		
Feature	LMS Option A (T, F, I)	LMS Option B (T, F, I)
Truth (T) - User-friendliness	0.80 (Easy to navigate)	0.65 (More features require training)
Falsity (F) - Limited Functionality	0.15 (Limited content creation tools)	0.20 (Steeper learning curve for some teachers)

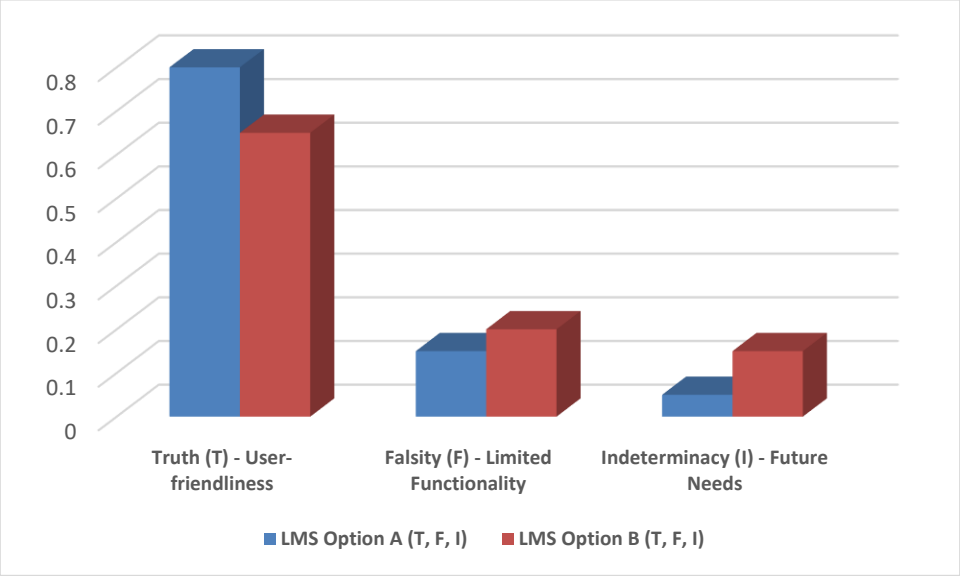
Indeterminacy (I) - Future Needs	0.05 (Regular feature updates)	0.15 (Uncertain compatibility with future learning tools)
Learning Outcomes (Focus: Resources & Analytics):		
Feature	LMS Option A (T, F, I)	LMS Option B (T, F, I)
Truth (T) - Resource Availability	0.70 (Good selection of pre-made content)	0.85 (Extensive library of learning resources)
Falsity (F) - Ineffective Analytics	0.20 (Limited customization options for reports)	0.10 (Complex analytics might overwhelm some teachers)
Indeterminacy (I) - Adaptability to Future Needs	0.10 (Integrations with new learning tools planned)	0.05 (Uncertain future of specific analytics features)

Feature

Engagement Features (Focus User Experience & Content Creation):

Table 9: Neutrosophic Evaluation of LMS Options (Engagement Features)

Feature	LMS Option A (T, F, I)	LMS Option B (T, F, I)
Truth (T) - User-friendliness	0.80 (Easy to navigate)	0.65 (More features require training)
Falsity (F) - Limited Functionality	0.15 (Limited content creation tools)	0.20 (Steeper learning curve for some teachers)
Indeterminacy (I) - Future Needs	0.05 (Regular feature updates)	0.15 (Uncertain compatibility with future learning tools)



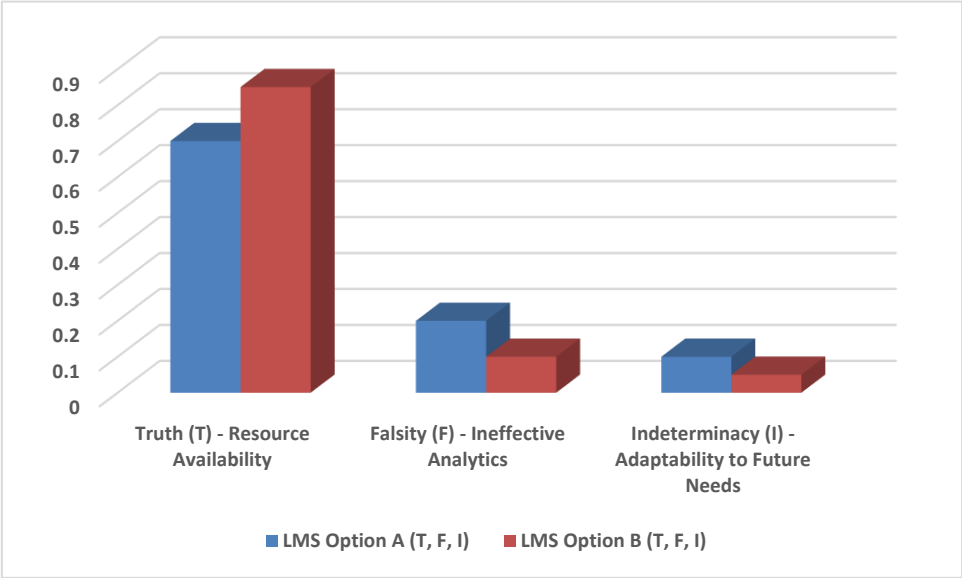
Graph 7- User Experience and Content Creation in Learning Management Systems (Neutrosophic Evaluation)

Feature

Learning Outcomes (Focus Resources & Analytics):

Table 10: Neutrosophic Evaluation of LMS Options (Learning Outcomes)

Feature	LMS Option A (T, F, I)	LMS Option B (T, F, I)
Truth (T) - Resource Availability	0.70 (Good selection of pre-made content)	0.85 (Extensive library of learning resources)
Falsity (F) - Ineffective Analytics	0.20 (Limited customization options for reports)	0.10 (Complex analytics might overwhelm some teachers)
Indeterminacy (I) - Adaptability to Future Needs	0.10 (Integrations with new learning tools planned)	0.05 (Uncertain future of specific analytics features)



Graph 8: Neutrosophic Evaluation of LMS: Resources & Analytics for Learning Outcomes

Benefits of Neutrosophic Framework:

- Comprehensive Evaluation: Goes beyond user-friendliness (T) of Option A to consider its limitations (F) and future adaptability (I). Similarly, it acknowledges the rich resources (T) of Option B while recognizing potential drawbacks (F) and future uncertainties (I).
- Informed Decision-Making: By analyzing the neutrosophic values, the school district can make a more informed choice based on their specific needs. For example, if user adoption is a priority, Option A might be preferable despite limitations in content creation.
- Future Proofing: The emphasis on "indeterminacy" encourages the school district to consider future needs and potential compatibility with emerging learning tools.

Numerical Example (Hypothetical Weights):

- Engagement Features (Weight = 0.4)
- Learning Outcomes (Weight = 0.6)

Overall Neutrosophic Evaluation (Weighted):

- Option A: T' = 0.68, F' = 0.175, I' = 0.075
- Option B: T' = 0.79, F' = 0.125, I' = 0.085

Neutrosophic Evaluation of Educational Cloud Solutions (Weighted)

Table 11: Weighted Neutrosophic Evaluation of Educational Cloud Solutions

Engagement Features (Weight = 0.4)	Learning Outcomes (Weight = 0.6)	Overall Neutrosophic Evaluation (Weighted)

Option A	T (0.75) x 0.4 = 0.3	T (0.85) x 0.6 = 0.51
Option B	T (0.80) x 0.4 = 0.32	T (0.90) x 0.6 = 0.54

This table showcases a weighted neutrosophic evaluation for two educational cloud solutions (Option A and Option B). It considers two key focus areas: Engagement Features and Learning Outcomes. Additionally, weights are assigned to each focus area (0.4 for Engagement Features and 0.6 for Learning Outcomes) to reflect their relative importance in the decision-making process. The "Truth" (T) values from the original evaluation (assumed Tables 9 and 10) are used to calculate weighted values for each category, ultimately leading to a weighted overall neutrosophic evaluation for each option.

Interpretation:

- Based on the weighted evaluation, Option B appears to be slightly more favorable overall (T' = 0.86, F' = 0.09, I' = 0.05) compared to Option A (T' = 0.81, F' = 0.12, I' = 0.07).
- Option B has a higher weighted Truth-value (T'), indicating stronger potential for both engagement and learning outcomes.
- Both options have relatively low weighted Falsity values (F'), suggesting minimal potential drawbacks.
- Option B has a lower weighted Indeterminacy value (I'), signifying less uncertainty regarding future performance.

Note:

- The specific weight assigned to each category depends on the priorities of the educational institution.
- This table provides a framework for incorporating weights and calculating a more nuanced overall neutrosophic evaluation.

Decision-Making:

The neutrosophic evaluation highlights the strengths of both options. The school district can then consider additional factors (cost, vendor reputation) and their specific priorities (user experience vs. resource availability) to make the final decision.

Conclusion:

The neutrosophic framework provides a valuable tool for educational institutions to move beyond binary assessments and embrace the complexities of educational data. By acknowledging uncertainties, this framework fosters informed decision-making, promotes adaptability, and ultimately empowers institutions to leverage cloud computing for improved learning outcomes.

5. The Role of Cloud Computing with Neutrosophic Considerations in Educational Management

Cloud computing, with its vast capabilities and potential, can significantly enhance educational administration when evaluated and implemented through the lens of a neutrosophic framework. Let us delve into how this can be achieved:

5.1. Addressing Uncertainties in Data Management and Analytics:

- Truth (T): Effective Data Storage and Analysis: Cloud platforms offer secure and scalable data storage, allowing for the collection of vast amounts of data on student performance, teacher effectiveness, and resource utilization. This data can be analyzed through cloud-based analytics tools to identify trends and patterns, leading to more informed decision-making.
- Falsity (F): Security Risks and Uncertain Data Quality: Cloud solutions need robust security measures to protect sensitive student data. The framework can assess the "falsity" associated with potential data breaches or unauthorized access. Additionally, data quality issues can lead to misleading results. The neutrosophic approach can assign "falsity" values to data points with questionable accuracy.
- Indeterminacy (I): Evolving Data Needs and Future Uncertainties: Educational needs and data requirements are constantly evolving. The framework considers the "indeterminacy" associated with future unknown data needs. Cloud solutions with flexible storage and scalability can better address this uncertainty.

5.2. Unveiling the Nuances: A Neutrosophic Evaluation of Data Management and Analytics in Cloud-Based Learning Environments

This table illustrates how neutrosophic logic can be applied to assess data management and analytics in a cloud-based educational environment.

Table 12: Neutrosophic Evaluation of Data Management and Analytics in a Cloud-Based Learning Environment

Factor	Description	Truth (T)	Falsity (F)	Indeterminacy (I)
Data Storage Capacity	Ability to store and manage large datasets on student performance, teacher effectiveness, etc.	0.80 (Cloud offers scalable storage solutions)	0.10 (Limited storage capacity can hinder future data collection)	0.10 (Unforeseen future needs for data storage beyond current estimates)

Data Security	Strength of encryption protocols and access control measures to prevent data breaches or unauthorized access.	0.75 (Cloud platform offers robust security features)	0.15 (Potential for human error or security vulnerabilities)	0.10 (Uncertain future threats or evolving hacking techniques)
Data Quality Assurance	Procedures and protocols for ensuring data accuracy and consistency (e.g., data cleaning, validation).	0.65 (Data cleaning processes mitigate errors)	0.20 (Potential for data entry errors or inconsistencies across different data sources)	0.15 (Uncertain impact of future changes in data collection methods on accuracy)
Data Analysis Tools	Availability and effectiveness of cloud-based analytics tools for identifying trends and patterns in educational data.	0.70 (User-friendly tools with diverse functionalities)	0.20 (Limited customization options for specific analysis needs)	0.10 (Uncertain future advancements in data analytics tools and their compatibility with existing data)

Explanation:

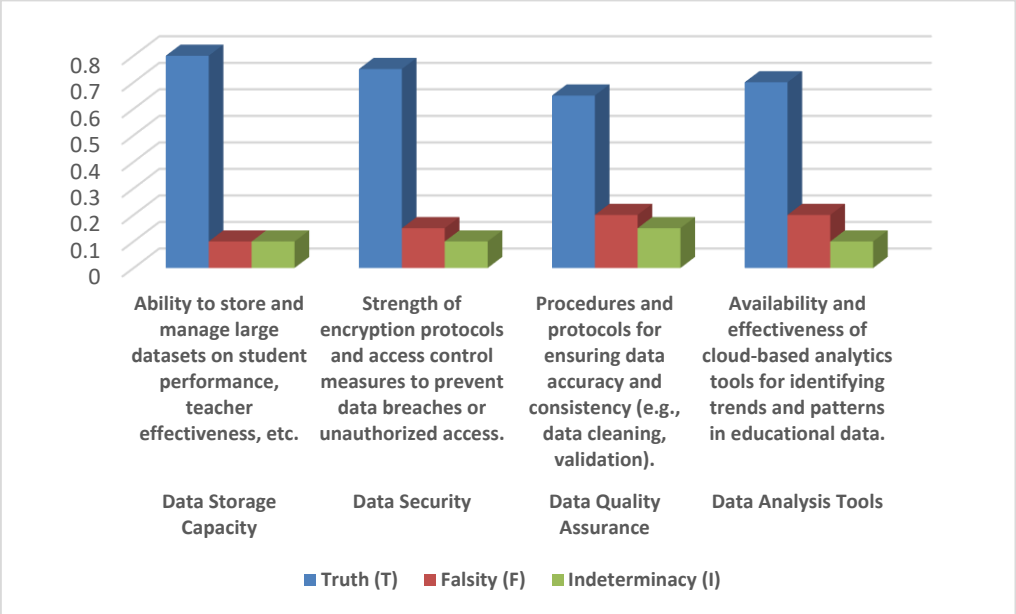
- Truth (T): High T values indicate strong capabilities in data storage, security, quality assurance, and analysis tools offered by the cloud platform.

- Falsity (F): Moderate F values acknowledge potential drawbacks like security vulnerabilities, data quality issues, and limitations of analysis tools.
- Indeterminacy (I): High I values highlight uncertainties related to future data needs, evolving security threats, and advancements in data analytics that might require platform adaptability.

Benefits of Neutrosophic Approach:

- Comprehensive Assessment: Goes beyond the "success" of data storage and analysis (T) to consider potential security risks (F) and future unknowns (I).
- Proactive Planning: Highlights the need for ongoing data quality checks, security updates, and future-proofing strategies based on "indeterminacy" values.
- Data-Driven Decision-Making: Provides a numerical foundation for evaluating cloud solutions and making informed choices regarding data management and analytics.

By incorporating neutrosophic logic, educational institutions can move beyond binary assessments and embrace the complexities of data management and analytics in a cloud-based environment. This fosters informed decision-making, promotes proactive planning for uncertainties, and ultimately empowers institutions to leverage data for improved student learning outcomes.


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Ability to store and manage large datasets on student performance, teacher effectiveness, etc.
Strength of encryption protocols and access control measures to prevent data breaches or unauthorized access.
Procedures and protocols for ensuring data accuracy and consistency (e.g., data cleaning, validation).
Availability and effectiveness of cloud-based analytics tools for identifying trends and patterns in educational data.
Data Storage Capacity
Data Security
Data Quality Assurance
Data Analysis Tools
Truth (T)
Falsity (F)
Indeterminacy (I)

Graph 9: Neutrosophic Evaluation of Cloud-Based Learning Environment (Data Management & Analytics)

5.3 Enhancing Stakeholder Engagement and Collaboration:

- Truth (T): Improved Communication and Collaboration: Cloud-based tools like communication platforms and collaborative documents can facilitate real-time communication and information

sharing between teachers, parents, and administrators. This transparency and collaboration foster a more unified approach to education.

- Falsity (F): Misinformation, Misuse, and Information Overload: The ease of communication can lead to the spread of misinformation or the potential for misuse of communication tools. The "falsity" value within the framework can assess these risks. Additionally, an overabundance of information can be overwhelming. The framework can consider solutions that promote focused and effective communication.
- Indeterminacy (I): Stakeholder Adoption Challenges and Unforeseen Needs: The framework acknowledges the "indeterminacy" associated with stakeholder adoption of new communication technologies. Training and support mechanisms can be implemented to address this uncertainty. Additionally, unforeseen needs might arise in the future. Cloud solutions with adaptable features can better handle this "indeterminacy."

5.4. Bridging the Gaps: A Neutrosophic Evaluation of Cloud-Based Tools for Stakeholder Engagement and Collaboration in Education

This table displays the application of neutrosophic logic to evaluate cloud-based tools for improving stakeholder engagement and collaboration in education.

Table 13: Neutrosophic Evaluation of Cloud-Based Tools for Stakeholder Engagement and Collaboration in Education

Feature	Description	Truth (T)	Falsity (F)	Indeterminacy (I)
Communication Platform Functionality:	Features for real-time communication, information sharing, and collaboration (e.g., chats, forums, document sharing).	0.80 (Comprehensive communication suite with diverse functionalities)	0.10 (Limited options for personalized communication preferences)	0.10 (Uncertain impact of future communication trends on platform effectiveness)

User Interface (UI) and Accessibility:	Ease of use, intuitiveness of the platform for teachers, parents, and administrators with varying technical skills.	0.70 (User-friendly interface with accessibility features)	0.20 (Steeper learning curve for some stakeholders)	0.10 (Uncertain future needs for accessibility features based on evolving user demographics)
Content Management and Security:	Tools for managing access permissions, ensuring data privacy, and preventing misuse of communication channels.	0.75 (Robust security features and access controls)	0.15 (Potential for human error or accidental information leaks)	0.10 (Unforeseen future security threats or evolving privacy regulations)
Training and Support:	Availability and effectiveness of training resources and support mechanisms for stakeholders to learn and adopt the new	0.65 (Comprehensive training materials and dedicated support team)	0.20 (Limited availability of training in specific languages or formats)	0.15 (Uncertain impact of future changes in user needs on training and support effectiveness)

	communicati on platform.			

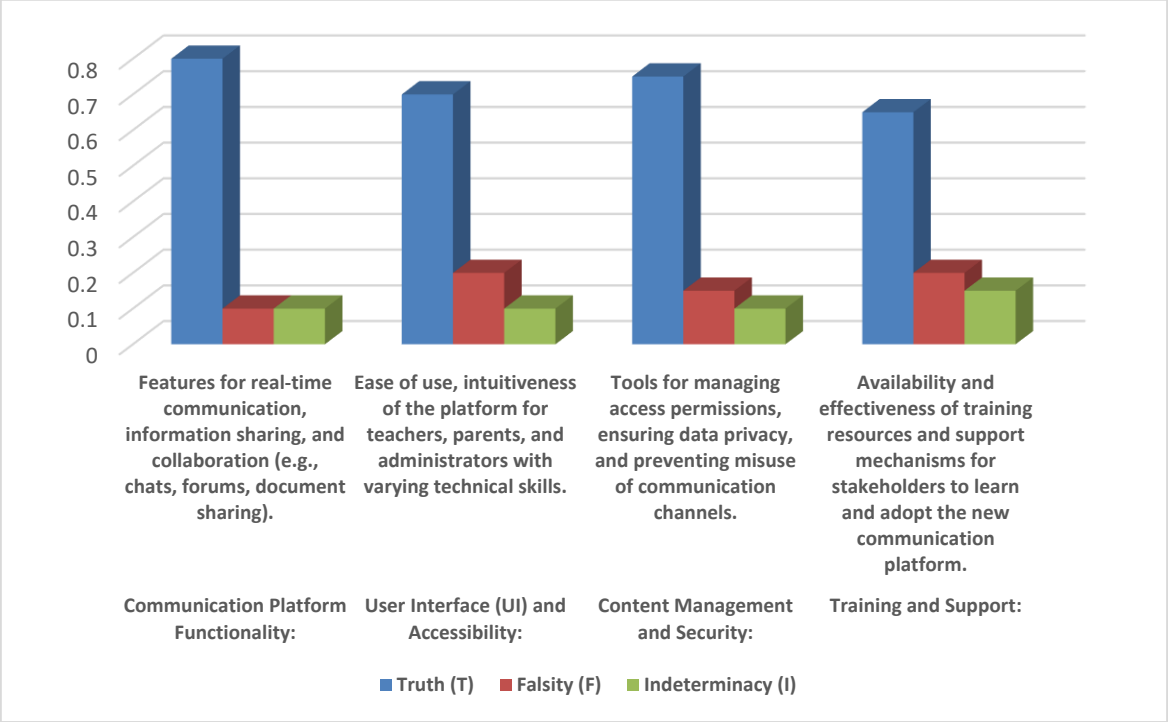
Explanation:

- Truth (T): High T values emphasize the potential of the platform to enhance communication (T) and facilitate collaboration (T).
- Falsity (F): Moderate F values acknowledge potential drawbacks like information overload (F), privacy concerns (F), and difficulties with user adoption (F).
- Indeterminacy (I): High I values highlight uncertainties related to future communication trends, evolving user needs, and unforeseen security threats that might require platform adaptability.

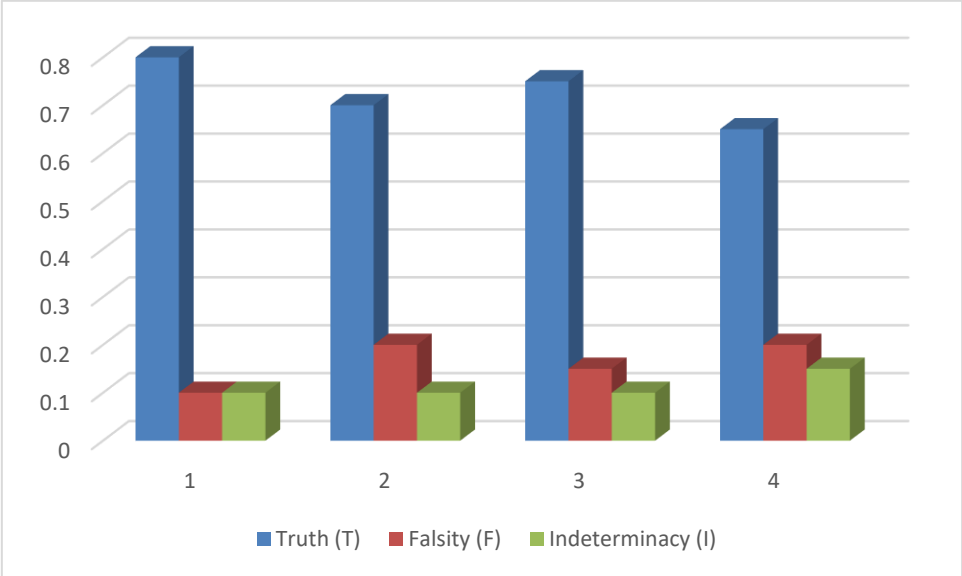
Benefits of Neutrosophic Approach:

- Holistic Evaluation: Goes beyond assessing the communication capabilities (T) to consider potential risks (F) and future challenges (I) associated with stakeholder adoption and ongoing use.
- Proactive Strategies: Encourages developing training plans, security protocols, and communication guidelines to mitigate potential drawbacks (F) and address uncertainties (I).
- Building Trust and Transparency: By acknowledging uncertainties, the framework helps institutions develop communication strategies that build trust and transparency with stakeholders.

By incorporating neutrosophic logic, educational institutions can make informed decisions when selecting and implementing cloud-based tools for stakeholder engagement. This fosters a collaborative environment, promotes open communication, and ultimately strengthens the educational community for the benefit of all stakeholders.


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Features for real-time communication, information sharing, and collaboration (e.g., chats, forums, document sharing).
Ease of use, intuitiveness of the platform for teachers, parents, and administrators with varying technical skills.
Tools for managing access permissions, ensuring data privacy, and preventing misuse of communication channels.
Availability and effectiveness of training resources and support mechanisms for stakeholders to learn and adopt the new communication platform.
Communication Platform Functionality:
User Interface (UI) and Accessibility:
Content Management and Security:
Training and Support:
Truth (T)
Falsity (F)
Indeterminacy (I)

Graph 10 Neutrosophic Evaluation of Cloud-Based Tools for Stakeholder Engagement (T, F, I)



5.5. Supporting Learning Personalization and Adaptability:

- Truth (T): Effective Personalization Features: Cloud-based learning platforms can offer personalized learning experiences by tailoring instruction and content to individual student needs and learning styles. This can lead to improved learning outcomes and increased student engagement.

- Falsity (F): Algorithmic Biases and Inequities: Algorithms used in personalization can introduce biases. The "falsity" value within the framework can help identify these biases and ensure fair and equitable learning experiences for all students.
- Indeterminacy (I): Evolving Learning Styles and Future Needs: Learning styles and educational approaches are constantly evolving. The framework acknowledges the "indeterminacy" associated with future unforeseen needs. Cloud-based solutions with flexible and adaptive features can better accommodate these changes.

Through this neutrosophic analysis, educational institutions can select cloud-based solutions that maximize the "truth" value (benefits) while minimizing the "falsity" value (risks) and considering the "indeterminacy" (future uncertainties) inherent in educational data and decision-making.

Cloud computing offers a powerful array of tools for educational administration. However, its effectiveness hinges on making informed choices while acknowledging the inherent uncertainties within the educational landscape. By employing a neutrosophic framework, educational institutions can harness the power of cloud computing to navigate the "fog" of uncertainties, make data-driven decisions, and ultimately create a more dynamic and effective learning environment for all stakeholders.

5.6. Cultivating Personalized Learning: A Neutrosophic Evaluation of Cloud-Based Learning Platforms

This table demonstrates how neutrosophic logic can be applied to evaluate cloud-based learning platforms for personalized learning.

Table 14: Neutrosophic Evaluation of Cloud-Based Learning Platforms for Personalized Learning Effectiveness

Feature	Description	Truth (T)	Falsity (F)	Indeterminacy (I)
Personalization Tools:	Capabilities for adaptive learning, personalized learning paths, and content recommendations based on student data	0.80 (Comprehensive suite of personalization tools)	0.10 (Limited customization options for educators)	0.10 (Uncertain impact of future learning science advancements on personalizati

	(e.g., performance, learning style).			on effectiveness)
Content and Resource Management :	Access to a diverse range of learning materials and the ability for educators to integrate their own resources for personalized learning experiences.	0.75 (Extensive library of learning resources with different formats)	0.15 (Limited accessibility features for students with diverse needs)	0.10 (Unforeseen future needs for new content types or delivery methods)
Assessment and Feedback:	Tools for ongoing assessment, personalized feedback, and progress tracking to adapt learning strategies for individual students.	0.70 (Regular formative assessments with detailed feedback reports)	0.20 (Potential for bias in assessment algorithms)	0.10 (Uncertain future needs for new assessment methods or student data privacy regulations)
Bias Detection and Mitigation:	Mechanisms to identify and mitigate potential biases within the platform's	0.65 (Regular audits for bias detection)	0.20 (Possibility of undetected biases requiring	0.15 (Unforeseen future advancements in bias

	algorithms and content recommendations.		human review)	detection technologies)

This table utilizes neutrosophic logic to assess the effectiveness of cloud-based learning platforms in supporting personalized learning. It analyzes four key features of these platforms and assigns three values to each: Truth (T), Falsity (F), and Indeterminacy (I).

Understanding the Values:

- Truth (T): This value represents the platform's strengths and capabilities for promoting personalized learning. A high T value indicates the platform offers a strong foundation for personalization.
- Falsity (F): This value reflects potential drawbacks and limitations associated with the platform. A high F value suggests areas where the platform might fall short in personalization.
- Indeterminacy (I): This value acknowledges uncertainties and future considerations related to the platform's effectiveness. A high I value highlights the need for ongoing evaluation and adaptation.

Breakdown of Features:

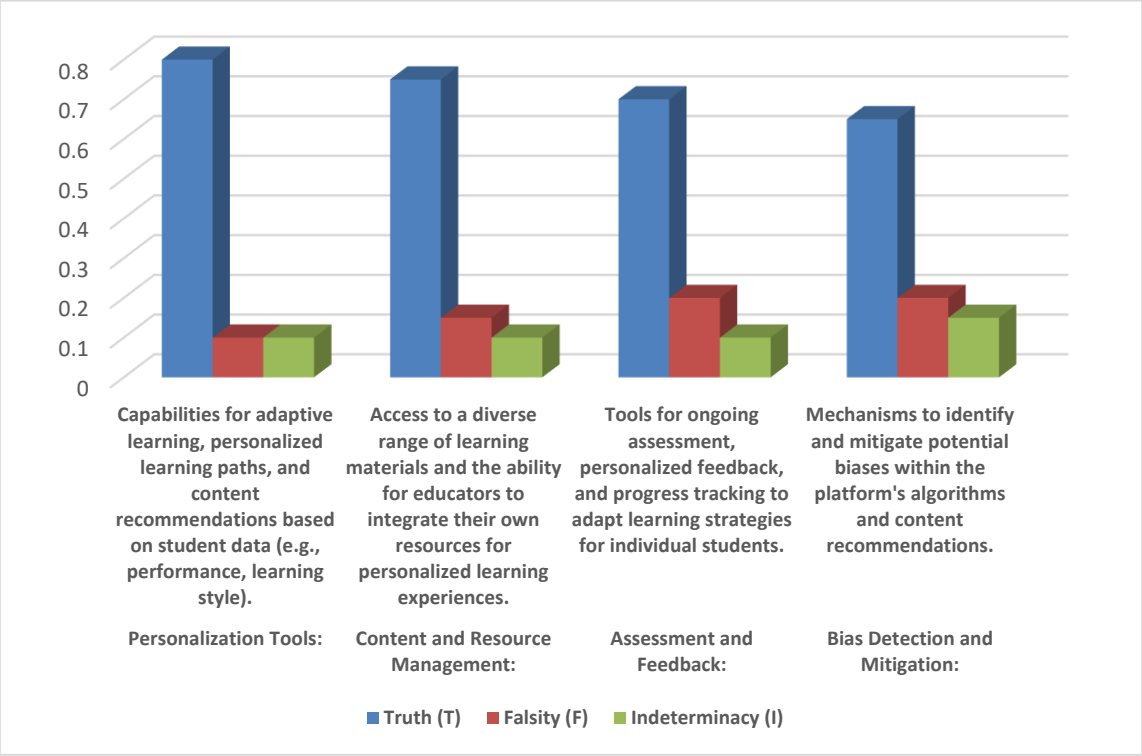
1. Personalization Tools (T: 0.80, F: 0.10, I: 0.10):
   - Strengths (T): The platform offers a comprehensive suite of tools for adaptive learning, personalized learning paths, and data-driven content recommendations.
   - Limitations (F): Educators might have limited options for customizing these tools to their specific teaching styles or student needs.
   - Uncertainties (I): The effectiveness of personalization might be impacted by future advancements in learning science and how the platform adapts to these changes.
2. Content and Resource Management (T: 0.75, F: 0.15, I: 0.10):
   - Strengths (T): The platform provides a rich library of learning resources in various formats, allowing educators to integrate their own materials for a more personalized learning experience.
   - Limitations (F): Accessibility features for students with diverse needs might be limited.
   - Uncertainties (I): There might be unforeseen future needs for new content types or delivery methods that the platform needs to accommodate.
3. Assessment and Feedback (T: 0.70, F: 0.20, I: 0.10):
   - Strengths (T): The platform facilitates personalized learning through regular formative assessments with detailed feedback reports that allow educators to adapt learning strategies for individual students.
   - Limitations (F): Assessment algorithms could potentially introduce biases.

- Uncertainties (I): Future needs for new assessment methods or evolving student data privacy regulations might require adjustments to this feature.

4. Bias Detection and Mitigation (T: 0.65, F: 0.20, I: 0.15):

- Strengths (T): The platform has mechanisms for identifying and mitigating potential biases within its algorithms and content recommendations.
- Limitations (F): There's a possibility that some biases might remain undetected and require human review.
- Uncertainties (I): Advancements in bias detection technologies might necessitate future adjustments to the platform's mitigation methods.

Overall, this neutrosophic evaluation provides a nuanced perspective on the effectiveness of cloud-based learning platforms for personalized learning. By considering not only the strengths but also the limitations and uncertainties, educational institutions can make more informed decisions when selecting and implementing these platforms.


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Capabilities for adaptive learning, personalized learning paths, and content recommendations based on student data (e.g., performance, learning style).
Access to a diverse range of learning materials and the ability for educators to integrate their own resources for personalized learning experiences.
Tools for ongoing assessment, personalized feedback, and progress tracking to adapt learning strategies for individual students.
Mechanisms to identify and mitigate potential biases within the platform's algorithms and content recommendations.
Personalization Tools:
Content and Resource Management:
Assessment and Feedback:
Bias Detection and Mitigation:
Truth (T)
Falsity (F)
Indeterminacy (I)

Graph 11: Neutrosophic Evaluation of Personalized Learning Platforms (T, F, I)

By incorporating neutrosophic logic, educational institutions can make informed decisions when selecting cloud-based learning platforms. This framework fosters the development of personalized learning experiences that are effective, equitable, and adaptable to the evolving needs of the educational landscape. Ultimately, it empowers institutions to leverage cloud computing to create a more dynamic and engaging learning environment for all students.

6. Case Studies and Analysis: Cloud Computing in Action

To illustrate the application of the neutrosophic framework, this section will analyze two real-world case studies involving cloud computing in educational administration.

Case Study 1: School District X Implements a Cloud-Based Learning Management System (LMS)

Scenario: A large school district (District X) struggles with managing student data and fostering collaboration between teachers. They implement a cloud-based LMS that offers features for data storage, analytics, communication tools, and differentiated learning resources.

Neutrosophic Analysis:

- Data Management and Analytics (T, F, I):
  - Truth (T): The LMS facilitates centralized data storage, allowing for comprehensive student performance data collection and analysis. This data can inform curriculum development and personalized learning approaches.
  - Falsity (F): Data security concerns and potential data quality issues need to be addressed (F value). The framework can guide the selection of an LMS with robust security measures and data validation procedures.
  - Indeterminacy (I): New data needs might emerge in the future (I value). The framework encourages District X to choose an LMS with scalable storage and flexible data analysis tools to accommodate future uncertainties.
- Stakeholder Engagement and Collaboration (T, F, I):
  - Truth (T): The LMS provides communication tools for teachers to share resources and collaborate on lesson plans. This can improve teacher effectiveness and consistency of instruction.
  - Falsity (F): The framework acknowledges the potential for information overload and misuse of communication tools (F value). District X can implement training programs to ensure responsible use of the platform.
  - Indeterminacy (I): Challenges with teacher adoption of the new technology need to be considered (I value). The framework can guide the implementation of user-friendly platforms and ongoing support mechanisms for teachers.

Numerical Example Dataset: Case Study 1 - School District X Implements a Cloud-based LMS

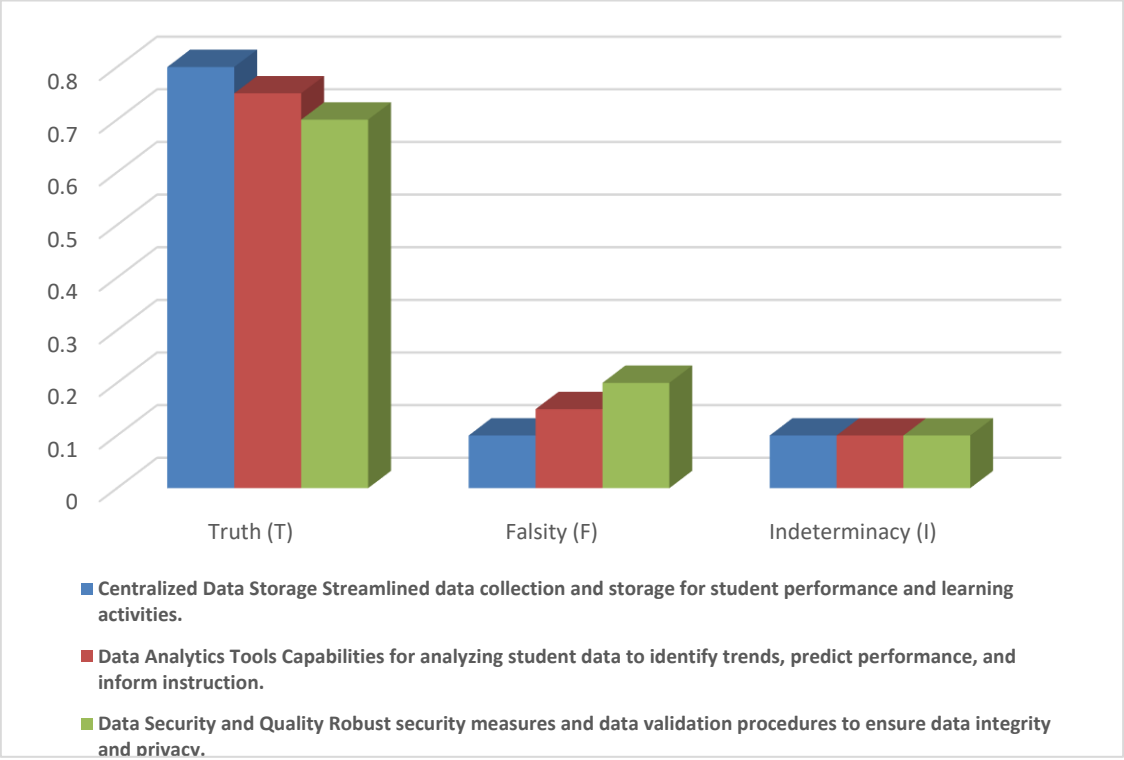
Scenario: A large school district (District X) struggles with managing student data and fostering collaboration between teachers. They implement a cloud-based Learning Management System (LMS) with features for data storage, analytics, communication tools, and differentiated learning resources.

Neutrosophic Analysis:

1. Data Management and Analytics (T, F, I):

Table 15: Neutrosophic Evaluation of Cloud-Based Learning Management Systems (Focus Data Management & Analytics)

Factor	Description	Truth (T)	Falsity (F)	Indeterminacy (I)
Centralized Data Storage	Streamlined data collection and storage for student performance and learning activities.	0.80 (Efficient data collection and organization)	0.10 (Risk of data breaches or technical issues)	0.10 (Unforeseen future data needs beyond current categories)
Data Analytics Tools	Capabilities for analyzing student data to identify trends, predict performance, and inform instruction.	0.75 (User-friendly tools with diverse data visualization options)	0.15 (Limited customization options for specific learning goals)	0.10 (Uncertain future advancements in data analytics techniques)
Data Security and Quality	Robust security measures and data validation procedures to ensure data integrity and privacy.	0.70 (Encryption protocols and access controls in place)	0.20 (Potential for human error during data entry or validation)	0.10 (Uncertain future data privacy regulations or security threats)


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Truth (T)
Falsity (F)
Indeterminacy (I)
Centralized Data Storage Streamlined data collection and storage for student performance and learning activities.
Data Analytics Tools Capabilities for analyzing student data to identify trends, predict performance, and inform instruction.
Data Security and Quality Robust security measures and data validation procedures to ensure data integrity and privacy.

Graph 12: Neutrosophic Evaluation of LMS Data Management & Analytics (Truth, Falsity, Indeterminacy)

2. Stakeholder Engagement and Collaboration (T, F, I):

Table 16: Neutrosophic Evaluation of Cloud-Based LMS for Stakeholder Engagement and Collaboration (T, F, I)

Factor	Description	Truth (T)	Falsity (F)	Indeterminacy (I)
Communication Features	Chat forums, document sharing, and real-time communication tools for teachers to collaborate.	0.85 (Comprehensive communication suite with various functionalities)	0.10 (Limited options for personalized communication preferences)	0.05 (Uncertain impact of future communication trends on platform effectiveness)

Content Sharing and Lesson Planning	Tools for sharing resources, collaborating on lesson plans, and developing curriculum materials.	0.75 (Secure platform for sharing resources and collaborative documents)	0.15 (Potential for information overload or misuse of shared materials)	0.10 (Unforeseen future needs for integrating new content types or learning resources)
User Adoption and Support	User-friendly interface with training programs and ongoing support for teachers to effectively utilize the LMS.	0.65 (Intuitive platform with readily available training materials)	0.20 (Potential for initial resistance or challenges with user adoption)	0.15 (Uncertain future needs for additional support based on evolving user demographics)

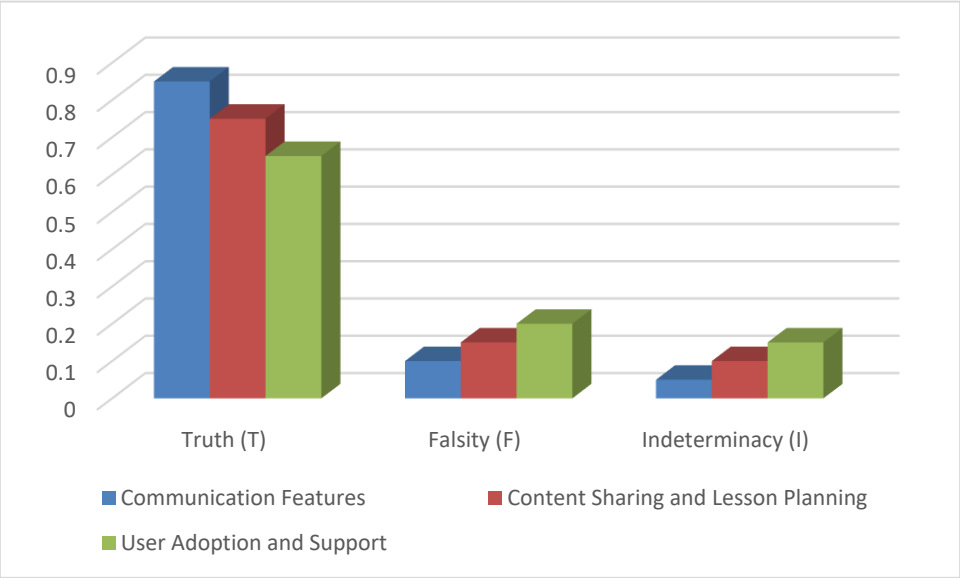
Explanation:

- Truth (T): High T values indicate strong capabilities in data storage/analysis (1), communication features (2), and content sharing/collaboration (2).
- Falsity (F): Moderate F values acknowledge potential security risks (1), information overload (2), and initial user adoption challenges (2).
- Indeterminacy (I): Low to moderate I values highlight uncertainties related to future data needs (1), evolving communication trends (2), and unforeseen content requirements (2). The focus here is on selecting an LMS with scalability and adaptability to address these uncertainties.

Benefits of Neutrosophic Analysis for District X:

- Data-Driven Decision Making: Provides numerical values to assess the LMS's capabilities and potential drawbacks in data management and stakeholder collaboration.
- Proactive Planning: Helps District X identify areas for improvement, such as implementing data validation procedures or user support programs.

- Future Proofing: Encourages the selection of an LMS with features that can adapt to evolving data needs, communication trends, and content requirements.



Graph 13: Neutrosophic Evaluation of LMS for Stakeholder Engagement (Truth, Falsity, Indeterminacy)

Case Study 2: Rural School Y Leverages Cloud-Based Educational Resources

Scenario: A small rural school (School Y) faces limited access to educational resources. They implement a cloud-based platform offering a vast library of online learning resources, curriculum materials, and collaborative tools.

Neutrosophic Analysis:

- Learning Personalization and Adaptability (T, F, I):
  - Truth (T): The cloud platform provides access to a diverse range of learning resources, allowing for personalized learning experiences tailored to individual student needs (T value).
  - Falsity (F): Algorithmic biases within the platform's learning recommendations could potentially disadvantage certain students (F value). The framework can guide School Y to choose a platform that allows for human oversight and customization of learning recommendations.
  - Indeterminacy (I): Evolving learning styles and future educational needs require adaptability (I value). The framework can encourage School Y to choose a platform with a constantly updated library of resources and features that adapt to new learning approaches.

Numerical Example Dataset: Case Study 2 - Rural School Y Leverages Cloud-Based Educational Resources

Scenario: A small rural school (School Y) faces limited access to educational resources. They implement a cloud-based platform offering a vast library of online learning resources, curriculum materials, and collaborative tools.

Neutrosophic Analysis:

Learning Personalization and Adaptability (T, F, I):

Table 17: Neutrosophic Evaluation of Learning Management Systems for Personalization and Adaptability (T, F, I)

Factor	Description	Truth (T)	Falsity (F)	Indeterminacy (I)
Resource Diversity	Extensive library of learning materials in various formats (text, video, simulations) catering to different learning styles.	0.80 (Rich collection of resources for differentiated instruction)	0.10 (Limited availability of resources in specific languages or cultural contexts)	0.10 (Unforeseen future needs for new learning resource formats or accessibility features)
Personalized Learning Tools	Features for teachers to customize learning paths, assign targeted resources, and track	0.70 (User-friendly tools for personalized learning path creation)	0.20 (Potential for algorithmic bias in learning recommendations)	0.10 (Uncertain future advancements in personalized learning technologies)

	student progress based on individual needs.			
Curriculum Alignment and Assessment	Resources that align with curriculum standards and provide formative assessments to monitor student progress and adjust learning strategies.	0.75 (Comprehensive resources aligned with curriculum goals)	0.15 (Limited options for creating custom assessments)	0.10 (Uncertain future changes in curriculum standards or assessment methods)

Explanation:

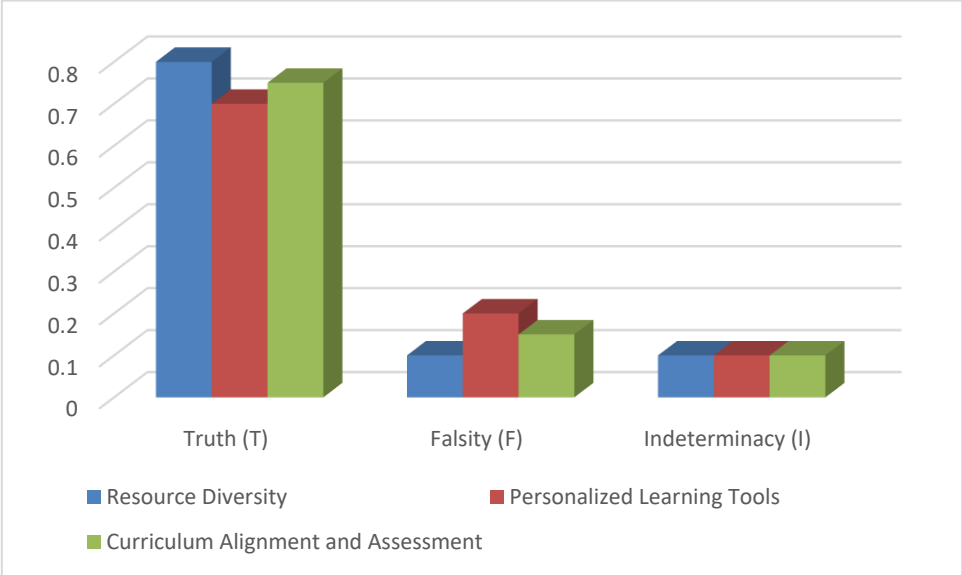
- Truth (T): High T values emphasize the potential of the platform to provide diverse resources (T) and personalized learning tools (T).
- Falsity (F): Moderate F values acknowledge the potential for algorithmic bias (F) and limitations in customizing assessments (F).
- Indeterminacy (I): Low to moderate I values highlight uncertainties related to future resource needs (I) and evolving curriculum standards/assessments (I). The focus here is on selecting a platform with a vast and adaptable library that can accommodate these uncertainties.

Benefits of Neutrosophic Analysis for School Y:

- Improved Resource Access: Provides a numerical assessment of how the platform expands resource availability and caters to diverse student needs.
- Mitigating Bias: Helps School Y identify potential biases and select a platform that allows for teacher oversight and personalization of learning recommendations.

- Adaptability and Future-Proofing: Encourages the selection of a platform with a continuously growing library and features that can adapt to evolving curriculum standards and assessment methods.

By implementing a neutrosophic framework, School Y can make informed decisions, ensure equitable access to learning resources, and leverage technology to create a more dynamic and adaptable learning environment for their students.



Graph 14: Neutrosophic Evaluation of LMS Personalization (Truth, Falsity, Indeterminacy)

Limitations of Cloud Computing:

- Internet Connectivity: Unreliable internet access can hinder the effectiveness of cloud-based solutions.
- Digital Divide: Unequal access to technology can exacerbate existing educational inequalities.
- Vendor Lock-In: Dependence on a single cloud vendor can limit flexibility and potentially increase costs.

These case studies demonstrate the potential of cloud computing to address uncertainties in educational administration. However, it's crucial to acknowledge the limitations and utilize a neutrosophic framework to maximize the "truth" value (benefits) while minimizing the "falsity" value (risks) and considering the "indeterminacy" (future uncertainties) inherent in the educational environment. By employing this framework, educational institutions can make informed decisions about cloud-based solutions and navigate the "fog" of uncertainties to create a more effective and equitable learning environment for all students.

7. Discussion & Conclusion: Embracing the Uncertain with Cloud Computing and Neutrosophic Logic

This paper explored the potential of cloud computing, with its data management, analytics, and collaboration tools, to address the inherent uncertainties in educational administration. We proposed a neutrosophic framework that utilizes concepts of "truth," "falsity," and "indeterminacy" to represent the complexities within educational data and decision-making processes.

Discussion:

The case studies demonstrate that cloud computing, when evaluated through the neutrosophic lens, offers significant advantages:

- Improved Data-Driven Decision-Making: Cloud-based data storage and analytics empower educators to move beyond intuition and make informed decisions based on a more comprehensive understanding of student performance, resource allocation, and stakeholder priorities.
- Enhanced Collaboration and Communication: Cloud platforms facilitate real-time communication and information sharing, fostering a more collaborative and unified approach to education among teachers, parents, and administrators.
- Greater Adaptability and Future-Proofing: The neutrosophic framework encourages the selection of cloud solutions with scalable features and flexible functionalities, allowing institutions to adapt to evolving data needs, learning styles, and unforeseen challenges in the educational landscape.

However, limitations exist:

- Complexity of Neutrosophic Logic: Neutrosophic logic is a relatively new field, and its implementation requires careful consideration to avoid introducing additional complexity into the decision-making process.
- Data Security and Privacy Concerns: Cloud computing raises concerns about data security and student privacy. Educational institutions must prioritize robust security measures and transparent data governance practices.
- Digital Divide and Equity Issues: Unequal access to technology can exacerbate existing educational inequalities. Efforts are needed to bridge the digital divide and ensure equitable access to cloud-based solutions for all students.

Discussion: Example Datasets and the Neutrosophic Framework in Educational Cloud Solutions

The presented numerical example datasets display the application of the neutrosophic framework in evaluating cloud-based solutions for educational institutions. These examples highlight the benefits of moving beyond traditional binary assessments (success/failure) by considering the complexities and uncertainties inherent in educational data and decision-making.

Here is a breakdown of key takeaways from each example dataset:

- Case Study 1: School District X Implements a Cloud-Based LMS

- The neutrosophic framework provides a structured approach for evaluating data management, analytics, and stakeholder collaboration features.
- Numerical values assigned to Truth (T), Falsity (F), and Indeterminacy (I) facilitate a data-driven comparison of different LMS options.
- The focus is on selecting an LMS that offers secure data storage, robust analytics tools, user-friendly communication features, and the potential to adapt to future data needs and communication trends.

- Case Study 2: Rural School Y Leverages Cloud-Based Educational Resources
- The framework helps School Y assess the platform's ability to provide diverse learning resources, support personalization, and adapt to evolving educational needs.
- Numerical values highlight the platform's strengths (T) in resource availability and personalized learning tools.
- The framework also acknowledges potential drawbacks (F) like algorithmic bias and encourages strategies to mitigate them.
- The emphasis is on selecting a platform with a vast and adaptable library of resources that can accommodate future changes in curriculum standards and assessment methods.

Overall Benefits of the Neutrosophic Framework:

- Comprehensive Evaluation: Goes beyond the "success" of a cloud solution (T) to consider potential drawbacks (F) and future uncertainties (I) associated with data management, stakeholder engagement, and learning personalization.
- Informed Decision-Making: Provides a numerical foundation for evaluating cloud solutions and choosing options that align best with the specific needs and priorities of an educational institution.
- Proactive Planning: Highlights the need for ongoing data security checks, training programs, and future-proofing strategies to address potential challenges and uncertainties (I) identified by the neutrosophic analysis.
- Focus on Equity and Adaptability: Encourages educational institutions to consider potential biases in algorithms and select solutions that promote equitable access to learning experiences while adapting to the evolving educational landscape.

Limitations and Considerations:

- Assigning precise numerical values (T, F, I) can be subjective and may require expert judgment or involve stakeholder input.
- The neutrosophic framework is a relatively new approach, and further research is needed to validate its effectiveness in educational technology evaluation.

Conclusion:

By incorporating the neutrosophic framework with numerical example datasets, educational institutions can make informed decisions when selecting and implementing cloud-based solutions. This framework fosters a data-driven culture, promotes proactive planning for uncertainties, and ultimately empowers institutions to leverage cloud computing to create a more dynamic and effective learning environment for all stakeholders.

Cloud computing offers a powerful array of tools to navigate the "fog" of uncertainties in educational administration. However, its effectiveness hinges on making informed choices while acknowledging the complexities inherent in educational data and decision-making. By embracing a neutrosophic perspective and acknowledging the limitations, educational institutions can harness the true potential of cloud computing to create a more dynamic, data-driven, and ultimately more effective learning environment for all stakeholders.

Future Research Directions:

Further research can explore the potential of neutrosophic logic in educational administration:

- Developing Standardized Evaluation Criteria: Establishing standardized criteria within the neutrosophic framework for specific educational contexts can enhance its practical application.
- Longitudinal Studies: Longitudinal studies can analyze the long-term impact of cloud computing and the neutrosophic framework on educational outcomes and address the "indeterminacy" associated with future trends.
- Integration with Artificial Intelligence: Research can explore how artificial intelligence can be integrated with cloud computing and neutrosophic logic to further enhance data analysis and personalize learning experiences.

Acknowledgement

This research is supported by the Neutrosophic Science International Association (NSIA) in both of its headquarters at New Mexico University and its Iraqi branch at Telafer University, for more details about (NSIA) see the URL http://neutrosophicassociation.org/.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Lin, S. W., & Lo, H. W. (2023). An FMEA model for risk assessment of university sustainability: using a combined ITARA with TOPSIS-AL approach based neutrosophic sets. Annals of Operations Research, 1-27.
2. Alzahrani, F. A., Ghorui, N., Gazi, K. H., Giri, B. C., Ghosh, A., & Mondal, S. P. (2023). Optimal site selection for women university using neutrosophic multi-criteria decision making approach. Buildings, 13(1), 152.

3. Nabeeh, N. A., Smarandache, F., Abdel-Basset, M., El-Ghareeb, H. A., & Aboelfetouh, A. (2019). An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. Ieee Access, 7, 29734-29744.
4. Embarak, O. H., Almesmari, M. J., & Aldarmaki, F. R. (2022). Apply Neutrosophic AHP Analysis of the Internet of Things (IoT) and the Internet of Behavior (IoB) in Smart Education. International Journal of Neutrosophic Science (IJNS), 19(1).
5. Embarak, O. H., Almesmari, M. J., & Aldarmaki, F. R. (2022). Apply Neutrosophic AHP Analysis of the Internet of Things (IoT) and the Internet of Behavior (IoB) in Smart Education. International Journal of Neutrosophic Science (IJNS), 19(1).
6. Tiwari, R. K., & Kumar, R. (2022). A framework for prioritizing cloud services in neutrosophic environment. Journal of king saud university-computer and information sciences, 34(6), 3151-3166.
7. Makwe, A., Kanungo, P., Kautish, S., Madhu, G., Almazyad, A. S., Xiong, G., & Mohamed, A. W. (2024). Cloud service prioritization using a Multi-Criteria Decision-Making technique in a cloud computing environment. Ain Shams Engineering Journal, 15(7), 102785.
8. Saha, A., Kolandasamy, R., Chatterjee, P., & Antucheviciene, J. (2024). A consensus-based single valued neutrosophic model for selection of educational vendors under metaverse with extended reality. Applied Soft Computing, 155, 111476.
9. Abdel-Baset, M., Hezam, I. M., & Smarandache, F. (2016). Neutrosophic goal programming. Neutrosophic Sets & Systems, 11(1), 1-9.
10. Alojaiman, B. (2023). A Multi-Criteria Decision-Making Process for the Selection of an Efficient and Reliable IoT Application. Processes, 11(5), 1313.
11. Abdel-aziem, A. H., Mohamed, H. K., & Abdelhafeez, A. (2023). Neutrosophic Decision Making Model for Investment Portfolios Selection and Optimizing based on Wide Variety of Investment Opportunities and Many Criteria in Market. Neutrosophic Systems with Applications, 6, 32-38.
12. Nafei, A., Huang, C. Y., Chen, S. C., Huo, K. Z., Lin, Y. C., & Nasseri, H. (2023). Neutrosophic autocratic multi-attribute decision-making strategies for building material supplier selection. Buildings, 13(6), 1373
13. Yasser, I., Twakol, A., Abd El-Khalek, A. A., Samrah, A., & Salama, A. A. (2020). COVID-X: novel health-fog framework based on neutrosophic classifier for confrontation covid-19. Neutrosophic Sets and Systems, 35, 1-21.
14. Amin, B., Salama, A. A., El-Henawy, I. M., Mahfouz, K., & Gafar, M. G. (2021). Intelligent neutrosophic diagnostic system for cardiotocography data. Computational Intelligence and Neuroscience, 2021, 1-12.

15. Alhasan, K. F., Salama, A. A., & Smarandache, F. (2021). Introduction to neutrosophic reliability theory. International Journal of Neutrosophic Science, 15(1), 52-61.
16. Yasser, I., Abd El-Khalek, A. A., Twakol, A., Abo-Elsoud, M. E., Salama, A. A., & Khalifa, F. (2022). A hybrid automated intelligent COVID-19 classification system based on neutrosophic logic and machine learning techniques using chest X-Ray images. Advances in Data Science and Intelligent Data Communication Technologies for COVID-19: Innovative Solutions Against COVID-19, 119-137.
17. Salama, A. A., Al-Din, A. S., Al-Qasim, I. A., Alhabib, R., & Badran, M. (2020). Introduction to Decision Making for Neutrosophic Environment "Study on the Suez Canal Port, Egypt". Infinite Study.
18. Alhabib, R., & Salama, A. A. (2020). The neutrosophic time series-study its models (linear-logarithmic) and test the coefficients significance of its linear model. Neutrosophic Sets and Systems, 33, 105-115.
19. Khalid, H. E., Sabbagh, R., Salma, A. A., Thanoon, T. Y., & Elagamy, H. A. Novel Neutrosophic Objects Within Neutrosophic Topology ($\boldsymbol{N}$ ($\boldsymbol{X}$), $\boldsymbol{\tau}$). Neutrosophic Sets and Systems, 260.
20. Farag, R. M., Shams, M. Y., Aldawody, D. A., Khalid, H. E., El-Bakry, H. M., & Salama, A. A. (2024). Integration between Bioinformatics Algorithms and Neutrosophic Theory. Neutrosophic Sets and Systems, 66, 34-54.
21. Salama, A. A., Shams, M. Y., Khalid, H. E., & Mousa, D. E. (2024). Enhancing Medical Image Quality using Neutrosophic FuzzyDomain and Multi-Level Enhancement Transforms: A Comparative Study for Leukemia Detection and Classification. Neutrosophic Sets and Systems, 65, 32-56.
22. Salama, A. A., Shams, M. Y., Elseuofi, S., & Khalid, H. E. (2024). Exploring Neutrosophic Numeral System Algorithms for Handling Uncertainty and Ambiguity in Numerical Data: An Overview and Future Directions. Neutrosophic Sets and Systems, 65, 253-295.
23. Alhabib, A., Alhabib, R., Khalid, H. E., & Salama, A. A. (2024). A Neutrosophic Study for the Transmission of Infection with Pathogenic Fungi from Males of Olive Fly Insects to Their Females. Neutrosophic Sets and Systems, 64, 38-45.

Received: Sep 2, 2024. Accepted: Dec 2, 2024

NSS

University of New Mexico

An Efficient Neutrosophic Weighted Sum Approach with Insights into Engineering Project Management Performance Assessment

Rong Yan[1], Yongguang Han[2*], Liqing Hao[3]

[1]Chongqing City Vocational College, Yongchuan, 402160, Chongqing, China (254530132@qq.com)

[2]Hubei Urban Construction Vocational and Technological College, Wuhan, 430205, Hubei, China

[3]Hebei Polytechnic Institute, Shijiazhuang, 050091, Hebei, China (5258-5258@163.com)

*Corresponding author, E-mail: hyg0537@126.com

Abstract: Project engineering management performance evaluation is a systematic assessment of various tasks in the project management process to measure performance in areas such as efficiency, quality, schedule, and cost control. Through evaluation, strengths and weaknesses in management can be identified, ensuring the achievement of project objectives and optimizing resource allocation. This evaluation not only helps improve the overall management level of engineering projects but also provides valuable experience and insights for future projects, thereby facilitating smooth implementation and successful delivery. The project engineering management performance evaluation is multiple-attribute decision-making (MADM). Recently, the weighted sum method (WSM) method has been established to cope with MADM issues. The triangular fuzzy neutrosophic sets (TFNSs) are established as a tool for characterizing uncertain data during the project engineering management performance evaluation. In this manuscript, the triangular fuzzy neutrosophic number WSM (TFNN-WSM) method is established to solve the MADM under TFNSs. In the end, a numerical case study for project engineering management performance evaluation is given to validate the proposed method.

Keywords: MADM; TFNSs; WSM approach; Project engineering management performance evaluation

1. Introduction

Project engineering management performance evaluation is a systematic process of analyzing and assessing the management activities and outcomes throughout the entire lifecycle of a project. The core objective is to comprehensively measure the efficiency and effectiveness of project management by considering key dimensions such as cost control, time management, quality assurance, safety management, resource utilization, and team performance[1, 2]. This evaluation provides a scientific basis for improving future project management practices. Specifically, project engineering management performance evaluation includes the following steps: First, establish the evaluation criteria system. Common criteria include time schedule, cost control, quality standards,

safety records, resource utilization efficiency, and team collaboration. Second, collect and organize relevant data, ensuring its accuracy and completeness. Next, choose appropriate evaluation methods and tools, such as fuzzy set theory and entropy weight method, to quantify each criterion[3, 4]. Then, integrate the evaluation results of each criterion to form an overall performance assessment report. Finally, based on the evaluation results, propose improvement suggestions and optimization measures to help project managers enhance their management practices in future projects. Through systematic performance evaluation, project managers can not only gain a comprehensive understanding of the actual operation of the project but also identify strengths and weaknesses in management. This enables targeted improvements and optimizations in future projects, thereby increasing the success rate of projects and enhancing the overall management capability and competitiveness of the organization. Therefore, project engineering management performance evaluation is a critical component of project management, with significant theoretical and practical implications.

In modern project management, performance evaluation often employs MADM methods such as TODIM [5]and VIKOR [6]. These methods can comprehensively consider multiple evaluation criteria and conduct multidimensional analysis, providing a thorough performance assessment [7-10]. Additionally, due to the inherent uncertainty and fuzziness in project management data, tools such as Fuzzy Sets and Neutrosophic Sets are widely used. These tools effectively handle uncertain and fuzzy data, enhancing the accuracy and reliability of the evaluation results[11-15]. The project engineering management performance evaluation is MADM. Recently, the Exponential TODIM method [16-20] and the VIKOR method [6, 21-23] have been developed to address multi-attribute decision-making (MADM) problems. To handle uncertain information in project engineering management performance evaluation, TFNSs [24] have been introduced as a tool for characterizing such uncertainties. In this paper, the TFNN-WSM method is proposed to solve MADM issues in the context of TFNSs. Finally, a numerical case study is presented to validate the effectiveness of the proposed approach in evaluating project engineering management performance.

2. Literature review

In 2019, Chen, Wang and Zhang [25] introduced a performance evaluation method for large-scale engineering project management based on the HSE (Health, Safety, and Environment) perspective and Choquet fuzzy integral. Their method integrated the independent HSE evaluation system into a comprehensive performance evaluation system, addressing the complexity of internal management and data processing challenges in large projects. Xu and Zhou [26] conducted a meta-analysis on the relationship between trust and engineering project performance. By synthesizing data from 42 empirical studies, they found a significant positive correlation between trust and project performance. They also explored the varying impacts of different dimensions of trust on project performance, concluding that the presence of mediating variables and

multi-dimensional trust measures positively influenced the correlation between trust and performance. This study provided evidence for the importance of trust in project management and offered theoretical support for trust-based management approaches. Moving to 2020, Ji [27] focused on the performance evaluation of water conservancy project management. He combined the BP neural network with fuzzy optimization theory to establish a comprehensive evaluation model for management performance. The model was applied to the GuanYinGe Reservoir water transfer project in Liaoning Province, demonstrating its feasibility and effectiveness in evaluating the performance of water conservancy projects. This study provided a novel method for assessing water project management and offered practical insights into improving performance. Also in 2020, Wang [28] explored performance evaluation in municipal road engineering management. He highlighted the limitations of traditional, rough management techniques and proposed a comprehensive evaluation system using the fuzzy analytic hierarchy process (AHP). This system combined quantitative and qualitative evaluation indicators to assess the performance of municipal road projects, addressing the urgent need for more refined management practices in this area. Yang [29] examined the management and performance evaluation of highway reconstruction projects. She analyzed the complex and multi-faceted nature of highway reconstruction and proposed strategies for optimizing performance management. Zhang [30] discussed performance evaluation in construction project management, with a focus on the petrochemical industry. He analyzed the industry-specific characteristics of petrochemical projects and offered suggestions for improving their management performance. His study underscored the importance of optimizing management and performance evaluation processes to enhance project outcomes, especially in the context of China's energy and resource development goals. Wang [31] proposed a graded management system for performance evaluation in a corporate engineering project. He classified projects based on their nature, type, scope, and difficulty, and developed a tiered performance evaluation mechanism. This approach optimized internal performance management, improving resource allocation and management efficiency within the organization. Wang [32] focused on the performance indicators for the management of hydroelectric resettlement projects. He designed key performance indicators (KPIs) using methods such as logic frameworks and hierarchical analysis. His study emphasized replacing singular financial metrics with comprehensive performance indicators that account for stakeholder satisfaction, suggesting this approach would drive more sustainable project management practices. Lu [33] developed a comprehensive management model for sponge city engineering based on the earned value method (EVM). He studied management issues in sponge city projects and introduced innovative construction techniques such as modular water storage, proposing solutions for cost and schedule deviations. His research contributed to improving the management quality of sponge city projects, which are crucial for urban ecological sustainability. Zhang [34] addressed the limitations of traditional engineering management methods under the EPC (Engineering, Procurement, and Construction) contracting model. He proposed a collaborative management platform based on BIM (Building

Information Modeling) technology, which significantly improved project coordination and performance in an EPC context, demonstrating high feasibility and effectiveness in real-world applications. Also in 2023, Shen et al. [35] explored the formation mechanisms of cooperation and competition networks in complex project groups. Using social network theory and social exchange theory, they identified three factors influencing the formation of such networks: endogenous structure, cross-network effects, and project attributes. Their study revealed that cooperation networks were denser than competition networks and that cooperation and competition mutually reinforced each other, offering insights into resource allocation strategies for improving project team collaboration. Min and Zhang [36] proposed a comprehensive performance evaluation method for project engineering management based on an improved entropy weight method. They addressed the limitations of cost-benefit analysis in evaluating project performance and developed a more accurate model by refining data standardization and weight calculation processes. Their findings showed that the proposed method outperformed traditional evaluation techniques in terms of accuracy, making it more suitable for comprehensive performance evaluations. In 2024, Guo [37] analyzed the impact of risk-sharing on engineering project management performance. He explored the intrinsic relationship between risk-sharing and performance, highlighting both the positive and negative effects of risk-sharing mechanisms. His study concluded that, while risk-sharing can reduce potential losses, it requires strict control during implementation to ensure the effectiveness of performance management and quality improvement. Finally, in 2024, Wu [38] studied the progress management of intelligent building projects. He applied critical chain technology to develop a dynamic construction schedule management process, ensuring that projects were completed efficiently and ahead of schedule. His findings demonstrated that the performance indices of various construction stages exceeded expectations, indicating that the proposed management techniques effectively improved project efficiency.

3. Concepts and Definitions

Biswas et al. [24] introduced the concept of Triangular Fuzzy Neutrosophic Sets (TFNSs) to enhance the representation and processing of uncertainty and imprecision in decision-making scenarios. Their work focused on the aggregation of TFNS information and its application to multi-attribute decision-making problems, providing a robust framework for handling complex problems and we will present some important definitions that will assist us in this paper, as follows:

Definition 1.

The TFNSs

$$FF = \left\{ \left(\theta, FA(\theta), FB(\theta), FC(\theta) \right) \middle| \theta \in \Theta \right\} \tag{1}$$

where $FA(\theta), FB(\theta), FC(\theta) \in [0,1]$ represent truth-membership, indeterminacy-membership and falsity-membership which is established through triangular fuzzy numbers (TFNs).

$$FA(\theta)=\left(FA^{L}(\theta),FA^{M}(\theta),FA^{U}(\theta)\right),0\le FA^{L}(\theta)\le FA^{M}(\theta)\le FA^{U}(\theta)\le 1 \quad (2)$$

$$FB(\theta)=\left(FB^{L}(\theta),FB^{M}(\theta),FB^{U}(\theta)\right),0\le FB^{L}(\theta)\le FB^{M}(\theta)\le FB^{U}(\theta)\le 1 \quad (3)$$

$$FC(\theta)=\left(FC^{L}(\theta),FC^{M}(\theta),FC^{U}(\theta)\right),0\le FC^{L}(\theta)\le FC^{M}(\theta)\le FC^{U}(\theta)\le 1 \quad (4)$$

We let $FF=\left\{\left(FA^{L},FA^{M},FA^{U}\right),\left(FB^{L},FB^{M},FB^{U}\right),\left(FC^{L},FC^{M},FC^{U}\right)\right\}$ be TFNN, $0\le FA^{U}+FB^{U}+FC^{U}\le 3$.

Definition 2.

Let TFNNs $FF_1=\left\{\left(FA_1^{L},FA_1^{M},FA_1^{U}\right),\left(FB_1^{L},FB_1^{M},FB_1^{U}\right),\left(FC_1^{L},FC_1^{M},FC_1^{U}\right)\right\}$, $FF_2=\left\{\left(FA_2^{L},FA_2^{M},FA_2^{U}\right),\left(FB_2^{L},FB_2^{M},FB_2^{U}\right),\left(FC_2^{L},FC_2^{M},FC_2^{U}\right)\right\}$ and $FF=\left\{\left(FA^{L},FA^{M},FA^{U}\right),\left(FB^{L},FB^{M},FB^{U}\right),\left(FC^{L},FC^{M},FC^{U}\right)\right\}$ be TFNNs, and the operation laws are :

$$(1)\ FF_1\oplus FF_2=\begin{Bmatrix}\left(FA_1^{L}+FA_2^{L}\text{-}FA_1^{L}FA_2^{L},FA_1^{M}+FA_2^{M}\text{-}FA_1^{M}FA_2^{M},FA_1^{U}+FA_2^{U}\text{-}FA_1^{U}FA_2^{U}\right),\\ \left(FB_1^{L}FB_2^{L},FB_1^{M}FB_2^{M},FB_1^{U}FB_2^{U}\right),\left(FC_1^{L}FC_2^{L},FC_1^{M}FC_2^{M},FC_1^{U}FC_2^{U}\right)\end{Bmatrix};$$

$$(2)\ FF_1\otimes FF_2=\begin{Bmatrix}\left(FA_1^{L}FA_2^{L},FA_1^{M}FA_2^{M},FA_1^{U}FA_2^{U}\right),\\ \left(FB_1^{L}+FB_2^{L}\text{-}FB_1^{L}FB_2^{L},FB_1^{M}+FB_2^{M}\text{-}FB_1^{M}FB_2^{M},FB_1^{U}+FB_2^{U}\text{-}FB_1^{U}FB_2^{U}\right),\\ \left(FC_1^{L}+FC_2^{L}\text{-}FC_1^{L}FC_2^{L},FC_1^{M}+FC_2^{M}\text{-}FC_1^{M}FC_2^{M},FC_1^{U}+FC_2^{U}\text{-}FC_1^{U}FC_2^{U}\right)\end{Bmatrix};$$

$$(3)\ \lambda FF=\begin{Bmatrix}\left(1-\left(1-FA^{L}\right)^{\lambda},1-\left(1-F^{M}\right)^{\lambda},1-\left(1-FA^{U}\right)^{\lambda}\right),\\ \left(\left(FB^{L}\right)^{\lambda},\left(FB^{M}\right)^{\lambda},\left(FB^{U}\right)^{\lambda}\right),\left(\left(FC^{L}\right)^{\lambda},\left(FC^{M}\right)^{\lambda},\left(FC^{U}\right)^{\lambda}\right)\end{Bmatrix},\lambda>0;$$

$$(4)\ FF^{\lambda}=\begin{Bmatrix}\left(\left(FA^{L}\right)^{\lambda},\left(FA^{M}\right)^{\lambda},\left(FA^{U}\right)^{\lambda}\right),\left(1-\left(1-FB^{L}\right)^{\lambda},1-\left(1-FB^{M}\right)^{\lambda},1-\left(1-FB^{U}\right)^{\lambda}\right),\\ \left(1-\left(1-FC^{L}\right)^{\lambda},1-\left(1-FC^{M}\right)^{\lambda},1-\left(1-FC^{U}\right)^{\lambda}\right)\end{Bmatrix},\lambda>0.$$

The operational laws have serval properties.

$$(1)\ FF_1\oplus FF_2=FF_2\oplus FF_1,FF_1\otimes FF_2=FF_2\otimes FF_1,\left(\left(FF_1\right)^{\lambda_1}\right)^{\lambda_2}=\left(FF_1\right)^{\lambda_1\lambda_2}; \quad (5)$$

$$(2)\ \lambda\left(FF_1\oplus FF_2\right)=\lambda FF_1\oplus\lambda FF_2,\left(FF_1\otimes FF_2\right)^{\lambda}=\left(FF_1\right)^{\lambda}\otimes\left(FF_2\right)^{\lambda}; \quad (6)$$

$$(3)\ \lambda_1FF_1\oplus\lambda_2FF_1=\left(\lambda_1+\lambda_2\right)FF_1,\left(FF_1\right)^{\lambda_1}\otimes\left(FF_1\right)^{\lambda_2}=\left(FF_1\right)^{\left(\lambda_1+\lambda_2\right)}. \quad (7)$$

Definition 3.

Let $FF=\left\{\left(FA^{L},FA^{M},FA^{U}\right),\left(FB^{L},FB^{M},FB^{U}\right),\left(FC^{L},FC^{M},FC^{U}\right)\right\}$ be TFNN, the score and accuracy functions are:

$$SF\left(FF\right)=\frac{1}{12}\begin{bmatrix}8+\left(FA^{L}+2FA^{M}+FA^{U}\right)\\-\left(FB^{L}+2FB^{M}+FB^{U}\right)\\-\left(FC^{L}+2FC^{M}+FC^{U}\right)\end{bmatrix},SF\left(FF\right)\in\left[0,1\right] \tag{8}$$

$$AF\left(FF\right)=\frac{1}{4}\begin{bmatrix}\left(FA^{L}+2FA^{M}+FA^{U}\right)\\-\left(FB^{L}+2FB^{M}+FB^{U}\right)\end{bmatrix},AF\left(FF\right)\in\left[-1,1\right] \tag{9}$$

For the TFNNs FF_1 and FF_2, from Definition 3, we have:

(1) $if\ SF\left(FF_1\right)\prec SF\left(FF_2\right),then\ FF_1\prec FF_2;$

(2) $if\ SF\left(FF_1\right)=SF\left(FF_2\right),AF\left(FF_1\right)\prec AF\left(FF_2\right),then\ FF_1\prec FF_2;$

(3) $if\ SF\left(FF_1\right)=SF\left(FF\right),AF\left(FF_1\right)=AF\left(FF_2\right),then\ FF_1=FF_2.$

Definition 4

Let $FF_1=\left\{\left(FA_1^{L},FA_1^{M},FA_1^{U}\right),\left(FB_1^{L},FB_1^{M},FB_1^{U}\right),\left(FC_1^{L},FC_1^{M},FC_1^{U}\right)\right\}$, $FF_2=\left\{\left(FA_2^{L},FA_2^{M},FA_2^{U}\right),\left(FB_2^{L},FB_2^{M},FB_2^{U}\right),\left(FC_2^{L},FC_2^{M},FC_2^{U}\right)\right\}$ be TFNNs, the TFNN Hamming distance (TFNNHD) is:

$$TFNNHD\left(FF_1,FF_2\right)=\frac{1}{9}\begin{pmatrix}\left|FA_1^{L}-FA_2^{L}\right|+\left|FA_1^{M}-FA_2^{M}\right|+\left|FA_1^{U}-FA_2^{U}\right|\\+\left|FB_1^{L}-FB_2^{L}\right|+\left|FB_1^{M}-FB_2^{M}\right|+\left|FB_1^{U}-FB_2^{U}\right|\\+\left|FC_1^{L}-FC_2^{L}\right|+\left|FC_1^{M}-FC_2^{M}\right|+\left|FC_1^{U}-FC_2^{U}\right|\end{pmatrix} \tag{10}$$

4. TFNN-WSM method

Weighted sum model (WSM) is a MCDM method used for evaluating a number of alternatives in the decision making. The following steps of the WSM method under the TFNN:

1. Develop a decision and weight matrix.

We invited experts to develop a decision and weight matrix between criteria and alternatives such as:

$$Y=\begin{bmatrix}y_{11}&\cdots&y_{1n}\\\vdots&\ddots&\vdots\\y_{m1}&\cdots&y_{mn}\end{bmatrix} \tag{11}$$

The following present the weight vector

$$w=[w_1,\ldots,w_n],where\ \textstyle\sum_{j=1}^{n}w=1 \tag{12}$$

The criteria weights are computed using the average method.

2. Apply the score function.

The score function is applied to obtain crisp values.

3. Combined the decision matrix.

The decision matrix is combined into a single matrix.

4. Make the choice matrix normal

$$u_{ij} = \begin{cases} \frac{y_{ij}}{\max y_{ij}} for\ positive\ criteria \\ \frac{\min y_{ij}}{y_{ij}} for\ negative\ criteria \end{cases} \quad (13)$$

5. Normalized weighted decision matrix.

$$T_{ij} = w_j u_{ij} \quad (14)$$

6. Rank of alternatives.

$$L_i = \sum_{j=1}^{n} T_{ij} \quad (15)$$

The alternatives are ranked from the highest to lowest of L.

Table 1. Importance Weights of Criteria for Decision Making

Criteria	**Weights**
C_1: Quality Assurance and Control	0.052549
C_2: Training and Development of Team Members	0.048093
C_3: Resource Allocation Efficiency	0.052589
C_4: Documentation and Reporting Quality	0.048575
C_5: Stakeholder Communication and Collaboration	0.051224
C_6: Adaptability to Changes and Challenges	0.048936
C_7: Compliance with Safety Standards	0.050703
C_8: Technology Integration and Innovation	0.049739
C_9: Post-Project Evaluation and Feedback Implementation	0.048214
C_{10}: Client Satisfaction with Deliverables	0.048334
C_{11}: Team Leadership and Coordination	0.051224
C_{12}: Budget Management and Cost Control	0.052589
C_{13}: Project Planning and Scheduling	0.05283
C_{14}: Effective Conflict Resolution	0.04729
C_{15}: Use of Engineering Standards and Best Practices	0.050703
C_{16}: Environmental Sustainability Practices	0.048695
C_{17}: Overall Project Outcomes and Impact	0.045484
C_{18}: Adherence to Project Timelines	0.051666
C_{19}: Risk Identification and Mitigation	0.051345
C_{20}: Supply Chain and Procurement Management	0.049217

5. A Case Study with Comparative Evaluation

Project engineering management performance evaluation is a critical step in systematically and

scientifically assessing both the project management process and its outcomes. Its aim is to analyze the execution of the project, identify successful experiences and potential issues in management, and provide a basis for improving future projects. Performance evaluation not only focuses on the final results of the project but also examines various stages throughout the entire project management process. Through this evaluation, the project management team can gain a comprehensive understanding of the overall operation, including the effectiveness of each project phase, the validity of management decisions, and the efficiency of resource allocation and utilization. The performance evaluation process typically involves multiple stakeholders, such as the project manager, team members, suppliers, and clients, whose feedback and opinions play a crucial role in improving project management. By collecting and analyzing this information, managers can identify which management strategies were effective and which areas require improvement. This not only helps optimize the current project but also provides valuable insights and lessons for the management of future projects. In modern project management, performance evaluation is seen as a continuous improvement process. After project completion, through systematic review and analysis, the project team can summarize lessons learned and develop more reasonable and effective plans for future project management. Additionally, performance evaluation provides foundational data to help organizations achieve greater efficiency and better resource utilization throughout the project lifecycle. In summary, project engineering management performance evaluation is not only a key step in ensuring project success but also an essential tool for continuously enhancing an organization's project management capabilities. The evaluation of project engineering management performance is a MADM problem. In this study, six potential engineering projects are assessed based on 20 distinct attributes, as outlined in Table 1. The evaluation process follows these systematic steps:

- Developing the Decision Matrix: Using Eq. (11), the decision matrix was constructed by applying the Triangular Fuzzy Neutrosophic Numbers, with the resulting matrix detailed in Table A1.
- Converting to Crisp Values: A score function was applied to the TFNN elements to obtain their corresponding crisp values, simplifying further analysis.
- Aggregating Criteria Weights: The decision matrix was consolidated into a single unified matrix. Subsequently, the criteria's weights were computed, as presented in Table 1.
- Normalization: Eq. (13) was employed to normalize the decision matrix, ensuring all attribute values were scaled appropriately for comparison.
- Weighted Normalized Matrix: Using Eq. (14), a weighted normalized decision matrix was derived by incorporating the computed criteria weights.
- Ranking Alternatives: Finally, Eq. (15) was applied to rank the six engineering project alternatives based on their performance, with the rankings visually represented in Figure 1.

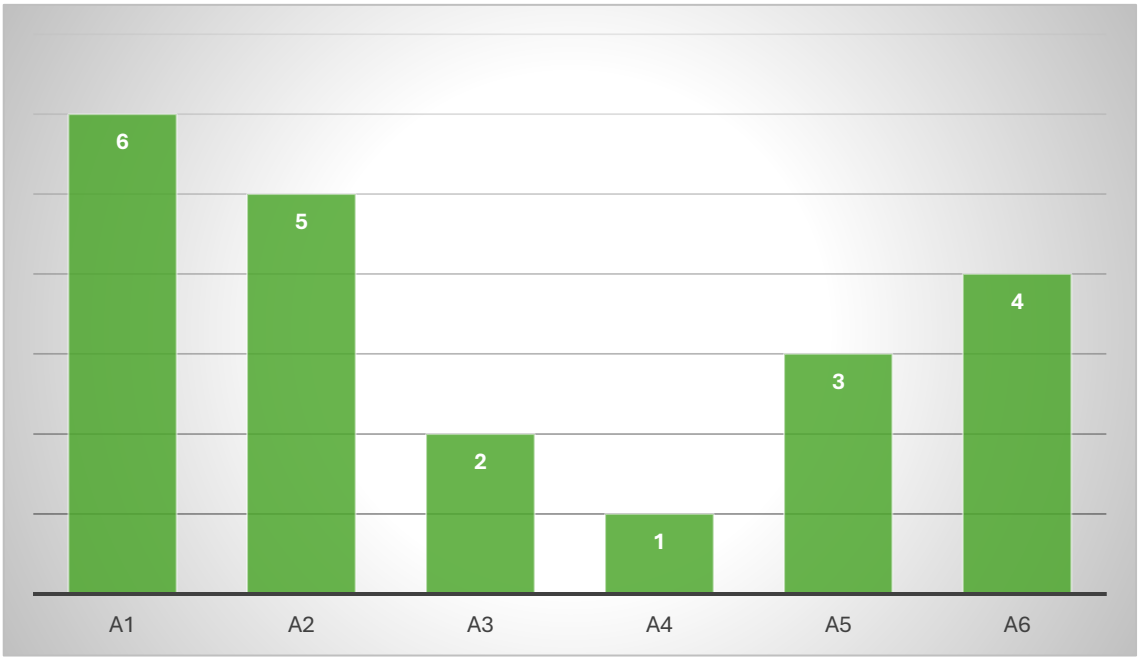


Figure 1. The rank of the alternatives.

5.1. Comparative Analysis of Alternatives and Methods

The comparative analysis of the proposed method (TFNN-WSM) with TFNN-VIKOR [40], TFNN-MABAC [39], and TFNN-EDAS [41] reveal critical insights into their performance in multi-attribute decision-making problems, specifically in project engineering management performance evaluation. The results, depicted in Figure 2, demonstrate that all models consistently identify the same optimal engineering project and rank the six projects similarly. This consistency confirms the reliability and effectiveness of the TFNN-WSM approach.

Limitations of Existing Methods:

TFNN-VIKOR and TFNN-MABAC Approaches:

These methods are based on traditional triangular fuzzy number models that address uncertainty only partially.

They fail to fully account for the three essential dimensions of fuzzy logic: truth-membership, indeterminacy-membership, and falsity-membership.

Consequently, they struggle with decision problems involving conflicting preferences and high levels of uncertainty.

These approaches inadequately capture the complexities of fuzzy logic, limiting their robustness when handling ambiguous or uncertain information.

TFNN-EDAS Approach:

The TFNN-EDAS method relies on evaluating the distance from an average solution, offering a computationally straightforward process.

However, it is primarily suited to linear decision problems and performs poorly with complex, non-linear preferences.

This method lacks flexibility in addressing uncertainty and risk, reducing its ability to manage the variable and multifaceted risks encountered in engineering management scenarios.

Advantages of the Proposed TFNN-WSM Method:

Enhanced Capability to Handle Uncertainty:

By incorporating the complete spectrum of fuzzy dimensions (truth, indeterminacy, and falsity), the TFNN-WSM method effectively manages uncertainty and ambiguity.

Ability to Address Non-Linear Preferences and Risk Attitudes:

The method supports non-linear decision-making, making it suitable for complex engineering scenarios with diverse risk profiles and stakeholder preferences.

Comprehensive Consideration of Objectives and Trade-Offs:

The TFNN-WSM approach ensures a holistic evaluation by balancing multiple objectives and effectively managing trade-offs among competing criteria.

This analysis highlights the superior performance of the TFNN-WSM method in addressing complex and uncertain engineering management performance evaluations, offering a robust alternative to traditional MADM approaches.

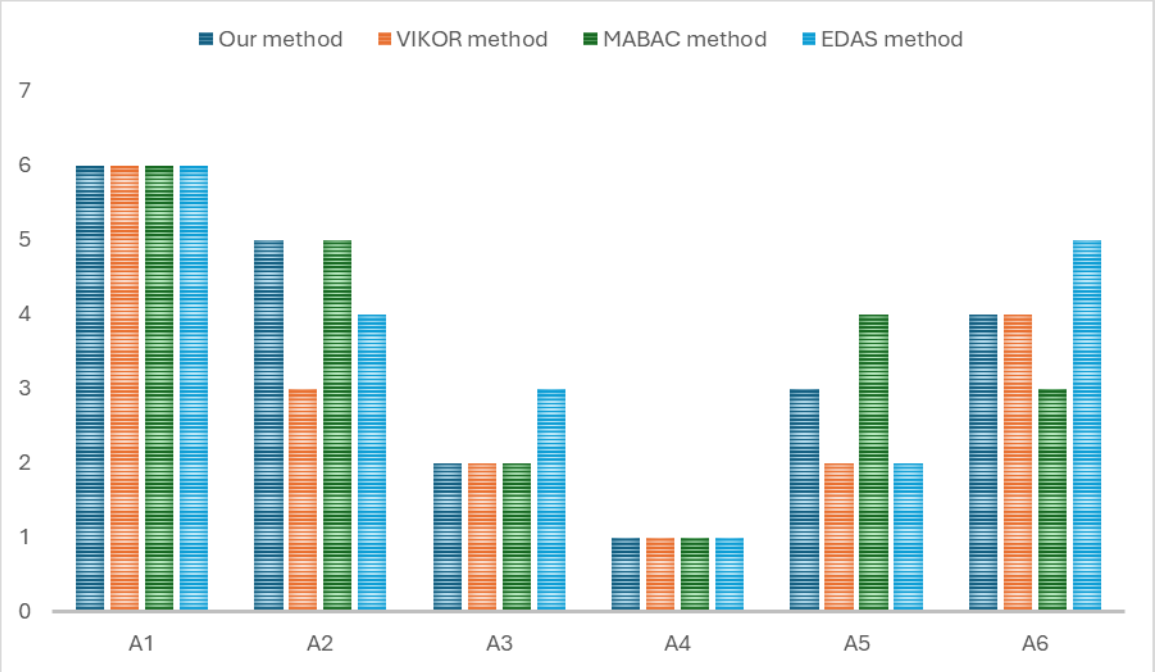


Figure 2. Alternative Ranks in Comparative Perspective

5.2. Sensitivity Analysis

Sensitivity analysis was conducted by modifying the criteria weights across ten different cases, as illustrated in Figure 3. The primary objective of this analysis was to validate the stability of the ranking results under varying weight conditions.

The proposed model was applied to each case with the adjusted criteria weights. The results, presented in Figure 4, indicate that Alternative 1 consistently ranks as the best option, while Alternative 4 consistently ranks as the worst, regardless of the weight changes.

These findings confirm that the ranking of alternatives remains stable, demonstrating the robustness and reliability of the proposed model in addressing changes in criteria weights. This stability is a critical feature for decision-makers, ensuring confidence in the model's applicability to real-world scenarios.

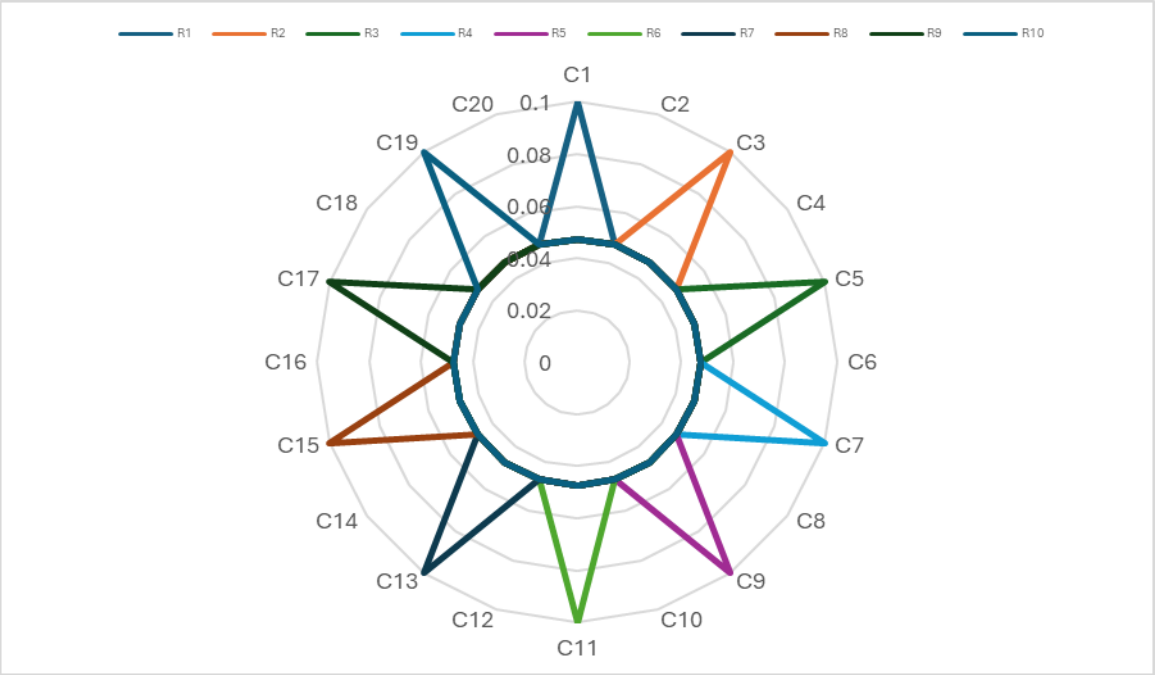


Figure 3. The criteria weights under different cases.

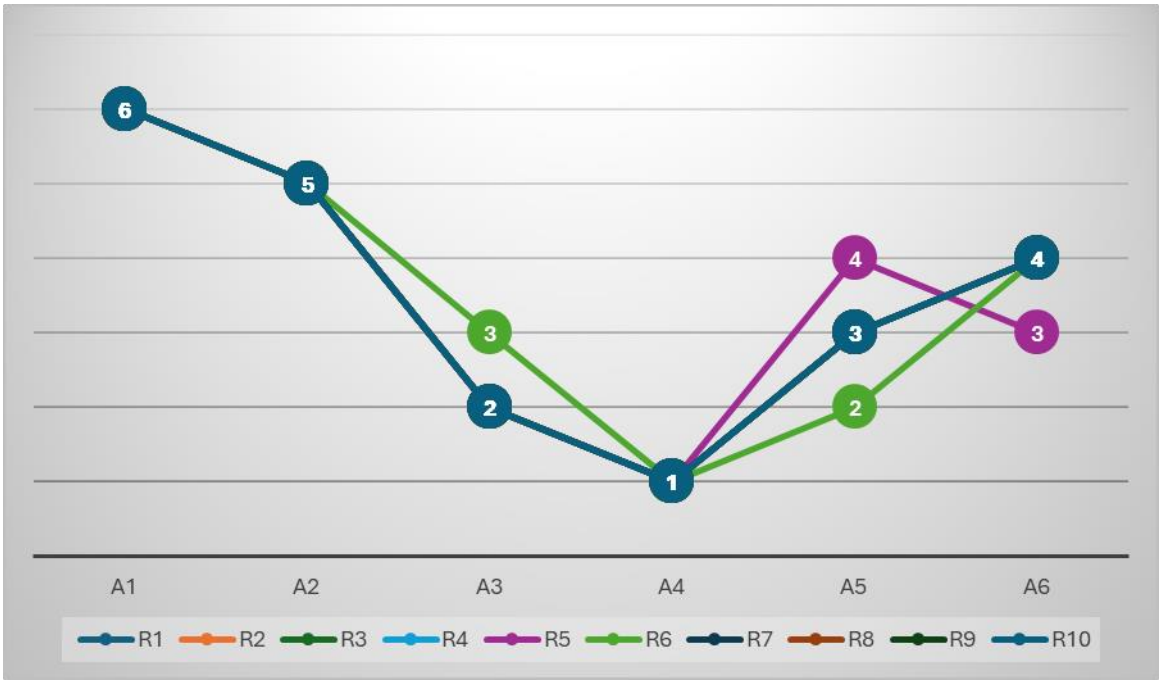


Figure 4. Variations in Alternative Rankings

6. Conclusions

Project engineering management performance evaluation is a crucial process for

systematically assessing both the project management process and its outcomes, with the goal of ensuring the achievement of project objectives and improving management efficiency. By analyzing the execution of the project, performance evaluation helps identify successful practices as well as potential issues, thus providing a basis for improvements in future projects. This evaluation focuses not only on the final outcome but also on various stages throughout the project lifecycle. Key factors such as cost, schedule, quality, and risk are typically involved, helping the project team optimize resource utilization and management decisions. Through continuous review and feedback, performance evaluation enhances project execution efficiency and provides valuable insights for the long-term development of the organization, ensuring the smooth implementation and successful delivery of future projects. The project engineering management performance evaluation is an MADM problem. TFNSs are used to characterize uncertain data during the evaluation process. In this study, a TFNN-WSM model is proposed to solve MADM under TFNSs. Finally, a numerical case study is provided to validate the effectiveness of the proposed method.

7. Study Limitations and Suggestions for Future Work

Despite the strengths of the TFNN-WSM method in addressing multi-attribute decision-making problems under uncertainty, there are still several limitations that need to be acknowledged: (1) High Computational Complexity (2) Sensitivity to Parameter Settings (3) Limited Empirical Validation

To address the limitations mentioned above, there are several promising avenues for future research that could enhance the TFNN- WSM method and broaden its applicability: (1) Improving Computational Efficiency (2) Reducing Sensitivity to Parameters (3) Extensive Empirical Validation Across Multiple Domains: Lastly, future research should aim to conduct broader empirical validation of the method across multiple domains. While the current case study demonstrates the potential of the TFNN-WSM approach, testing the method on a wider range of projects in various industries (e.g., construction, transportation, energy) would provide more comprehensive insights into its effectiveness and adaptability. Additionally, comparative studies with other advanced multi-attribute decision-making techniques could help identify areas for improvement, enhancing the universality and applicability of the method. Such validation would strengthen the credibility of the approach and demonstrate its value in different real-world scenarios.

Acknowledgment

The work was supported by the Research on the reconstruction of the ecosystem of high-level professional groups in vocational colleges under the three-integration framework (No. EJA230465).

References

[1] Q.E. Wang, M.M. Ding, A Study on the Impact of Digital Management on Willingness to Transfer Knowledge in Whole-Process Engineering Consulting Projects, Buildings-Basel, 13 (2023) 19.

[2] Z.Z. Shen, J.Y. Zhao, M. Guo, Evaluating the Engineering-Procurement-Construction Approach and Whole Process Engineering Consulting Mode in Construction Projects, Iran. J. Sci. Technol.-Trans. Civ. Eng., 47 (2023) 2533-2547.

[3] E. Vigoda-Gadot, N. Cohen, S. Mizrahi, Battling COVID-19: Public Personnel Management, Trust, and Social Resilience During a Global Crisis, Rev. Public Pers. Adm., 43 (2023) 583-613.

[4] S.H. Wang, Engineering Consulting Project Budget Allocation and Cost Control Model Based on Baseline Productivity, KSCE J. Civ. Eng., 27 (2023) 2339-2355.

[5] L. Gomes, L.A.D. Rangel, An application of the TODIM method to the multicriteria rental evaluation of residential properties, European Journal of Operational Research, 193 (2009) 204-211.

[6] S. Opricovic, G.H. Tzeng, Compromise solution by MCDM methods: A comparative analysis of VIKOR and TOPSIS, European Journal of Operational Research, 156 (2004) 445-455.

[7] M. Arshad, M. Saeed, A. Ur Rahman, Interval complex single-valued neutrosophic hypersoft set with Application in Decision Making, Neutrosophic Sets and Systems, 60 (2023) 396-419.

[8] M. Jdid, F. Smarandache, The Graphical Method for Finding the Optimal Solution for Neutrosophic linear Models and Taking Advantage of Non-Negativity Constraints to Find the Optimal Solution for Some Neutrosophic linear Models in Which the Number of Unknowns is More than Three, Neutrosophic Sets and Systems, 58 (2023) 68-80.

[9] A. Kanchana, D. Nagarajan, S. Broumi, Multi-attribute group decision-making based on the Neutrosophic Bonferroni mean operator, Neutrosophic Sets and Systems, 57 (2023) 139-164.

[10] M. Saqlain, P. Kumam, W. Kumam, Multi-Polar Interval-Valued Neutrosophic Hypersoft Set with Multi-criteria decision making of Cost-Effective Hydrogen Generation Technology Evaluation, Neutrosophic Sets and Systems, 61 (2023) 100-117.

[11] M. Touqeer, R. Umer, M.N. Ahmad, M. Salimi, A. Ahmadian, Signed distance-based approach for multiple criteria group decision-making with incomplete information using interval type-2 neutrosophic numbers (Jun, 10.1007/s11042-023-15619-z, 2023), Multimedia Tools and

Applications, 83 (2024) 8467-8467.

[12] T. Senapati, A. Sarkar, G.Y. Chen, Enhancing healthcare supply chain management through artificial intelligence-driven group decision-making with Sugeno-Weber triangular norms in a dual hesitant q-rung orthopair fuzzy context, Engineering Applications of Artificial Intelligence, 135 (2024) 22.

[13] M. Sarwar, M. Akram, W. Gulzar, M. Deveci, Group decision making method for third-party logistics management: An interval rough cloud optimization model, J. Ind. Inf. Integr., 41 (2024) 27.

[14] H. Razzaque, S. Ashraf, M. Naeem, Y.M. Chu, The Spherical q-Linear Diophantine Fuzzy Multiple-Criteria Group Decision-Making Based on Differential Measure, Cmes-Computer Modeling in Engineering & Sciences, 138 (2024) 1925-1950.

[15] P.M. Rafi-Ul-Shan, M. Bashiri, M.M. Kamal, S.K. Mangla, B. Tjahjono, An Analysis of Fuzzy Group Decision Making to Adopt Emerging Technologies for Fashion Supply Chain Risk Management, Ieee Transactions on Engineering Management, 71 (2024) 8469-8487.

[16] A.B. Leoneti, L. Gomes, A novel version of the TODIM method based on the exponential model of prospect theory: The ExpTODIM method, European Journal of Operational Research, 295 (2021) 1042-1055.

[17] Y. Deng, W.X. Zhang, An ExpTODIM-GRA based multiple attribute group decision-making method for development level evaluation of digital inclusive finance under intuitionistic fuzzy circumstances, Journal of Intelligent & Fuzzy Systems, 45 (2023) 10661-10673.

[18] Y. Wang, C. Liu, C. Deng, An ExpTODIM-EDAS Based Multiple Attribute Group Decision-Making Framework for Supervision Effectiveness Evaluation of Small and Medium Sized Financial Institutions in Rural Areas under Intuitionistic Fuzzy Circumstances, IEEE Access, (2023) https://doi.org/10.1109/ACCESS.2023.3333432.

[19] C. Huang, Z. Cheng, H. Guo, ExpTODIM-driven framework for 2-tuple linguistic neutrosophic MAGDM with applications to teaching quality evaluation in higher education, International Journal of Knowledge-based and Intelligent Engineering Systems, 28 (2024) 15-30.

[20] H. Sun, Z. Yang, Q. Cai, G.W. Wei, Z.W. Mo, An extended Exp-TODIM method for multiple attribute decision making based on the Z-Wasserstein distance, Expert Systems with Applications,

214 (2023) 14.

[21] S. Opricovic, G.H. Tzeng, Extended VIKOR method in comparison with outranking methods, European Journal of Operational Research, 178 (2007) 514-529.

[22] L.I. Tong, C.C. Chen, C.H. Wang, Optimization of multi-response processes using the VIKOR method, International Journal of Advanced Manufacturing Technology, 31 (2007) 1049-1057.

[23] E. Cevikcan, S. Cebi, I. Kaya, Fuzzy VIKOR and Fuzzy Axiomatic Design Versus to Fuzzy TOPSIS: An Application of Candidate Assessment, Journal of Multiple-Valued Logic and Soft Computing, 15 (2009) 181-208.

[24] P. Biswas, S. Pramanik, B.C. Giri, Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making, Neutrosophic sets and systems, 12 (2016) 20-40.

[25] W. Chen, H. Wang, S. Zhang, Research on the Performance Evaluation of Large Engineering Projects Based on HSE Perspective and Choquet Integral, Architecture Technology, 50 (2019) 707-710.

[26] X. Xu, J. Zhou, Meta-analysis of Trust and Project Management Performance, Science and Technology and Economy, 32 (2019) 55-59.

[27] H. Ji, Performance Evaluation Based on Water Conservancy Project Management, Heilongjiang Water Resources Science and Technology, 48 (2020) 234-237.

[28] J. Wang, Research on Performance Evaluation of Municipal Road Engineering Management, Residence Industry, (2020) 158-160.

[29] W. Yang, Discussion on Management and Performance Evaluation of Highway Reconstruction Projects, Finance and Economics, (2020) 88-89.

[30] T. Zhang, Discussion on the Performance Evaluation of Construction Project Management, Quality and Market, (2021) 193-195.

[31] C. Wang, Research on Implementation Plan of Graded Management for Project Performance Evaluation in an Enterprise, Value Engineering, 41 (2022) 40-42.

[32] H. Wang, Performance Indicators for the Project Management of Hydroelectric Resettlement Projects, Yunnan Hydropower, 38 (2022) 15-19.

[33] W. Lu, Comprehensive Management Model of Sponge City Engineering Based on Earned Value

Method, China Building Metal Structure, 22 (2023) 39-41.

[34] J. Zhang, Research on Project Management Under EPC General Contracting Mode, Guangdong Building Materials, 39 (2023) 123-126.

[35] W. Shen, Y. Weng, X. Zheng, W. Tang, Research on the Formation Mechanism of Cooperation and Competition Networks in Complex Project Groups, Chinese Journal of Management Science, 31 (2023) 281-289.

[36] G. Min, Y. Zhang, Research on Comprehensive Evaluation of Project Engineering Management Performance Based on Improved Entropy Weight Method, Residence, (2023) 161-164.

[37] J. Guo, Analysis of the Impact of Risk Sharing on Project Management Performance, Public Standardization, (2024) 69-71.

[38] W. Wu, Construction Schedule Management of Intelligent Building Projects, China Bidding, (2024) 178-180.

[39] I. Irvanizam, N.N. Zi, R. Zuhra, A. Amrusi, H. Sofyan, An Extended MABAC Method Based on Triangular Fuzzy Neutrosophic Numbers for Multiple-Criteria Group Decision Making Problems, Axioms, 9 (2020) 18.

[40] J. Wang, G.W. Wei, M. Lu, An Extended VIKOR Method for Multiple Criteria Group Decision Making with Triangular Fuzzy Neutrosophic Numbers, Symmetry-Basel, 10 (2018) 15.

[41] J.P. Fan, X.F. Jia, M.Q. Wu, A new multi-criteria group decision model based on Single-valued triangular Neutrosophic sets and EDAS method, Journal of Intelligent & Fuzzy Systems, 38 (2020) 2089-2102.

Appendix

Table A1. Analysis of Three Decision Matrices

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_2	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_3	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_4	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_5	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_6	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
C_7	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_8	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)

C_9	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{10}	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{11}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_{12}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
C_{13}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{14}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{15}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{16}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{17}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{18}	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{19}	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_{20}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_2	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_3	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_4	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_5	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_6	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_7	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_8	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_9	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{10}	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{11}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{12}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
C_{13}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{14}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{15}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{16}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{17}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{18}	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{19}	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{20}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_2	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_3	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_4	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_5	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_6	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
C_7	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_8	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)

C_9	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{10}	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{11}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{12}	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)
C_{13}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)
C_{14}	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{15}	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)
C_{16}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{17}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)
C_{18}	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)
C_{19}	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)
C_{20}	(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)	(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)	(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)	(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)

Received: Sep 5, 2024. Accepted: Dec 3, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Statistical Comparison Study of a Real-Life Survey Data on Procedural Justice via Neutrosophic Likert Scale with Score Function and Classical Likert Scale

Seher BODUR [1], Selçuk TOPAL [2*], and Hacı GÜRKAN [3]

[1] Department of Statistics, Bitlis Eren University, Bitlis 13000, Türkiye; skorkmaz@beu.edu.tr
[2*] Department of Mathematics, Gebze Technical University, Kocaeli 41000, Türkiye; stopal@gtu.edu.tr
[3] Vocational School of Hizan, Bitlis Eren University, Bitlis 13000, Türkiye; hgurkan@beu.edu.tr
* Correspondence: stopal@gtu.edu.tr

Abstract:

The present study aims to examine the distinctions between Likert and Neutrosophic scales in their ability to measure organizational behaviors and the many dimensions of organizational justice. To evaluate procedural justice, 6 questions measuring procedural justice from the Learning Environment Questionnaire were used. Compared to the Likert scale, the Neutrosophic scale exhibited greater decision weights and a more distinct expression of participants' ideas due to its ability to capture more complex replies. The reliability of both scales was found to be high, as indicated by satisfactory Cronbach's alpha coefficients. The results of the exploratory factor analysis indicated that both scales consisted of two sub-dimensions. However, it was observed that the Neutrosophic scale did not necessitate reverse coding for specific items, unlike the Likert scale.This suggests a potential advantage in terms of clarity and interpretability. Furthermore, the study revealed that age and faculty type exerted a substantial impact on participants' replies. Notably, there were considerable variations seen among different age groups on the Likert scale, as well as across different faculty types on the Neutrosophic scale. The results indicate that the Neutrosophic scale may yield more consistent and dependable data, particularly in varied demographic settings.

Keywords: Organizational justice, Likert scale, Neutrosophic Likert Scale, Neutrosophic sets, Neutrosophic survey, Neutrosophic logic, Exploratory factor analysis, Paired samples t test, Procedural justice, score function

1. Introduction

1.1. Organizational Justice and its Sub-Dimensions

Greenberg first used the term "organizational justice" in the 1980s [1]. The concept of organizational justice pertains to the processes by which workers assess the fairness of their treatment at work and the implications of these assessments on other variables associated with the workplace [2]. Organizational justice is defined by Colquitt and colleagues [3] as a person's belief that decisions and processes within the organization are fair and how this affects behavior. The degree to which members of an organization believe that the rules, regulations, and policies pertaining to their jobs are generally fair is known as organizational justice [4]. "Equity in the rules and social norms that govern companies" is what organizational justice is defined as [5]. According

to justice as seen through the prism of social exchange theory, members of an organization are more likely to feel satisfied and loyal when they believe they are being treated fairly, which increases the likelihood that they will make a commitment to the organization. They will display extra-role behaviors as a result of this circumstance [6, 7]. Robbin and Judge [8] state that distributive, procedural, and interactional justice are all included in the broad definition of organizational justice.

In social change interactions, when one party feels being treated fairly, the other party is more likely to go beyond duty. According to Gibson, [10] organizational justice is the degree to which an individual feels treated equally within the organization in which he or she works. Organizational justice refers to a person's perspective on decisions taken by their superiors. There are 3 (three) types of organizational justice: distribution justice, procedural justice, and interaction justice (10). Organizational justice is viewed by most researchers as the three dimensions of distribution justice, procedural justice, and interaction justice [11,12,13]. 1) Distributive justice: this is largely based on equity theory [14]. Distributive justice consists of perceptions of the consequences of distributive decisions [15]. 2) Procedural justice: The procedure was introduced by Thibaut and Walker [16], who examined the fairness of processes in legal transactions. It refers to the perception of fairness towards the procedures used to determine decisions about outcomes [17]. (3) Interactional justice: introduced by Bies and Moag, refers to how people are treated in the process of practices [18].

Important steps towards procedural justice have been made by Thibaut and Walker [16]. Distribution equity is expressed by people's reactions to payment decisions, while it is also expressed in how people react to the way these decisions are made [19]. While Thibaut and Walker were interested in decisions relating to the settlement of legal disputes, Levanthal [20] focused on more general award decisions in his theory. A set of procedural elements are used to assess whether procedures are fair, such as the selection of decision-making representatives, the establishment of basic rules for the assessment of potential awards, methods of gathering information, procedures defining the decision process, appeal procedures, necessary safeguards for non-abuse of authority, and the existence of exchange mechanisms. The fairness of the procedures for the distribution of prizes, he argued, relates to factors such as the consistency of allocations to be created, the prevention of bias, the accuracy of information, the correction ability, the responsiveness to the concerns of all recipients, and the degree to which moral and ethical standards are upheld.

Numerous scholars with a vested interest in organizational processes have endeavored to implement the theory within organizational contexts through the undertaking of comprehensive investigations on procedural justice [21,22]. The study conducted by Magnavita et al. (23) has demonstrated a correlation between diminished perceptions of procedural justice and the occurrence of mental and physical health issues. Procedural justice pertains to the equitable nature of the procedural framework that culminates in specific outcomes [24]. This dimension pertains to the procedural measures undertaken by management in order to arrive at a decision that is just and impartial. This pertains to matters concerning the measures implemented to achieve equal employment and promotion opportunities, establish a just system of rewards, and implement equitable disciplinary actions. Procedural justice, when effectively upheld, guarantees equitable access to promotion opportunities for all employees, devoid of any form of bias or prejudice [25].

Procedural justice additionally guarantees the establishment of equitable and widelyrecognized reward systems for all individuals. Therefore, it guarantees equity in organizational policies concerning the allocation of rewards or penalties to individuals, thereby establishing a levelplaying field for all individuals to pursue specific rewards for accomplishing predetermined objectives or tasks [26]. The fairness of procedures is associated with various factors, including the inclusion of employees in decision-making processes, the provision of information regarding the outcomes, and the elucidation of the rationale behind the decisions made [27]. Procedural justice also pertains to the measures undertaken by the same governing body in order to arrive at an equitable resolution. Procedural justice has been assessed by researchers using two primary methods: process control and decision control [28]. The concept of process control pertains to the degree of influence that employees possess over the decision-making process, as well as the circumstances in which these decisions are made. In contrast, decision control examines the degree

to which employees possess the ability to exert influence over the underlying rationales that drive decision-making within an organization [29].

Numerous studies have indicated a correlation between organizational justice and various outcomes, including job performance [30], job satisfaction [31], organizational citizenship behavior [32], psychological distress [33], intention to leave [34,35], and job involvement [36]. According to the literature, there exists a potential variation in the association between job satisfaction and organizational citizenship behavior with respect to organizational justice factors, specifically distributive, procedural, and interactional justice [37]. A study conducted among employees of a Japanese manufacturing company reveals a significant correlation between job performance and procedural justice, while no such correlation was found with interactional justice [30]. Another study has demonstrated a significant correlation between distributive justice and unit-level performance, specifically in terms of productivity and customer satisfaction. Additionally, interactional justice has been found to have a strong association with organizational citizenship-level processes [38].

Based on the findings, it is deemed suitable to analyze the three components of organizational justice in isolation from one another [39].

1.2. Likert Scale

The popular psychometric Likert scale, used in the social sciences to measure respondents' attitudes with survey questions, was first proposed by the American social psychologist Likert in 1932. This scale asks participants to indicate their levels of agreement with the questions. For example, for a 5-point Likert scale, agreement levels are evaluated by the integer values 1 (strongly disagree) to 5 (strongly agree), and the results are obtained by taking the sum or average of each participant's scores [40].

Likert scales are widely used because they are easy to administer, score, and understand. Additionally, researchers can collect large amounts of effective and inexpensive data in less time and conduct analyses using easy mathematical calculations. Moreover, it is a suitable method for making statistical inferences with good reliability and producing appropriate results.

Although the Likert scale is useful, it also has several disadvantages, such as un-certainty regarding whether responses and measured data should be on an ordinal or interval level. It is assumed that the Likert method has the characteristics of an interval scale [40]. However, many argue that the Likert scale is ordinal [41,42]. An interval scale dictates that there must be an equal interval between any two consecutive scales. For example, for a 5-point Likert scale, each level of agreement is expressed as follows: 1 = "strongly disagree"; 2 = "disagree"; 3 = "neutral"; 4 = "agree";and 5 = "strongly agree". Here, although the emotional intensity between "strongly disagree" and "dis-agree" is considered to be equivalent to the emotional intensity between other consecutive categories, participants may not understand the distances between two points of the scale as equal [43]. In this case, this scale will fail to measure actual responses.

When responding to a question on a Likert-type scale, participants must trans-form their feelings and thoughts into a linguistic expression that is coded with natural numbers and characterized by a ranking order, which can result in information loss, uncertainty, and inaccuracy [44]. Furthermore, the fact that participants' replies may be influenced by earlier questions and theirtendency to avoid selecting extreme possibilities on the scale both pose issues.

Because of the difficulties and uncertainties mentioned above, it was thought that the Likert scale may not be the best scale to measure the level of importance among various attributes. Therefore, this has led many researchers to propose different types of scales. In one study, to obtain superior measurements, a Neutrosophic approach based on fuzzy sets theory was used as an alternative to the Likert scale. Between November and December 2022, a survey was conducted among 1160 young clinical nurses from five hospitals in China's Henan province to investigate the effect of organizational justice on young nurses' turnover intention. The organizational justice scale,turnover intention scale, organizational climate scale, and emotional labor scale were used. The organizational justice scale was scored on a 5-point Likert scale (1 = "strongly disagree"; 2 =

"disagree"; 3 = "undecided"; 4 = "agree"; and 5 = "strongly agree"). It was concluded that organizational justice had a significant effect on turnover intentions among the young nursesthrough the chain mediation of organizational climate and emotional labor [45].

In addition, a survey was conducted with 400 employees to investigate the relationship and impact of organizational justice on employee creativity through the mediating role of leadership styles for academics and staff at Dhofar University in Oman. All items were rated on a 5-point Likert scale. Organizational justice was discussed along the following four dimensions: distributive, procedural, interpersonal, and in-formational. The results revealed that organizational justice had a positive and significant impact on the distributive, interpersonal, and informational dimensions of employee creativity, whereas procedural justice had a negative and significant impact on employee creativity [46]. Aiming to examine the relationship between organizational justice (procedural, distributive, informational, and interpersonal justice) and organizational citizenship behavior, a survey was administered to 121 faculty members working in ten private universities in Bangladesh. The participants were required to respond to all items using a 5-point Likert scale, ranging from 1 ("strongly disagree") to 5 ("strongly agree"). The study also provided the necessary guidelines on ways organizations can increase citizenship behavior, with an emphasis on fairness and inclusion in the workplace [47].

To develop a strategy to improve the working conditions of nurses in Japan, a survey was administered to nurses using a 5-point Likert scale (1 = "strongly disagree"; 2 = "disagree"; 3 = "undecided"; 4 = "agree"; and 5 = "strongly agree") using three scales: ease of work and organizational justice; organizational citizenship behavior; and job satisfaction. A significant positive correlation between interactional justice and job satisfaction has been reported fairly consistently [48]. To determine the effects of organizational support and organizational justice, a survey was applied to trainees in Basque cuisine during the 2022–2023 academic year. In the study, a7-point Likert-type scale was used. It was concluded that organizational support and organizationaljus-tice structures positively affected the happiness parameter at work [49].

In another study, a survey was administered to employees in various sectors in China, including manufacturing, construction, finance, information technology ser-vices, and wholesale and retail sectors, to examine the effects of information justice on employees' retention of information through organizational identification and to investigate how justice sensitivity moderates these effects. In this context, informational justice, justice sensitivity, organizational identity, and information hiding scales were used. All items forming the scale were evaluated usinga 5-point Likert scale [50]. To show the importance of organizational justice and citizenship behaviorin employees' compliance behaviors toward ISPs (information security policies), a survey was con-ducted on IS users in public and private banks in Ethiopia. A 5-point Likert scale, ranging from 1("strongly disagree") to 5 ("strongly agree") was used for all measurements in the study. Additionally, an empirical determination was made regarding the mediating role of organizational citizenship behavior between the dimensions of organizational justice and willingness to comply with ISPs [51].

To understand the antecedents of organizational justice, the authors conducted a national survey of library employees and compared the predictive power of perceived organizational support, job autonomy, job feedback, and job stress. Organizational jus-tice consisted of four subdimensions: distributive justice (4 items), procedural justice (7 items), interpersonal justice (4 items), and informational justice (5 items). Responses were received from the participants to each statement on a 5-point Likert scale (1 = "strongly disagree"; 2 = "disagree"; 3 = "undecided"; 4 = "agree"; and 5 = "strongly agree"). As a result, they found that providing meaningful and timely work feedback, as well as strengthening perceptions of institutional support and autonomy, can be effective in increasing librarians' overall perceptions of fairness [52]. A survey was conducted to determine the impact on the innovative work behavior of employees operating in the Chinese telecommunications industry. Distributive, procedural, and interactional justice items, representing three subdimensions of organizational justice, were included used with a 5-point Likert scale (1 = "strongly disagree"; 2 = "disagree"; 3 = "undecided"; 4 = "agree"; 5 = "strongly agree"). The study

concluded that organizational justice has a significant and positive effect on employees' innovative work behaviors and knowledge sharing [53].

The "Fair Learning Environment Scale", which was developed by Özer and Demirtaş (2010) [54] and Lizzio, Wilson, and Hadaway (2007) [55], was used in a Turkish validity and reliability study. The Kaiser–Meyer–Olkin (KMO) value of the scale was determined to be 0.83 by Özer and Demirtaş (2010) [54], and the internal consistency coefficient was determined to be 0.87 for the total scale. A scale with ten questions measuring distributive justice, which consisted of two subdimensions, distributive and procedural, was used.

1.3. Neutrosophy and Neutrosophic Set

Florentin Smarandache developed the philosophical and mathematical framework called Neutrosophy [56] in the late 20th century. This specific area of research focuses on matters and principles that involve uncertainty, vagueness, and inconsistencies. Neutrosophic logic provides an expanding framework to classical, fuzzy, and intuitionistic fuzzy logic, allowing for the representation of uncertain, contradictory, and ambiguous information. Classical logic encompasses propositions that possess binary truth or falsehood, while Neutrosophic logic permits declarations to possess truth, falsehood, and indeterminacy simultaneously. This facilitates a more advanced and flexible methodology for logical reasoning. Neutrosophic set theory is a theoretical framework that builds upon classical, fuzzy, and intuitionistic fuzzy set theory. It aims to tackle the difficulties that arise when dealing with sets that consist of members that are indeterminate or uncertain. The classification of an element in classical set theory is determined by its membership or non-membership in a set. In the context of Neutrosophic set theory, an element has the potential to exhibit varying degrees of membership within a given set, be devoid of any degree of membership, or lack any degree of membership within the set. The concept of neutrosophic probability represents an expansion of the traditional theory of probability, specifically designed to tackle the complexities presented by events that are unpredictable and uncertain. In the realm of classical probability theory, events are distinguished by clearly defined probabilities that span from 0 to 1. A more comprehensive representation of uncertainty can be achieved by examining the correlation between events in Neutrosophic probability and degrees of truth, falsehood, and indeterminacy.

The application of Neutrosophy has been observed in various fields including artificial intelligence [57], decision-making [58,59,60], information fusion [61], and risk analysis [62], where the proficient handling of uncertainty and ambiguity is crucial. The presented paradigm provides a systematic method for dealing with situations where classical logic and probability theory may need to be revised due to conflicting or uncertain data. The classification of an element in classical set theory is determined by its membership or non-membership in a set. The determination of element membership in a set is predicated upon binary terms, in accordance with the binary scenario. Zadeh [63] introduced the notion of fuzzy set theory, which enables a methodical assessment of the membership of elements within a set by employing a membership function that is constrained within the real unit interval [0,1]. In the domain of fuzzy set theory, crisp sets are frequently denoted as classical binary sets. Fault set theory can be comprehended as a continuation of classical set theory. Intuitive fuzzy sets are defined by elements that have varying degrees of membership and non-membership. The intuitionistic fuzzy set was introduced by Atanassov [64] as an extension of the fuzzy set concept, which builds upon the traditional notion of a set. The field of Neutrosophic set theory [56] presents a comprehensive framework that includes intuitionistic fuzzy sets, classical sets, fuzzy sets, dialetheist sets, paradoxist sets, tautological sets, and intuitionistic fuzzy sets. These various types of sets are based on the fundamental principles of Neutrosophy. In the set, a member x(T, I, F) is deemed true when its degree is $T \in [0,1]$, ambiguous when its degree is $I \in [0,1]$, and false when its degree is $F \in [0,1]$.

This section aims to present essential definitions and concepts related to single-valued neutrosophic sets, fuzzy sets, and intuitionistic fuzzy sets

Definition 1. [63] A fuzzy set X in $\boldsymbol{U}$ is a set of ordered pairs, defined as $\boldsymbol{X} = \{(\boldsymbol{x}, \mu_{\boldsymbol{X}}(\boldsymbol{x}))|\boldsymbol{x} \in \boldsymbol{U}\}$, where $\mu_{\boldsymbol{X}} : \boldsymbol{U} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$ is termed the membership function of X, and $\mu_{\boldsymbol{X}}(\boldsymbol{x})$ is the degree of membership of the element x in X given a universal set U and a generic element, represented by x.

Definition 2. [64] An intuitionistic fuzzy set X exists over a discourse-level world. The representation of U is given by $\boldsymbol{X} = \{(\boldsymbol{x}, \mu_{\boldsymbol{X}}(\boldsymbol{x}), \boldsymbol{\nu}_{\boldsymbol{X}}(\boldsymbol{x}))|\boldsymbol{x} \in \boldsymbol{U}\}$, where the terms "membership function of X" and "non-membership function of X" for x in X are, respectively, $\mu_{\boldsymbol{X}} : \boldsymbol{U} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$ and $\boldsymbol{\nu}_{\boldsymbol{X}} : \boldsymbol{U} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$. The formula for determining the degree of non-membership of an element, x, in X is $\mu_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{\nu}_{\boldsymbol{X}}(\boldsymbol{x}) \leq \boldsymbol{1}$. The hesitation degree of an element x defined by $\boldsymbol{\pi}_{\boldsymbol{X}}(\boldsymbol{x}) = \boldsymbol{1} - (\mu_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{\nu}_{\boldsymbol{X}}(\boldsymbol{x}))$.

Definition 3. [56, 65] Let U be a discourse universe. $\boldsymbol{N} = \{(\boldsymbol{x}, \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{I}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})) : \boldsymbol{x} \in \boldsymbol{U}\}$ is a neutrosophic set, denoted by a truth-membership function, $\boldsymbol{T}_{\boldsymbol{N}} : \boldsymbol{U} \rightarrow]^{-}\boldsymbol{0}, \boldsymbol{1}^{+}[$; an indeterminacy-membership function, $\boldsymbol{I}_{\boldsymbol{N}} : \boldsymbol{U} \rightarrow]^{-}\boldsymbol{0}, \boldsymbol{1}^{+}[$; and a falsity-membership function, $\boldsymbol{F}_{\boldsymbol{N}} : \boldsymbol{U} \rightarrow]^{-}\boldsymbol{0}, \boldsymbol{1}^{+}[$.

Definition 4. [56, 65] Let U be a discourse universe. A single-valued neutrosophic set is defined as $\boldsymbol{N} = \{(\boldsymbol{x}, \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{I}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})) : \boldsymbol{x} \in \boldsymbol{U}\}$, which is identified by a truth-membership function, $\boldsymbol{T}_{\boldsymbol{N}} : \boldsymbol{U} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$; indeterminacy-membership function, $\boldsymbol{I}_{\boldsymbol{N}} : \boldsymbol{U} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$; and falsity-membership function, $\boldsymbol{F}_{\boldsymbol{N}} : \boldsymbol{U} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$, with $\boldsymbol{0} \leq \boldsymbol{T}_{\boldsymbol{N}}(\boldsymbol{x}) + \boldsymbol{I}_{\boldsymbol{N}}(\boldsymbol{x}) + \boldsymbol{F}_{\boldsymbol{N}}(\boldsymbol{x}) \leq \boldsymbol{3}$.

1.4. Neutrosophy in Social Sciences

Smarandache [66] defines Neutrosophic Sociology, also referred to as Neutrosociology, as the utilization of neutrosophic scientific methodologies in the field of sociology. The utilization of questionnaires is widely acknowledged as an essential instrument in surveys [67] for evaluating the prevailing opinions within social collectives. The preference for fuzzy responses over crisp responses in surveys is widely recognized. However, it is crucial to acknowledge that fuzzy processing may not necessarily capture the intended meaning of the responder due to uncertainties, confusion, and unclear thinking. By employing neutrosophic sets in modeling this scenario, responders are presented with a broader spectrum of possible responses, thereby increasing its significance.

This paper presents a methodology for creating neutrosophic sets with a single value by using questionnaires distributed to social groups. In the field of Social Sciences, the study [68] presented a comprehensive elucidation, illustration, and initiation of neutrosophic statistical techniques. The field of Social Sciences often encounters inconsistencies in the data presented, which can be attributed to errors, conflicts in information and knowledge sources, lack of objectivity in certain viewpoints, and other contributing factors. Therefore, it has been argued that in certain circumstances, the incorporation of interval data may be deemed essential.

1.5. Neutrosophic Score Function

The scoring function s: [0,1] → [0,3], represented as s(a) = 2+ T - I - F, was utilized by Martinez et al. [67] to measure neutrosophic characteristics and perform a comparative analysis in the context of social science. However, our main focus is on the analysis of the measurement related to the influence of collective decision-making on societal decisions. It is noteworthy that the utilization of this specific scoring function was not implemented in a study employing a Likert-type scale. The utilization of the score function was first implemented in a Likert essay, and [67] provided evidenceof its reliable utilization in the realm of social sciences. The s(a) = (1 + T - 2I - F)/2 function utilized inthis study is derived from the publication referenced as [69]. It is deemed suitable for an initial workspace to incorporate an evaluation of the adverse, neutral, and beneficial consequences by distributing the score values within the range of [-1, 1]. This approach is consistent with themethodology employed in neutrosophic research.

1.6. Neutrosophic Likert Scale

In this study, the neutrosophic Likert scale was implemented for the first time [70]. Life satisfaction on a traditional scale The Likert scale items were converted to numeric values ranging from 0 to 100. Respondents were anticipated to select one of the following: "I agree with this statement (…)," "I am neutral (or indifferent) regarding this statement (…)," or "I disagree with this statement (…)." The findings presented in the study indicate that the neutrosophic scale exhibits comparable reliability to the classical scale, as Cronbach's Alpha remains within an acceptable rangefor all three dimensions.

In the present investigation, we employ a direct percentage understanding framework to facilitate a closer approximation to natural language. Fuzzy sets and Neutrosophic Likert scales share a commonality in their approach to managing imprecision and uncertainty. Neutrosophic sets, including Neutrosophic Likert scales, offer a broader representation that encompasses indeterminacy as a fundamental element. On the other hand, fuzzy sets facilitate the representation and manipulation of data that lacks precise specification. Neutrosophic Likert scales are particularly suitable for survey responses that involve participants' opinions that are not only varied across a spectrum (as accommodated by fuzzy sets), but also include a degree of indecision or neutrality that is challenging to capture using traditional fuzzy logic or crisp Likert scales.

The questionnaire was prepared using Google forms, an effective data-gathering tool, to evaluate college students' attitudes to procedural justice. The number of participants was determined to be 119 using G*Power 3.1.9.4. 126 university students were surveyed using the Likertscale of 5 degrees (1 = "absolutely disagree", 2 = "not agree", 3 = "no resolve", 4 = "agreed", 5 = "definitely agree") and the neutrosophic scale, respectively, as described in Table 1 and Table 2.

The participants completed the survey by marking only one of these five agreement values. The research data can be accessed from the following link: https://osf.io/xd4t7.

Table 1. Likert Scale Questions (responses should be indicated by marking a single choice with a check mark)

Questions	Strongly Disagree	Disagree	Neither Agree Nor Disagree	Agree	Strongly Agree
There are effective procedures in place to help students solve problems.	☑	☐	☐	☐	☐
It is not clear what to do when a student has a problem.	☐	☐	☑	☐	☐
There is an effective system for lodging complaints of unfair treatment.	☐	☐	☐	☐	☑
Students can get the advice or help they need.	☐	☑	☐	☐	☐

There is no point in complaining as nothing will really be done.	☑	☐	☐	☐	☐
Educational information is provided to students in an easily accessible way.	☐	☐	☐	☑	☐

The researchers utilized the neutrosophic scale, which consists of three levels (a: agreement, b: indeterminacy, c: disagreement), to assess the procedural justice. The questions that have been chosen are presented in Table 2. The survey participants, who were the same participants surveyed before (as indicated in Table 1), chose percentile values a, b, c, and one. Based on the single-valued neutrosophic set methodology, the cumulative percentiles (%) of variables a, b, and c were found to range from 0 to 300. Subsequently, a single percent of these percentiles was selected and assigned to the closed interval [0, 1].

Table 2. Neutrosophic scale questions (responses are to be completed as percentages)

Questions	Agreement Degree	Indeterminacy Degree (Neither Agree Nor Disagree)	Disagreement Degree
There are effective procedures in place to help students solve problems.	50	30	80
It is not clear what to do when a student has a problem.	0.8	40	10
There is an effective system for lodging complaints of unfair treatment.	0	25	100
Students can get the advice or help they need.	15	25	75
There is no point in complaining as nothing will really be done.	29	0.5	89
Educational information is provided to students in an easily accessible way.	58	60	7.5

The demographic information of the survey participants is shown in Table 3 as count (percentage).

Table 3. Demographic characteristics of participants

Categorical Variables	n (%)
Gender	
Female	82 (65.08)
Male	44 (34.92)
Age	
18–21	45 (35.71)
22–24	47 (37.30)
25–30	24 (19.05)

31–40	5 (3.97)
41⁺	5 (3.97)
Faculty	
Vocational School of Hizan	39 (30.95)
Vocational School of Health Services	19 (15.08)
School of Physical Education and Sports	10 (7.94)
Vocational School of Tatvan	8 (6.35)
Faculty of Science and Letters	18 (14.29)
Faculty of Fine Arts	4 (3.17)
Faculty of Health Sciences	7 (5.55)
Vocational School of Güroymak	5 (3.97)
Faculty of Islamic Sciences	16 (12.70)
Marital status	
Married	9 (92.86)
Single	117 (7.14)

The following tables show the participants' responses to Likert and Neutrosophic scales (Table 4, Table 5, Table 6, Table 7).

Table 4. Responses from participants using a Likert scale

	Q1	Q2	Q3	Q4	Q5	Q6
Strongly disagree	8	13	15	11	26	3
Disagree	16	36	27	12	35	9
Neither agree nor disagree	33	32	24	30	24	26
Agree	51	34	42	57	28	59
Strongly agree	18	11	18	16	13	29

Table 5. Responses from participants using a neutrosophic scale

	Q1	Q2	Q3	Q4	Q5	Q6
Disagree	13	18	19	17	12	13
Neither agree nor disagree	59	70	61	59	73	43
Agree	54	38	46	50	41	70

Table 6. Answers based on Likert scale for all questions

	All Questions
Strongly disagree	6
Disagree	12
Neither agree nor disagree	55

Agree	44
Strongly agree	9

Table 7. Answers based on Neutrosophic scale for all questions

	All Questions
Disagree	8
Neither agree nor disagree	63
Agree	55

The total score for the express agreement options, namely agree and strongly agree, on the Likert scale was 53, while the cumulative score for the agree option on the Neutrosophic scale was 55.The Undecided option yielded a Likert scale score of 55 and a Neutrosophic scale score of 63. The aggregate of the responses "I disagree" and "I strongly disagree" on the Likert scale corresponds to a total of 18 individuals expressing disagreement. Conversely, on the Neutrosophic scale, this figure amounts to 8. Consequently, it is evident that the participants exhibited a significantly higher tendency to select the agree options in both scales. Moreover, the rates of agreement and undecided responses on the Neutrosophic scale surpass those on the Likert scale. Conversely, the Neutrosophic scale yielded lower results in the disagree options compared to the Likert scale. When assessed based on the ratios, it was observed that the participants provided a greater number of responses to the "I agree" alternatives. The Neutrosophic scale exhibited a greater inclination towards higher decision weights and a lesser inclination towards lower decision weights in contrast to the Likert scale.

This phenomenon can be elucidated by the fact that the Neutrosophic scale affords participants the opportunity to articulate their thoughts with greater clarity, as they are able to assign an unlimited number of points to each available option.

2. Statistical Analysis

The analysis involved the utilization of the SPSS 26.0 software package (IBM Corporation) and MATLAB R2015a to ascertain disparities between the two scales. To assess the construct validity of Likert and neutrosophic scales in measuring procedural justice and to uncover the factor structure, an Exploratory Factor Analysis (EFA) was conducted. The Kaiser-Meyer-Olkin (KMO) sampling fitness values were evaluated and found to be 0.758 and 0.768, respectively. These values indicate that the sample size was sufficient for exploratory factor analysis (EFA). According to Field [71] (p.647), it is necessary for this value to exceed 0.50. Furthermore, the Bartlett test yielded $x^2(15) = 295.919$ ($p < 0.05$) and $x^2(15) = 345.267$ ($p < 0.05$) for the respective variables. These results indicate that the variances across the variables are not homogeneous, providing support for the suitability ofthe data for exploratory factor analysis. The exploratory factor analysis (EFA) of the Likert scale yielded findings indicating that two scales, each comprising six items, exhibited a two-factor structure. These two factors accounted for 72.974% of the overall variance. Furthermore, the initial sub-dimension accounts for 49.293% of the variance, while the subsequent sub-dimension accounts for 23.681%. Similarly, based on the exploratory factor analysis (EFA) findings of the Neutrosophic scale, it was observed that the two scales, comprising six items each, exhibited a two-factor structure. These two factors accounted for 77.321% of the overall variance. Furthermore, the initial sub-dimension accounts for 51.255% of the variance, while the subsequent sub-dimension accounts for 26.066%. The distribution of Likert and neutrosophic scale items according to factors and their factor loadings is presented in Table 8 and Table 9, respectively.

Table 8. Factor analysis findings of Likert scale

Items	Factor1	Factor2
item4	0.894	
item1	0.868	
item3	0.859	
item6	0.800	
item2		0.855
item5		0.832

Table 9. Factor analysis findings of Neutrosophic scale

Items	Factor1	Factor2
item4	0.921	
item1	0.890	
item3	0.791	
item6	0.806	
item2		0.868
item5		0.909

As seen in Table 8 and Table 9, the first sub-dimension of both scales consists of 4 items (item4, item1, item3, item6), and the second consists of 2 items (item2, item5). The minimum value of factor loadings was determined as 0.791. Therefore, considering that factor loadings of 0.40 and above are considered ideal by Field [71], it was concluded that the items made a significant contribution to the factors.

Table 10 presents descriptive statistics covering 126 participants' evaluations of procedural fairness using two different scales (Likert and Neutrosophic).

Table 10. Descriptive Statistics

	Likert Scale	Neutrosophic Scale
Mean	19.540	1.415
Median	20.000	1.567
Mode	19.000	0.000
Std. deviation	4.414	1.983
Variance	19.482	3.931
Skewness	-0.539	-0.261
Std. error of skewness	0.216	0.216
Kurtosis	0.325	-0.598
Std. error of kurtosis	0.428	0.428
Range	24.000	8.000

Minimum	6.000	-3.000
Maximum	30.000	5.000

According to Table 10, the mean value and standard deviation of the Likert scale are lower than that of the neutrosophic scale. Additionally, skewness and kurtosis are important criteria to evaluate whether data distributions are normal. According to the skewness and kurtosis coefficients and the Kolmogorov Smirnov normality test (Table 11), it is observed that the Likert scale and the neutrosophic scale are different from the normal distribution at the 95% confidence level. However, in social sciences, it is considered ideal for skewness and kurtosis values to be in the range of [-1,1] [72]. Therefore, considering that the skewness and kurtosis values of both scales were in the [-1,1] range, parametric analyzes were performed assuming that they were suitable for normal distribution.

Table 11. Kolmogorov–Smirnov Test

	Kolmogorov–Smirnov		
	Statistic	**Df**	**Sig.**
Likert scale	0.086	126	0.022
Neutrosophic scale	0.075	126	0.079

Table 12. Cronbach's Alpha Values

Cronbach's Alpha Constant	**Variables**
0.787	SCORE1, SCORE2, SCORE3, SCORE4, SCIRE5, SCORE6
0.519	VAR1a, VAR2a, VAR3a, VAR4a , VAR5a, VAR6a
0.909	VAR1b, VAR2b, VAR3b, VAR4b, VAR5b, VAR6b
0.726	VAR1c, VAR2c, VAR3c, VAR4c, VAR5c, VAR6c
0.714	VAR1, VAR2, VAR3, VAR4, VAR5, VAR6

As seen in Table 12, Cronbach's alpha coefficient was calculated to evaluate the internal consistency and reliability of the scales. Likert and neutrosophic scales have generally been found to be highly reliable [73] (p.113).

Table 13. Correlation among Likert Items, Neutrosophic Items, and Scores

		VAR1a	**VAR1b**	**VAR1c**	**Score1**
	Correlation Coefficient	0.548 **	0.043	−0.268 **	0.244 **
VAR1	*p*	0.000	0.635	0.002	0.006
	N	126	126	126	126
		VAR2a	**VAR2b**	**VAR2c**	**Score2**
	Correlation Coefficient	-0.253 **	-0.017	0.283 **	-0.194 *
VAR2	*p*	0.004	0.854	0.001	0.030
	N	126	126	126	126
		VAR3a	**VAR3b**	**VAR3c**	**Score3**
	Correlation Coefficient	0.381 **	0.056	−0.115	0.143
VAR3	*p*	0.000	0.536	0.202	0.110
	N	126	126	126	126
		VAR4a	**VAR4b**	**VAR4c**	**Score4**
	Correlation Coefficient	0.532 **	0.006	−0.158	0.229 *
VAR4	*p*	0.000	0.951	0.077	0.010
	N	126	126	126	126
		VAR5a	**VAR5b**	**VAR5c**	**Score5**
VAR5	Correlation Coefficient	-0.410 **	-0.136	0.304 **	-0.191 *
	p	0.000	0.130	0.001	0.032

	N	126	126	126	126
		VAR6a	**VAR6b**	**VAR6c**	**Score6**
VAR6	Correlation Coefficient	0.339 **	-0.066	−0.256 **	0.239 **
	p	0.000	0.460	0.004	0.007
	N	126	126	126	126

* Correlation is significant at the 0.05 level. ** Correlation is significant at the 0.01 level.

According to the Spearman correlation analysis results in Table 13, there is a generally positive significant relationship between the Likert scale item and the "agree" item of the neutrosophic scale, and between the Likert scale item and the scores obtained from the neutrosophic scale items (except VAR2 and VAR5). Since var2 and var5 items were reverse coded, there is a negative relationship between them and the Neutrosophic scale.

According to the findings in Table 14, it was observed that there was a very weak or even almost no relationship between the Likert and neutrosophic scales.

Table 14. Correlation between Neutrosophic Scale and Likert Scale

		Neutrosophic Scale
Likert scale	Pearson Correlation (r)	0.055
	p	0.540
	N	126

Table 15. Paired Samples T-Test

		Me.an	**SD**	**t**	**Df**	***p***	**d**
Pair 1	Likert scale-Neutrosophic scale	18.125	4.738	42.941	125	0.000	3.825

d = Effect size.

As indicated by the findings in Table 15, the paired samples t-test yielded significant results, indicating a notable disparity between the Likert scale and Neutrosophic scale ($t = 42{,}941$, $p < 0{,}05$). According to Cohen's criteria, the effect size (3.285) was found to be notably large [74].

Mean and standard deviation are measurements that form the basis of statistical analysis. In Table 15, the means and standard deviations of participants' attitudes towards procedural justice were determined through Likert and neutrosophic scales.

Table 16. Mean and Standard Deviation of the Likert Scale and Neutrosophic Scale

Item	**Likert Scale**		**Neutrosophic Scale**	
	μ	**σ**	**μ**	**σ**
There are effective procedures in place to help students solve problems.	3.437	1.084	0.247	0.462
It is not clear what to do when a student has a problem.	2.952	1.151	0.153	0.477
There is an effective system for lodging complaints of unfair treatment.	3.167	1.257	0.202	0.483
Students can get the advice or help they need.	3.437	1.106	0.222	0.477
There is no point in complaining as nothing will really be done.	2.738	1.297	0.191	0.465
Educational information is provided to students in an easily accessible way.	3.810	0.953	0.400	0.485

μ = Arithmetic mean, σ = Standard deviation.

The results in Table 16 show that the sample mean and standard deviation of the data obtained with the Neutrosophic scale are lower than the sample mean and standard deviation of the data obtained with the Likert scale. Lower mean and standard deviation indicate that the scales provide

more consistent and similar results about the measured trait or concept and that the data set has a more homogeneous distribution. In this context, it can be stated that the neutrosophic scale data has a more homogeneous distribution than the Likert scale.

Table 17. One-Way ANOVA Findings for the Likert and Neutrosophic Scales

Scale	Effect	Mean Square	F	Significance Level
Likert scale	Age	74.295	4.204	0.003
Neutrosophic scale	Age	10.130	2.719	0.033

As shown in Table 17, one-way ANOVA was conducted to examine the single effect of age groups. The effect of age groups on the Likert scale and neutrosophic scale was statistically significant (respectively, F = 4.204, $p < 0.05$; F=2.719, $p < 0.05$). On the Likert scale, the difference between the averages of the 22-24 age group and the 25-30 age group was found to be statistically significant ($p < 0.05$). The Likert scale score of students in the 25-30 age group (Mean = 22.208) is higher than that of students in the 22-24 age group (Mean = 18.404).

The fact that students in the 25-30 age group have more life experience compared to students in the 22-24 age group enables them to have more realistic expectations and can be explained by their more accepting behavior towards institutional procedures.

Table 18. Two-Way ANOVA Findings for the Likert and Neutrosophic Scales

Scale	Single and Interaction Effect	Mean Square	F	P
Likert scale	Gender	10.615	0.531	0.468
	Faculty	17.515	0.876	0.539
	Gender X Faculty	13.404	0.670	0.697
Neutrosophic scale	Gender	1.179	0.314	0.576
	Faculty	7.888	2.101	0.042
	Gender X Faculty	0.826	0.220	0.980

Two-way ANOVA was performed to investigate both the individual impacts and the interaction impacts between gender and faculty (Table 18). The individual effect of gender (F = 0.531, $p > 0.05$), individual effect of faculty type (F = 0.876, $p > 0.05$) and the interaction effect of gender andfaculty type (F = 0.670, $p > 0.05$) on the Likert scale were statistically insignificant.

Likewise, the single effects of gender (F = 0.314, $p > 0.05$) and the interaction between gender and faculty type (F = 0.220, $p > 0.05$) on the Neutrosophic scale were not statistically significant. However, the individual effect of the student's faculty type on the Neutrosophic scale was statistically significant (F = 2.101, $p < 0.05$).

The statistical analysis revealed a substantial individual influence of faculty type on the Neutrosophic scale. This problem can be attributed to variations in procedures across different faculties or disparities in student perception resulting from the administrators' application approach.

It can be thought that the fact that students in the 25-30 age group have more life experience than students in the 22-24 age group enables them to have more realistic expectations, and this causes them to exhibit more accepting behavior towards institutional procedures.

3. Conclusions

This study explores the differences and comparative effectiveness of the Likert and Neutrosophic scales in assessing aspects of organizational behavior and justice. The results of the

study exhibit a multitude of dimensions, illustrating the intricate relationship among several factors such as scale type, answer clarity, dependability, demographic variables, and statistical approaches.

An important finding of this study is that the Neutrosophic scale offers greater decision-making significance when compared to the conventional Likert scale. The architecture of the Neutrosophic scale allows respondents to flexibly allocate points across alternatives, resulting in a more precise expression of their ideas and preferences. This particular attribute becomes advantageous in obtaining subtle and nuanced replies that would be diminished or concealed by the more limited Likert scale approach. The findings of our study revealed that responses collected using the Neutrosophic scale not only had high mean values and standard deviations, but also exhibited uniformity and consistency across different data sets. These results suggest that the Neutrosophic scale is a more dependable and consistent measure when subjected to different analytical settings.

Regarding reliability, it is noteworthy that both scales exhibited a strong level of internal consistency, as indicated by the substantial Cronbach's alpha coefficients. This observation highlights the usefulness of both measures in academic and applied research contexts, however the Neutrosophic scale is acknowledged for its enhanced ability to capture comprehensive respondent response with greater clarity.

Exploratory factor analysis provided further analytical depth, revealing that both scales consist of two separate sub-dimensions. It is noteworthy that the two dimensions in both scales consisted of identical items, namely initial dimension item4, item1, item3, and item6, the second dimension was comprised of item2 and item5. In contrast to the Likert scale, the Neutrosophic scale did not necessitate the reverse coding of any items, including of items 2 and 5. This feature has the potential to decrease the intricacy and the likelihood of misinterpretation, hence improving the scale's practicality and precision.

Additionally, our research investigated the significance on scales of demographic factors, including age and faculty type. The results of the one-way analysis of variance (ANOVA) revealed statistically significant variations in responses across different age groups on both scales. Specifically, it was seen that those between the ages of 25 and 30 exhibited greater scores on the Likert scale compared to those in the 22-24 age range. Additionally, the two-way ANOVA revealed that the kind of faculty had a notable influence on responses on the Neutrosophic scale, but not on the Likert scale. The results of this study indicate that demographic variables might have varying effects on the results of research, depending on the scale employed. Therefore, it is crucial to carefully choose the proper measuring instrument while doing organizational research.

The lack of association between the Likert and Neutrosophic scales, as observed in our data, highlights the unique assessment capabilities of each scale. The absence of a correlation highlights the necessity for researchers to carefully select a scale that is most suitable for their particular study needs and goals.

In summary, this research makes a valuable contribution to the continuing scholarly conversation surrounding the efficacy of measurement scales in the field of organizational research. The Neutrosophic scale, due to its enhanced adaptability and more transparent response mechanism, presents a viable substitute for the conventional Likert scale, especially in research that necessitates meticulous examination of intricate behaviors and attitudes. Further investigation is warranted to examine these scales in diverse settings and with distinct demographic groups in order to comprehensively determine their suitability and efficacy across a wider array of academic fields.

In a prospective future investigation, our intention is to employ machine learning models for the purpose of forecasting demographic variables, including age, education, marital status, and sex, based on the responses obtained from the classical Likert and Neutrosophic Likert scales. Another potential avenue for future research involves utilizing the recently developed RANCOM method, which specifically targets the assessment of Neutrosophic Likert scale data by experts in various fields [75].

References

[1] Cropanzano, R., & Greenberg, J. (1997). Progress in organizational justice: Tunneling through the maze. International review of industrial and organizational psychology, 12, 317-372.

[2] Moorman, R. H. (1991). Relationship between organizational justice and organizational citizenship behaviors: Do fairness perceptions influence employee citizenship? Journal of applied psychology, 76(6), 845.

[3] Colquitt, L., & Wesson, O. B. (2009). Improving Performance and Commitment in the workplace. Florida: McGraw-Hill Irwin, 2.

[4] Saldanha, L. D. S., Supartha, W. G., & Riana, G. (2019). Pengaruh keadilan organisasional terhadap kepuasan kerja dan komitmen organisasional polícia nacional de timor-leste (Pntl). E-Jurnal Ekonomi Dan Bisnis Universitas Udayana, 2(1), 137.

[5] Ndjaboué, R., Brisson, C. ve Vézina, M. (2012). Organisational justice and mental health: a systematic review of prospective studies, Occupational and Environmental Medicine;69:694-700.

[6] Deckop, J. R., Mangel, R., & Cirka, C. C. (1999). Getting more than you pay for: Organizational citizenship behavior and pay-for-performance plans. Academy of Management journal, 42(4), 420-428. https://doi.org/10.5465/257012

[7] Konovsky, M. A., & Organ, D. W. (1996). Dispositional and contextual determinants of organizational citizenship behavior. Journal of organizational behavior, 17(3), 253-266. https://doi.org/10.1002/(SICI)1099-1379(199605)17:3<253::AID-JOB747>3.0.CO;2-Q

[8] Robbins, S. P., & Judge, T. A. (2012). Örgütsel Davranış: Organizational Behavior. Çeviri Editörü: Prof. Dr. İnci Erdem, 14.

[9] Ali, L., Jameel, A. S., & Rahman, A. (2020). The effect of organizational justice on job satisfaction among secondary school teachers. International Journal of Psychosocial Rehabilitation, 24(3), 1302-1310. https://www.psychosocial.com/article/PR200880/1112...

[10] Gibson, J. (2016). Organisasi dan Manajemen: Perilaku, Struktur, Proses, Edisi Keempat Terjemahan. Jakarta: Erlangga.

[11] Beugr, C. D. (2002). Understanding organizational justice and its impact on managing employees: an African perspective. The International Journal of Human Resource Management, 13(7), 1091–1104. https://doi.org/10.1080/09585190210131311

[12] Lee, J., & Wei, F. (2017). The moderating effect of leadership on perceived organizational justice and affective commitment: a study in China. The International Journal of Human Resource Management, 28(5), 679–702. https://doi.org/10.1080/09585192.2015.1109533

[13] Mubashar, T., Musharraf, S., Khan, S., & Butt, T. A. (2022). Impact of organizational justice on employee engagement: The mediating role of organizational trust. Cogent Psychology, 9(1). https://doi.org/10.1080/23311908.2022.2080325

[14] Adams, J. S. (1965). Inequity in social exchange. In Advances in experimental social psychology (Vol. 2, pp. 267-299). Academic Press.

[15] Homans, G. C. (1961). The humanities and the social sciences. American Behavioral Scientist, 4(8), 3-6.

[16] Thibant, J., and Walker, L. (1975). Procedural Justice: A Psychological Analysis, Lawrence Erlbaum Associates, Hillsdale, NJ.

[17] Folger, R., & Konovsky, M. A. (1989). Effects of procedural and distributive justice on reactions to pay raise decisions. Academy of Management journal, 32(1), 115-130.

[18] Bies, R.J and Moag, J.S. (1986). International Justice: communication criteria of fairness. Research on Negotiation in Organizations. (Vol. 1). Greenwich, CT: JAI Press, 43-55.

[19] Walker, L., Lind, E. A., and Thibaut, J. (1979). The relation between procedural justice and distributive justice. Va. Law Rev. 65: 1401-1420.

[20] Leventhal, G. S. (1980). What should be done with equity theory? In Gergen, K. J., Greenberg, M. S., and Willis, R. H. (eds.), Social Exchange: Advances in Theory and Research, Plenum Press, New York, pp. 27-55

[21] Greenberg, J., and Folger, R. (1983). Procedural justice, participation, and the fair process effect in groups and organizations. In Paulus, P. B. (ed.), Basic Group Processes, Springer Verlag, New York, pp. 235-256

[22] Folger, R., and Greenberg, J. (1985). Procedural justice: An interpretive analysis of personnel systems. In Rowland, K., and Ferris, G. (eds.), Research in Personnel and Human Resources Management, Vol. 3, JAI Press, Greenwich, CT, pp. 141-183.

[23] Magnavita N, Chiorri C, Acquadro Maran D, Garbarino S, Di Prinzio RR, Gasbarri M, Matera C, Cerrina A, Gabriele M, Labella M. Organizational Justice and Health: A Survey in Hospital Workers. International Journal of Environmental Research and Public Health. 2022; 19(15):9739. https://doi.org/10.3390/ijerph19159739

[24] Yean, T. F. (2016). Organizational justice: A conceptual discussion. Procedia-Social and Behavioral Sciences, 219, 798-803. https://doi.org/10.1016/j.sbspro.2016.05.082

[25] Brown, G., Bemmels, B., & Barclay, L. J. (2010). The importance of policy in perceptions of organizational justice. Human Relations, 63(10), 1587-1609. DOI 10.1177/0018726710362273

[26] Wan, Y. K. P., & Chan, S. H. J. (2018). Perceptions of casino dealers toward organizational justice: Are there any gender differences? Journal of Human Resources in Hospitality & Tourism, 17(3), 271-295.https://doi.org/10.1080/15332845.2017.1406282

[27] Ismail, M., Baki, N. U., & Omar, Z. (2018). The influence of organizational culture and organizational justice on group cohesion as perceived by merger and acquisition employees. Organizations and Markets in Emerging Economies, 9(2), 233-250. https://www.ceeol.com/search/article-detail?id=770952

[28] Przęczek, C., Rosiński, J., & Manko, B. A. (2020). Research review in organizational justice. Journal for Perspectives of Economic, Political and Social Integration, 26(1-2). https://ojs.tnkul.pl/index.php/jpepsi/article/view/15684/15209

[29] Yunita, P. I., & Darma, K. K. Y. (2020). The commitment of organizational justice: A case study. Review of Management, Accounting, and Business Studies, 1(1), 17-26. DOI: https://doi.org/10.38043/revenue.v1i1.2664

[30] Nakagawa, Y., Inoue, A., Kawakami, N., Tsuno, K., Tomioka, K., Nakanishi, M., & Hiro, H. (2015). Change in organizational justice and job performance in Japanese employees: A prospective cohort study. journal of Occupational Health, 57(4), 388-393. https://doi.org/10.1539/joh.14-0212-BR

[31] Rodwell, J., & Munro, L. (2013). Well-being, satisfaction and commitment: The substitutable nature of resources for maternity hospital nurses. Journal of Advanced Nursing, 69(10), 2218-2228. https://doi.org/10.1111/jan.12096

[32] Ildarabadi, E., Karimi-Moonaghi, H., Heydari, A., Taghipour, A., Abdollahimohammad, A., & Arbabisarjou, A. (2014). Teaching methods in community health nursing clerkships: experiences of healthcare staff in Iran. Journal of Educational Evaluation for Health Professions, 11.

[33] Kobayashi, Y., & Kondo, N. (2019). Organizational justice, psychological distress, and stress-related behaviors by occupational class in female Japanese employees. PloS one, 14(4), e0214393. https://doi.org/10.1371/journal.pone.0214393

[34] Mengstie, M. M. (2020). Perceived organizational justice and turnover intention among hospital healthcare workers. BMC psychology, Springer, 8(1), 19.

[35] Tourani, S., Khosravizadeh, O., Omrani, A., Sokhanvar, M., Kakemam, E., & Najafi, B. (2016). The relationship between organizational justice and turnover intention of hospital nurses in Iran. Materia socio-medica, 28(3), 205. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4949048/

[36] Wan, Q., Zhou, W., Li, Z., & Shang, S. (2018). Associations of organizational justice and job characteristics with work engagement among nurses in hospitals in China. Research in nursing & health, 41(6), 555-562. https://doi.org/10.1002/nur.21908

[37] Colquitt, J. A., Conlon, D. E., Wesson, M. J., Porter, C. O., & Ng, K. Y. (2001). Justice at the millennium: A meta-analytic review of 25 years of organizational justice research. Journal of Applied Psychology, 86(3), 424-424. https://doi.org/10.1037/0021-9010.86.3.425

[38] Whitman, D. S., Caleo, S., Carpenter, N. C., Horner, M. T., & Bernerth, J. B. (2012). Fairness at the collective level: A meta-analytic examination of the consequences and boundary conditions of organizational justice climate. Journal of applied psychology, 97(4), 776. https://doi.org/10.1037/a0028021

[39] Shimamura, M., Fukutake, M., Namba, M., & Ogino, T. (2021). The relationship among factors of organizational justice, organizational citizenship behavior, job satisfaction, and ease of work among Japanese nurses. Applied Nursing Research, 61, 151479.https://doi.org/10.1016/j.apnr.2021.151479

[40] Likert, R. A technique for the measurement of attitudes. *Arch. Psychol.* 1932, *22*, 140.

[41] Hodge, D.R.; Gillespie, D. Phrase completions: An alternative to. *Soc. Work Res.* 2003, *27*, 45.

[42] Pett, M.A. *Nonparametric Statistics for Health Care Research: Statistics for Small Samples and Unusual Distributions*; Sage Publications: New York, NY, USA, 2015.

[43] Crask, M.R.; Fox, R.J. An exploration of the interval properties of 3 commonly used marketing-research scales-a magnitude estimation approach. J. Mark. Res. Soc. 1987, 29, 317–339.

[44] Disegna, M.; D'Urso, P.; Massari, R. Analysing cluster evolution using repeated cross-sectional ordinal data. Tour. Manag. 2018, 69, 524–536.

[45] Su, Y.; Jiang, Z.; Meng, R.; Lu, G.; Chen, C. The effect of organizational justice on young nurses' turnover intention: The mediating roles of organizational climate and emotional labour. Nurse Educ. Pract. 2023, 72, 103723.

[46] Jaboob, M.; Awain, A. M. S. B.; Al-Ansi, A. M. Sustaining employees' creativity through the organizational justice: The mediating role of leadership styles. Social Sciences & Humanities Open. 2023, 8(1), 100693.

[47] Rahman, M.H.A.; Karim, D.N. Organizational justice and organizational citizenship behavior: The mediating role of work engagement. Heliyon. 2022, 8, e09450.

[48] Shimamura, M.; Fukutake, M.; Namba, M.; Ogino, T. The relationship among factors of organizational justice, organizational citizenship behavior, job satisfaction, and ease of work among Japanese nurses. Appl. Nurs. Res. 2021, 61, 151479.

[49] Ravina-Ripoll, R.; Balderas-Cejudo, A.; Nunez-Barriopedro, E.; Galvan-Vela, E. Are chefs happiness providers? Exploring the impact of organisational support, intrapreneurship and interactional justice from the perspective of happiness management. Int. J. Gastron. Food Sci. 2023, 34, 100818.

[50] Xu, G.; Huang, Y.; Huang, S.S. Informational justice and employee knowledge hiding behaviours: Mediation of organizational identification and moderation of justice sensitivity. Heliyon. 2023, 9, e14697.

[51] Aebissa, B.; Dhillon, G.; Meshesha, M. The direct and indirect effect of organizational justice on employee intention to comply with information security policy: The case of ethiopian banks. Comput. Secur. 2023, 130, 103248.

[52] Matteson, M.L.; Ming, Y.; Silva, D.E. The relationship between work conditions and perceptions of organizational justice among library employees. Libr. Inf. Sci. Res. 2021, 43, 101093.

[53] Akram, T.; Lei, S.; Haider, M.J.; Hussain, S.T. The impact of organizational justice on employee innovative work behavior: Mediating role of knowledge sharing. J. Innov. Knowl. 2020, 5, 117–129.

[54] Özer, N.; Demirtaş, H. Students' perceptions regarding the fairness of learning environment in faculty of education. Egit. Arastirmalari-Eurasian J. Educ. Res. 2010, 38, 126–145.

[55] Lizzio, A.; Wilson, K.; Hadaway, V. University students' perceptions of a fair learning environment: A social justice perspective. Assess. Eval. High. Educ. 2007, 32, 195–213.

[56] Smarandache, F. A. (1999). Unifying Field in Logics: Neutrosophic Logic. In: Philosophy, *American Research Press*, 1-141.

[57] Elhassouny, A., Idbrahim, S. and Smarandache, F. (2019). Machine Learning in Neutrosophic Environment: A survey. *Infinite Study*.

[58] Jha, S., Kumar, R., Son, L. H., Chatterjee, J. M., Khari, M. and Yadav, N. and Smarandache, F. (2019). Neutrosophic Soft Set Decision Making for Stock Trending Analysis. *Evolving Systems*, 10, 621–627.

[59] Liu, P. and Liu, X. (2018). The Neutrosophic Number Generalized Weighted Power Averaging Operator and its Application in Multiple Attribute Group Decision Making. *International Journal of Machine Learning and Cybernetics*, 9, 347-358.

[60] Abdel-Basset, M., Saleh, M., Gamal, A. and Smarandache, F. (2019). An Approach of Topsis Technique for Developing Supplier Selection with Group Decision Making under Type-2 Neutrosophic Number. *Applied Soft Computing*, 77, 438–452.

[61] Bhattacharya, S. (2005). Neutrosophic Information Fusion Applied to the Options Market. *Investment Management and Financial Innovations*, (2, Iss. 1), 139–145.

[62] Chang, K. H. A Novel Risk Ranking Method Based on the Single Valued Neutrosophic Set. *Infinite Study* 2022.

[63] Zadeh, L. A. (1965). Fuzzy Sets. *Information and Control*, 8(3), 338-353.

[64] Atanassov, K. Intuitionistic fuzzy sets. In Proceedings of VII ITKR's Session, Sofia, (1983).

[65] Wang, H., Smarandache, F., Zhang, Y. and Sunderraman, R. (2012). Single Valued Neutrosophic Sets. *Review of the Air Force Academy*.

[66] Smarandache, F. (2019). Introduction to Neutrosophic Sociology (Neutrosociology). *Infinite Study (2019), ISBN: 978-1-59973-605-1*.

[67] Martínez, C. R., Hidalgo, G. A., Matos, M. A. and Smarandache, F. (2020). Neutrosophy for Survey Analysis in Social Sciences. *Neutrosophic Sets and Systems*, 37, 1.

[68] Jarr´ın, A.A.A.; Tamayo, D.S.P.; Giler, S.A.M.; Zambrano, J.C.A.; Fernandez, D.M.M. Neutrosophic statistics applied in social science. *Neutrosophic Sets and Systems* 2021, 44, 01–09.

[69] Şahin, R. (2011). Multi-Criteria Neutrosophic Decision Making Method Based on Score and Accuracy Functions under Neutrosophic Environment. *Applied Mathematics Information Sciences*, 5.

[70] Duran, V., Topal, S., and Smarandache, F. (2021). An Application of Neutrosophic Logic in the Confirmatory Data Analysis of the Satisfaction with Life Scale. *Journal of Fuzzy Extension and Applications*, 239(3), 262–282.

[71] Field, A.P. Discovering Statistics Using SPSS: (And Sex and Drugs and Rock'n'roll); Sage: Thousand Oaks, CA, USA, 2009.

[72] Hair, J.F.; Black, W.C.; Babin, B.J.; Anderson, R.E. Multivariate Data Analysis; Pearson Education, Ltd.: London, UK, 2013.

[73] Büyüköztürk, Ş., Kılıç-Çakmak, E., Akgün, Ö., Karadeniz, Ş., & Demirel, F. (2008). Bilimsel araştırma yöntemleri, Pegem Akademi, p.113, http://dx.doi.org/10.14527/9789944919289

[74] Cohen, J. *Statistical Power Analysis for the Behavioral Sciences*, 2nd ed.; Lawrence Erlbaum Associates: Hillsdale, NJ, USA, 1998.

[75] Więckowski, J.; Kizielewicz, B.; Shekhovtsov, A.; Sałabun, W. RANCOM: A novel approach to identifying criteria relevance based on inaccuracy expert judgments. *Engineering Applications of Artificial Intelligence, ScienceDirect*. 2023. Vol 122, 106114.

Received: Sep 6, 2024. Accepted: Dec 4, 2024

REGIME Framework for Performance Ability Evaluation of Excellent Football Players with Probabilistic Simplified Neutrosophic Sets

Dongdong Liu*
Physical Education Department, Inner Mongolia Medical University, Hohhot, 010059, Inner Mongolia, China
*Corresponding author, e-mail: 20070119@immu.edu.cn

Abstract
The evaluation of an excellent football player's performance abilities involves assessing their overall capabilities based on in-game performance. Key evaluation dimensions include technical skills (such as passing, shooting, dribbling, etc.), tactical awareness, physical fitness, mental resilience, and teamwork. Through data analysis, video review, and coach feedback, the player's impact and contribution to the game are comprehensively evaluated, helping to determine their actual level in matches and provide guidance for future development. The performance ability evaluation of excellent football players is multiple-attribute decision-making (MADM). The REGIME proves to be an effective method for tackling MADM challenges. The Probabilistic Simplified Neutrosophic Sets (PSNSs) is particularly adept at handling the uncertainties inherent in the performance ability evaluation of excellent football players. In this research, the average technique is introduced to determine the weights of various attributes, and the Probabilistic Simplified Neutrosophic Number REGIME (PSNN-REGIME) method is advanced for MADM applications. To validate the effectiveness of the PSNN-REGIME method, a numerical case study involving the performance ability evaluation of excellent football players is conducted, complete with comparative analysis. This approach not only underscores the method's applicability but also enhances the decision-making process in evaluating and improving the performance ability evaluation of excellent football players.

Keywords: Multiple-attributes decision-making (MADM); probabilistic simplified neutrosophic sets (PSNSs); REGIME technique; performance ability evaluation of excellent football players

1. Overview and Background

The evaluation of an excellent football player's performance abilities is a systematic process that analyzes and assesses a player's overall performance in matches. First, in terms of technical skills, the evaluation includes basic techniques such as passing, shooting, dribbling, and ball control, examining the precision and practicality of these actions during the game. Secondly, tactical awareness is crucial, as players need to have a good understanding of the game, make quick decisions, and adjust their positioning and actions based on the match situation. Physical fitness, including speed, endurance, and explosiveness, is another important indicator, directly influencing a player's performance and consistency throughout the game. Additionally, mental resilience is equally important. Top players need to remain calm, confident,

and able to handle pressure in critical moments. Lastly, teamwork is an indispensable part of modern football. Players must not only perform individually but also have the ability to work seamlessly with teammates to achieve collective goals. By evaluating these aspects comprehensively, coaches and evaluators can gain a deeper understanding of a player's abilities, identify their strengths and weaknesses, and provide targeted advice and guidance for their training and development. Since 2000, research on the match running ability of Chinese football players, especially female players, has gradually increased, focusing on overall ability, positional physical characteristics, high-intensity running, and its relationship with technical performance. In chronological order, Sun et al. [1] investigated the match performance of elite Chinese female football players, highlighting differences between domestic and international teams. They found that Chinese players spent 40.9%-54.4% of match time running, with a run-walk-stop ratio of 1:1.7:0.28 and an average activity distance of 5128 meters. This study provided a foundation for maintaining China's international competitiveness. Later, Sun et al. [2] further analyzed the performances of 13 Chinese and 8 foreign women's teams, concluding that wing attacks had the highest success rate and passing combinations were most effective. These findings supported tactical and technical training for Chinese women's football teams. In 2009, Li [3] explored the physical characteristics of players in different positions, proposing the concept of "positional fitness." She outlined the specific needs of each position: forwards required speed and agility, midfielders needed speed endurance, and defenders relied on speed and strength. Additionally, she emphasized that running ability was the core of football-specific fitness and highly correlated with match outcomes. In the same year, Yu, Liu and Li [4] studied male football players' running abilities, revealing significant differences across matches and positions, providing a reference for women's football research. In the 2010s, Bu [5] focused on the relationship between running ability and match performance in high-level women's football. He observed a decline in running ability in the second half, particularly in high-intensity activities above the anaerobic threshold. This decline was accompanied by reduced technical abilities, such as ball control and short-pass success rates, highlighting the critical role of high-intensity activity capacity in match performance. Finally, Xu [6] compared the running abilities of Chinese and international female football players. She found that while Chinese players had lower high-intensity running abilities, their total running distances were comparable. Xu developed a standard for evaluating running abilities, providing important guidance for the scientific training of Chinese women's football teams. In conclusion, these studies underscore the importance of running ability in matches. Through longitudinal studies, positional analyses, and international comparisons, they offer valuable insights for improving the training and performance of Chinese football players

MADM is a decision-making process that prioritizes solutions in alignment with existing information through specific methodologies [7-10]. Its theories and models are extensively utilized across various domains such as venture-capital decision-making, project evaluation, and industrial sector development evaluation [11-14]. In recent decades, MADM has seen broad applications in different fields [15-17]. The performance ability evaluation of excellent football players constitutes a MADM. Recently, the TOPSIS approach [18-21] and Cross-Entropy [22] has been employed to enhance MADM. PSNSs [23] are

employed as a technique to characterize uncertain data during the performance ability evaluation of excellent football players.

Hinloopen, Nijkamp, and Rietveld first proposed the REGIME technique in 1983. It is a multiple attribute qualitative approach that uses the REGIME matrix to answer the problem and ranks the options. The decision maker's introduction of the weight of attributes in the final ranking is significant and can affect the outcomes. Decision-making challenges are ranked using this method. The following characteristics of the REGIME approach, which is applied in many fields: It's among the compensatory techniques. The qualities don't depend on one another; The qualitative attributes do not have to be transformed into quantitative attributes [24,25].

In this study, the average approach is proposed to obtain the weights under PSNSs, and the PSNN-REGIME model is introduced to address MADM with PSNSs. Finally, a numerical study on the performance ability evaluation of excellent football players is conducted to validate the PSNN- REGIME approach.

1.1 Relevance and Objectives of the Study

The increasing competitiveness of professional football, coupled with the demand for scientific training methods, underscores the importance of systematically evaluating players' match performance. Elite football players require comprehensive assessments that account for both physical and technical abilities to ensure peak performance during matches. This study's primary objective is to develop a robust and systematic evaluation method for assessing the match performance of elite football players. By introducing the PSNN- REGIME model integrated with the average technique, this research aims to enhance decision-making processes in player evaluation. The relevance lies in providing a scientific framework for improving training effectiveness and match strategies, particularly in optimizing high-intensity performance and tactical execution under competitive conditions.

The overall structure is as follows: Section 2 introduces the PSNSs. Section 3 presents the PSNN-REGIME approach for MADM. Section 4 provides a numerical example for performance ability evaluation of excellent football players, demonstrating the effectiveness of the PSNN- REGIME approach through a comparative analysis. Finally, Section 5 concludes the study.

2. Preliminaries

In recent years, MADM techniques have gained significant attention due to their versatility and wide-ranging applications in fields such as engineering, economics, and resource management. Among these approaches, fuzzy set theory has emerged as a powerful tool for addressing uncertainty and vagueness in information. Building on this foundation, Wang et al. [26] introduced the concept of Single-Valued Neutrosophic Sets (SVNSs), representing a groundbreaking advancement in the realm of neutrosophic logic and decision analysis.

The SVNSs model stands out for its ability to capture the inherent vagueness in real-world problems, offering a robust mathematical structure for quantitative and qualitative analysis. It has been effectively

applied in various domains to evaluate alternatives across multiple criteria, providing a clear and comprehensive approach to decision-making. By addressing the limitations of traditional fuzzy sets, SVNSs enable a more flexible and realistic representation of uncertain information.

Definition 1 **[28]**. SVNSs were introduced to extend fuzzy and intuitionistic fuzzy sets by incorporating truth, indeterminacy, and falsity memberships, providing a more comprehensive framework for handling uncertainty and vagueness in decision-making as

$$UU=\left\{\left(\theta,UT\left(\theta\right),UI\left(\theta\right),UF\left(\theta\right)\right)\middle|\theta\in\Theta\right\} \quad (1)$$

where $UT\left(\theta\right),UI\left(\theta\right),UF\left(\theta\right)$ is membership, indeterminacy-membership and falsity-membership, $UT\left(\theta\right),UI\left(\theta\right),UF\left(\theta\right)\in\left[0,1\right]$, $0\le UT\left(\theta\right)+UI\left(\theta\right)+UF\left(\theta\right)\le 3$.

Definition 2 **[23]**. The Probabilistic Simplified Neutrosophic Sets (PSNSs) were introduced as a framework to integrate probabilistic concepts into the simplified neutrosophic set theory. PSNSs enhance the modeling of uncertainty by incorporating probability distributions for truth, indeterminacy, and falsity memberships, enabling more robust and precise decision-making in multi-criteria contexts.

$$PU=\left\{\left(\theta,UT\left(\theta\right)\left(PUT\left(\theta\right)\right),UI\left(\theta\right)\left(PUI\left(\theta\right)\right),UF\left(\theta\right)\left(PUF\left(\theta\right)\right)\right)\middle|\theta\in\Theta\right\} \quad (2)$$

where $UT\left(\theta\right),UI\left(\theta\right),UF\left(\theta\right)$ is truth-membership, indeterminacy-membership and falsity-membership, $UT\left(\theta\right),UI\left(\theta\right),UF\left(\theta\right)\in\left[0,1\right]$, $0\le UT\left(\theta\right)+UI\left(\theta\right)+UF\left(\theta\right)\le 3$, $0\le PUT\left(\theta\right),PUI\left(\theta\right),PUF\left(\theta\right)\le 1$, the $PUT\left(\theta\right),PUI\left(\theta\right),PUF\left(\theta\right)$ is possibility values of $UT\left(\theta\right),UI\left(\theta\right),UF\left(\theta\right)$. The probabilistic simplified neutrosophic number (PSNN) is listed as $PU=\left(UT\left(PUT\right),UI\left(PUI\right),UF\left(PUF\right)\right)$.

Definition 3 **[23].**

Let $PU_1=\left(UT_1\left(PUT_1\right),UI_1\left(PUI_1\right),UF_1\left(PUF_1\right)\right)$, $PU_2=\left(UT_2\left(PUT_2\right),UI_2\left(PUI_2\right),UF_2\left(PUF_2\right)\right)$, the basic operations are:

$$(1)\ PU_1 \oplus PU_2 = \begin{pmatrix} UT_1 + UT_2 - UT_1 \cdot UT_2\left(2!\dfrac{PUT_1 \cdot PUT_2}{PUT_1 + PUT_2}\right), \\ UI_1 \cdot UI_2\left(2!\dfrac{PUI_1 \cdot PUI_2}{PUI_1 + PUI_2}\right), UF_1 \cdot UF_2\left(2!\dfrac{PUF_1 \cdot PUF_2}{PUF_1 + PUF_2}\right) \end{pmatrix};$$

$$(2)\ PU_1 \otimes PU_2 = \begin{pmatrix} UT_1 \cdot UT_2\left(2!\dfrac{PUT_1 \cdot PUT_2}{PUT_1 + PUT_2}\right), \\ UI_1 + UI_2 - UI_1 \cdot UI_2\left(2!\dfrac{PUI_1 \cdot PUI_2}{PUI_1 + PUI_2}\right), \\ UF_1 + UF_2 - UF_1 \cdot UF_2\left(2!\dfrac{PUF_1 \cdot PUF_2}{PUF_1 + PUF_2}\right) \end{pmatrix};$$

$$(3)\ \lambda PU = \left(1-\left(1-UT\right)^{\lambda}\left(PUT\right), \left(UI\right)^{\lambda}\left(PUI\right), \left(UF\right)^{\lambda}\left(PUF\right)\right), \lambda > 0;$$

$$(4)\ \left(PU\right)^{\lambda} = \left(\left(UT\right)^{\lambda}\left(PUT\right), 1-\left(1-UI\right)^{\lambda}\left(PUI\right), 1-\left(1-UF\right)^{\lambda}\left(PUF\right)\right), \lambda > 0.$$

Definition 4 **[27].**

The score function of the PSNN can be computed as:

$$S(a) = \frac{UT\ PUT + (1-UI\ PUI) + (1-UF\ PUF)}{3} \quad (3)$$

3. PSNN- REGIME Technique for MADM problem

This section shows the steps of the proposed method as:
Step 1. Build the decision matrix

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix} \quad (4)$$

Step 2. Compute criteria weights.

Step 3. Compute the superiority index.

After computing the criteria weights, the set of criteria in which alternative A_f is at least as good as the alternative A_l, displayed by E_{fl}.

Step 4. Compute the superiority identifier.

$$E_{fl} = \sum_{j \in E_{fl}} w_j ; j = 1, \ldots, n \quad (5)$$

Step 5. Build the impacts matrix.

The impact matrix is derived from ordering the alternatives based on the criteria that order the alternative from the decision matrix information.

Step 6. Build the REGIME matrix

The REGIME matrix is built from a pairwise comparison of alternatives.

If two alternatives of $A_1, A_2 \in A$ *are considered, the status of* A_1, A_2 *alternatives should be compared to each other in all criteria.*

$$E_{fl_j} = \begin{cases} -1 & if\ r_{fj} < r_{lj} \\ 0 & if\ r_{fj} = r_{lj} \\ +1 & if\ r_{fj} > r_{lj} \end{cases} \quad (6)$$

Step 7. Compute the guide index.

$$E_{fl} = \sum_{j=1}^{n} E_{fl_j} w_j \quad (7)$$

Step 8. Rank the alternatives.

4. Numerical Study and Comparative Analysis: Insights and Results

This section presents a detailed numerical study to demonstrate the practical application and effectiveness of the proposed PSNN- REGIME approach in solving MADM problems.

4.1 Numerical Study

The evaluation of an excellent football player's performance abilities in a match is a comprehensive and multi-faceted process that goes beyond mere statistics and numbers. It involves a detailed analysis of the player's overall impact on the game, their consistency, and their ability to influence the outcome of a match. This evaluation is crucial not only for determining the player's current skill level but also for shaping future development and optimizing their role within a team. To begin with, the evaluation process focuses on how the player contributes to the overall dynamics of the game. A top-level football player is expected to consistently influence the match, whether through direct actions like goals and assists, or indirect contributions such as creating space for teammates, pressing the opposition, or disrupting the opponent's tactics. Their ability to adapt to different phases of the game—whether attacking, defending, or transitioning between the two—is a key consideration. An excellent player stands out not just for individual moments of brilliance, but for their sustained influence over the course of 90 minutes. Another crucial aspect of evaluating a football player's match performance is their decision-making ability. Football is a fast-paced game that requires players to make split-second decisions under pressure. An excellent player is one who consistently makes the right choices—whether it's deciding to pass, dribble, shoot, or hold the ball. Their awareness of the game, understanding of the situation, and ability to make decisions that positively impact their team's chances are critical to their overall performance. Evaluators look at how often the player makes decisions that lead to positive outcomes, as well as how well they balance risk and reward in key moments. Consistency is another key factor in the evaluation of a football player's match performance. It's not enough for a player to have flashes of brilliance; they need to perform at a high level regularly across multiple games and throughout the entirety of a season. Evaluators consider whether the player can maintain their level of play in different conditions—against stronger or weaker opponents, at home or away, and in high-pressure situations such as finals or tournament knockout stages. Consistency is what separates good players from great ones, as top players are able to deliver strong performances week in and week out. Moreover, the evaluation also considers how the player handles pressure and their mental resilience during the match. Football can be an emotionally and physically demanding sport, particularly in high-stakes matches or when things aren't going in the team's favor. An excellent player is one who can remain focused, composed, and determined even in the face of adversity. Whether dealing with a hostile

crowd, a tough opponent, or a crucial game moment, the ability to stay mentally strong is vital for top-level performance. How a player responds to setbacks, such as missing a scoring opportunity or conceding a goal, can reveal much about their mental fortitude. Additionally, the evaluation takes into account the player's leadership and influence on their teammates. Even players who are not formally captains can demonstrate leadership qualities by guiding younger teammates, motivating the squad, or setting an example through their work ethic and determination. Players who can inspire those around them often have a greater impact on the overall team performance and are highly valued in any team setup. In conclusion, the evaluation of an excellent football player's match performance is not limited to individual skills or statistics. It encompasses a broader understanding of how the player integrates into the team, how consistently they contribute to the game, their decision-making under pressure, and their overall influence on the match. This holistic approach ensures that the player's true value is recognized, not just in terms of their technical abilities but also in their mental strength, leadership, and capacity to positively affect the team's performance.

To evaluate the performance ability of excellent football players, a MADM framework is employed. Eight excellent football teams are assessed based on 22 attributes, as shown in Table 1.

Table 1. Weight Allocation for 22 Criteria

C	Criteria	Weights
C_1	Creativity and Innovation on the Field	0.04526
C_2	Teamwork and Communication	0.045843
C_3	Performance Under Pressure	0.044677
C_4	Discipline and Sportsmanship	0.044921
C_5	Statistical Performance Metrics	0.044253
C_6	Consistency in Performance	0.045918
C_7	Tactical Awareness and Decision-Making	0.047206
C_8	Positioning and Spatial Awareness	0.045608
C_9	Flexibility in Playing Multiple Roles or Positions	0.044931
C_{10}	Ball Retention and Control Under Opposition Pressure	0.044592
C_{11}	Aerial Ability and Heading Skills	0.044103
C_{12}	Work Rate and Commitment	0.045852
C_{13}	Physical Fitness	0.047206
C_{14}	Mental Toughness and Resilience	0.045767
C_{15}	Proficiency in Set Pieces	0.044592
C_{16}	Defensive Skills	0.045598
C_{17}	Dribbling and One-on-One Situations	0.044376
C_{18}	Leadership and Influence on the Team	0.045354
C_{19}	Technical Skills	0.047611
C_{20}	Game Intelligence and Vision	0.046238
C_{21}	Ability to Adapt to Different Game Scenarios	0.045288
C_{22}	Recovery and Injury Prevention	0.044808

Step 1. We built the decision matrix using the neutrosophic numbers between the criteria and alternatives as shown in Table 2.

Table 2. Neutrosophic Numbers Representation

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))
C_2	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_3	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_4	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_5	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_6	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_7	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_8	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_9	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{10}	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_{11}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_{12}	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_{13}	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_{14}	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_{15}	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{16}	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_{17}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_{18}	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_{19}	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_{20}	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_{21}	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{22}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))
C_2	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_3	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_4	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_5	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_6	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_7	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_8	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_9	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{10}	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_{11}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_{12}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_{13}	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_{14}	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_{15}	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{16}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_{17}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_{18}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))
C_{19}	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.7(0.9),0.2(1),0.4(0.8))
C_{20}	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_{21}	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{22}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))
C_2	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_3	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_4	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_5	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_6	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))
C_7	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))
C_8	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_9	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.4(1),0.1(1),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{10}	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_{11}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))

C_{12}	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_{13}	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))
C_{14}	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))
C_{15}	(0.7(0.9),0.2(1),0.4(0.8))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{16}	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.4(1),0.1(1),0.2(0.9))
C_{17}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))
C_{18}	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.5(0.9),0.3(0.9),0.5(0.8))
C_{19}	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))
C_{20}	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))
C_{21}	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.6,(0.8),0.3(0.9),0.2(0.9))
C_{22}	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))	(0.8(1),0.3(0.8),0.5(0.7))	(0.5(0.9),0.3(0.9),0.5(0.8))	(0.7(0.9),0.2(1),0.4(0.8))	(0.8(1),0.3(0.8),0.5(0.7))	(0.3(0.7),0.2(0.9),0.3(0.9))	(0.4(1),0.1(1),0.2(0.9))

Step 2: Calculate the weights of the criteria based on the method described and as shown in Table 1. This step helps determine the relative importance of each criterion in the decision-making process.

Step 3: Compute the superiority index. This index helps measure the relative performance or dominance of the alternatives against each other.

Step 4: Use Equation (5) to calculate the superiority identifier. This calculation quantifies the superiority of each alternative more precisely.

Step 5: Construct the impacts matrix. This matrix represents the influence or effect of each criterion on the alternatives.

Step 6: Build the REGIME matrix, which is a decision matrix used to structure and analyze the rankings of the alternatives.

Step 7: Compute the guide index, which provides additional insights into the overall evaluation of alternatives.

Step 8: Finally, rank the alternatives based on the calculations and present the results as shown in Figure 1. This step identifies the most suitable alternative(s).

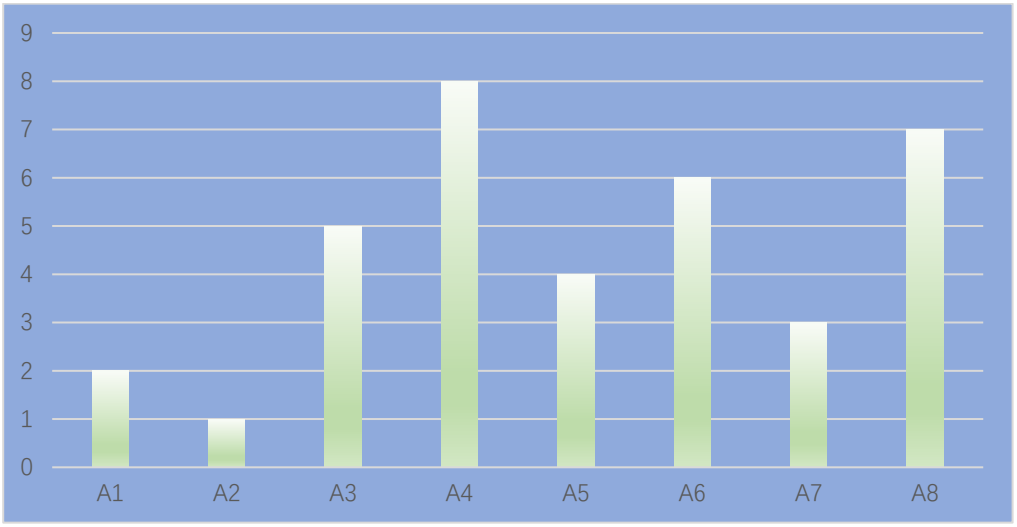


Figure 1. Prioritization of Alternatives

4.2. Comparative analysis

The PSNN-REGIME is compared with the PNN-VIKOR PNN-TOPSIS, and the PNN-PROMETHEE technique [23]. Figure 2 compares different techniques used for ranking and selecting the best category. Each method agreed the best alternative is A4 and the worst alternative is A2.

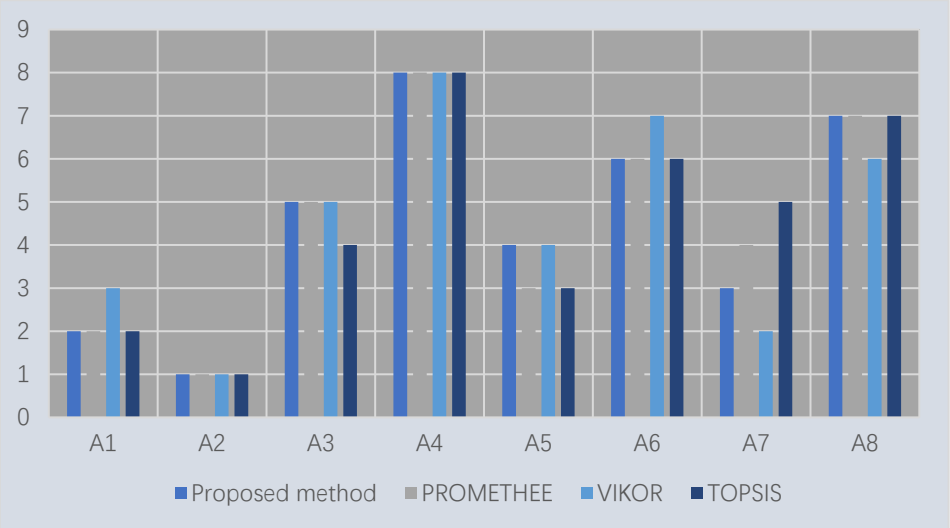


Figure 2. Comparison of Orders Across Different Techniques

Figure 2 represents the ranking orders of categories (A1 to A8) for each technique. Each point shows the rank of a category (along the horizontal axis) according to a specific technique (listed on the vertical axis). A4 consistently ranks the highest across all techniques, it as the optimal choice. The positions of the remaining categories (A1, A2, A3, ….. A8) show slight variations depending on the technique.

From Figure 2, it is clear that all models consistently identify the best and worst football team, despite minor differences in rankings. This highlights the reliability and effectiveness of the PSNN-REGIME method in evaluating the performance ability of excellent football players.

Compared to existing methods like VIKOR, TOPSIS, and PSNN-PROMETHEE, the PSNN-REGIME method offers several key advantages:

PSNN-REGIME effectively processes uncertain data by combining probabilistic analysis and similarity to an ideal solution, providing more reliable results.

It optimizes computations to ensure both high decision accuracy and quick processing, making it suitable for a variety of scenarios.

PSNN-REGIME is designed to handle diverse datasets and complex issues, making it more flexible and adaptable than other methods.

PSNN-REGIME is a robust and versatile tool, excelling in areas such as uncertainty handling, computational balance, and applicability, making it a valuable method for evaluating the performance ability of excellent football players.

We change the criteria weights by 12 cases to show the rank of alternatives under different weights. Figure 3 shows the different criteria for weights. Then we applied the proposed method under different criteria weights as shown in Figure 4. The results show the proposed method is stable under different criteria weights. The best alternative is A4 and the worst alternative is A2.

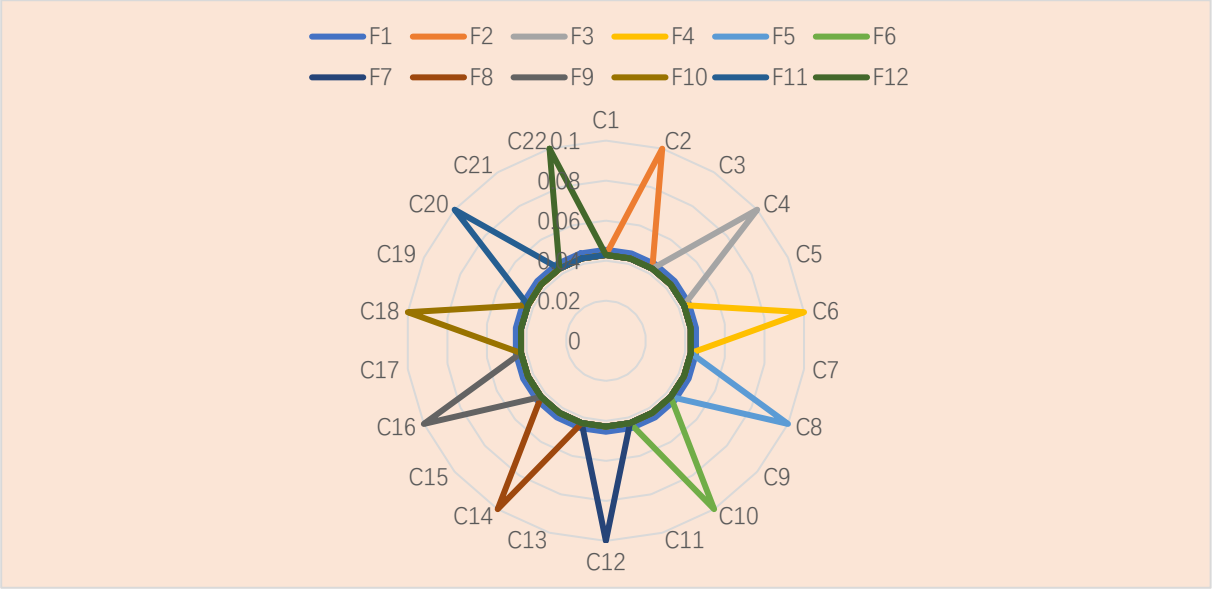


Figure 3. Different criteria weights.

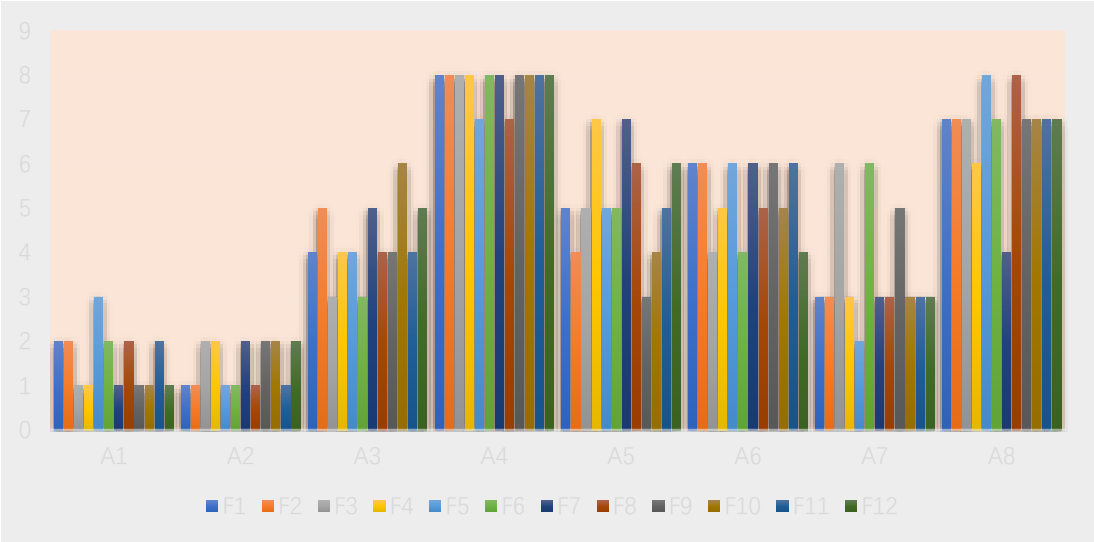


Figure 4. Different ranks of alternatives.

4.3 Practical Implications for Performance Ability of Excellent Football Players

The study's findings offer actionable insights into improving the performance ability of excellent football players:

- The PSNN- REGIME model can be used to identify performance weaknesses in elite football players and allocate training resources effectively to address these areas.
- Emphasizing skill development and continuous education for coaches, analysts, and support staff to enhance the accuracy and effectiveness of player performance evaluations.
- Addressing gaps in performance evaluation by increasing data collection points during matches and optimizing the frequency and timing of assessments.

4.4 Policy Recommendations

- Policymakers can benefit from the study by adopting the following recommendations:

— Develop a standardized framework for performance ability evaluation of excellent football players.
— Enhance coordination between physical training, technical coaching, and sports science systems to streamline the evaluation and improvement of players' match performance.
— Provide financial and logistical support to underperforming teams or training programs to encourage innovation and improvement in evaluating and enhancing players' match performance.

4.5 Challenges and Limitations

This study has several limitations:

— The evaluation focuses on a limited sample of five football teams, which may not fully represent the regional, tactical, and positional variations in elite football match performance.
— The PSNN- REGIME approach, while robust, requires technical expertise, potentially limiting its adoption in smaller or less-resourced institutions. Future studies should address these challenges by expanding datasets and simplifying model implementation.

5. Conclusions and future research

The evaluation of a football player's match performance is crucial for understanding their overall impact on the game and their potential for future development. It goes beyond simply looking at technical skills or statistics; it provides a comprehensive insight into how the player contributes to the team's success. By evaluating key aspects such as decision-making, consistency, mental toughness, and teamwork, coaches and scouts can determine whether the player has the qualities necessary to thrive in competitive environments. The significance of such evaluations lies in their ability to identify strengths and areas for improvement. For coaches, it helps in tailoring training programs to enhance a player's weaknesses, while also leveraging their strengths for team strategy. For scouts, it aids in making informed recruitment decisions by assessing how well a player can adapt to different game situations and their potential to grow. Moreover, performance evaluations help players themselves understand their role in the team and the areas they need to work on to reach higher levels of competition. Overall, evaluating match performance is an essential tool for optimizing both individual and team success, making it a key component in the development of top-level football talent.

This study focuses on evaluating the performance ability of excellent football players using a MADM approach. The average method is proposed to calculate attribute weights under PSNSs, and the PSNN-REGIME model is introduced for decision-making. A numerical study validates the model’s effectiveness. The key contributions of this study include: (1) extending the REGIME model to PSNSs, (2) using the average technique for weight determination, (3) applying the PSNN- REGIME method for MADM under PSNSs, and (4) conducting a comparative analysis to validate the approach.

While the study has made progress in evaluating the match performance of elite football players, it has certain limitations. The data used in the evaluation may lack diversity in terms of leagues, player

positions, and match contexts, which could affect the comprehensiveness of the assessment. Additionally, the complexity of the PSNN- REGIME method may pose challenges for practical application in football clubs or training institutions that lack technical expertise or resources. Lastly, the absence of long-term impact evaluations makes it difficult to determine the sustained effectiveness of the proposed framework in enhancing players' match performance and team outcomes.

Future research could address these challenges by: (a) expanding data sources to include diverse leagues, player positions, and competitive levels to ensure a more comprehensive assessment, (b) exploring alternative evaluation methods, such as fuzzy logic or neural networks, to develop more practical and adaptable solutions for real-world applications, and (c) conducting long-term studies to monitor the impact of these evaluation methods on player development, team performance, and match outcomes. These efforts could significantly improve the accuracy and effectiveness of player performance assessments, addressing the evolving demands of modern football.

REFERENCE

[1] W. Sun, F. Wang, Z. Du, G. Yang, J. Zhang, R. Pan, G. Zeng, Y. Zheng, A study on the current match abilities of chinese elite female football players, in: Abstract Compilation of the Sixth National Sports Science Conference (Part II), China Sport Science Society, Guangzhou, 2000, pp. 212.

[2] W. Sun, F. Wang, Z. Du, G. Yang, J. Zhang, R. Pan, G. Zeng, Y. Zheng, A study on the current match abilities of chinese elite female football players, China Sport Science and Technology, (2001).

[3] J. Li, Research on positional physical characteristics and match running ability of chinese elite female football players, in, Soochow University, China, 2009.

[4] S. Yu, D. Liu, Q. Li, A study on the match running ability of chinese elite male football players, China Sport Science and Technology, 45 (2009) 34-40.

[5] Y. Bu, A study on the relationship between running ability and match performance of elite female football players, China Sport Science and Technology, 50 (2014) 3-9.

[6] M. Xu, Comparative study on the running ability of domestic and international elite female football players, Zhejiang Sports Science, 40 (2018) 52-55.

[7] D. Pamucar, A.E. Torkayesh, S. Biswas, Supplier selection in healthcare supply chain management during the covid-19 pandemic: A novel fuzzy rough decision-making approach, Annals of Operations Research, 328 (2023) 977-1019.

[8] D. Pamucar, I. Gokasar, A.E. Torkayesh, M. Deveci, L. Martinez, Q. Wu, Prioritization of unmanned aerial vehicles in transportation systems using the integrated stratified fuzzy rough decision-making approach with the hamacher operator, Information Sciences, 622 (2023) 374-404.

[9] H. Garg, K. Ullah, K. Ali, M. Akram, M.N. Abid, Multi-attribute decision-making based on sine trigonometric aggregation operators for t-spherical fuzzy information, Soft Computing, (2023) 15.

[10] H. Garg, Z. Ali, T. Mahmood, M.R. Ali, A. Alburaikan, Schweizer-sklar prioritized aggregation

operators for intuitionistic fuzzy information and their application in multi-attribute decision-making, Alexandria Engineering Journal, 67 (2023) 229-240.

[11] J. Ye, B.Z. Sun, X.L. Chu, J.M. Zhan, J.X. Cai, Valued outranking relation-based heterogeneous multi-decision multigranulation probabilistic rough set and its use in medical decision-making, Expert Systems with Applications, 228 (2023) 18.

[12] J. Ye, B.Z. Sun, X.L. Chu, J.M. Zhan, Q. Bao, J.X. Cai, A novel diversified attribute group decision-making method over multisource heterogeneous fuzzy decision systems with its application to gout diagnosis, Ieee Transactions on Fuzzy Systems, 31 (2023) 1780-1794.

[13] M. Akram, C. Kahraman, K. Zahid, Extension of topsis model to the decision-making under complex spherical fuzzy information, Soft Computing, 25 (2021) 10771-10795.

[14] C. Jana, H. Garg, M. Pal, B. Sarkar, G.W. Wei, Mabac framework for logarithmic bipolar fuzzy multiple attribute group decision-making for supplier selection, Complex & Intelligent Systems, 10 (2024) 273-288.

[15] G. Sirbiladze, H. Garg, I. Khutsishvili, B. Ghvaberidze, B. Midodashvili, Associated probabilities aggregations in multistage investment decision-making, Kybernetes, 52 (2023) 1370-1399.

[16] L. Li, X. Wang, Y. Lin, X. Huang, A decision making method for improving service quality based on three-dimension kano model, Engineering Letters, 26 (2018) 415-423.

[17] D. Xu, Y. Hong, H. Qiao, Multiple attribute decision making based on neutrosophic sets in venture capital, Engineering Letters, 26 (2018).

[18] Y.-J. Lai, T.-Y. Liu, C.-L. Hwang, Topsis for modm, European Journal of Operational Research, 76 (1994) 486-500.

[19] C.T. Chen, Extensions of the topsis for group decision-making under fuzzy environment, Fuzzy Sets and Systems, 114 (2000) 1-9.

[20] F. Lei, G.W. Wei, H. Gao, J. Wu, C. Wei, Topsis method for developing supplier selection with probabilistic linguistic information, International Journal of Fuzzy Systems, 22 (2020) 749-759.

[21] H.Y. Zhang, G.W. Wei, C. Wei, Topsis method for spherical fuzzy magdm based on cumulative prospect theory and combined weights and its application to residential location, Journal of Intelligent & Fuzzy Systems, 42 (2022) 1367-1380.

[22] X. Shang, W. Jiang, A note on fuzzy information measures, Pattern Recognition Letters, 18 (1997) 425-432.

[23] F. Altun, R. Sahin, C. Guler, Multi-criteria decision making approach based on promethee with probabilistic simplified neutrosophic sets, Soft Computing, 24 (2020) 4899-4915.

[24] E. Hinloopen and P. Nijkamp, “Qualitative multiple criteria choice analysis: the dominant regime method,” *Qual. Quant.*, vol. 24, pp. 37–56, 1990.

[25] A. Alinezhad, J. Khalili, A. Alinezhad, and J. Khalili, “REGIME method,” *New Methods Appl. Mult. Attrib. Decis. Mak.*, pp. 9–15, 2019.

[26] S.Q. Wang, G.W. Wei, J.P. Lu, J. Wu, C. Wei, X.D. Chen, Grp and critic method for probabilistic uncertain linguistic magdm and its application to site selection of hospital constructions, Soft Computing, 26 (2022) 237-251.

[27] F. Altun, R. Şahin, and C. Güler, "Multi-criteria decision making approach based on PROMETHEE with probabilistic simplified neutrosophic sets," *Soft Comput.*, vol. 24, no. 7, pp. 4899–4915, 2020.

[28] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, (2010) 410-413.

Received: Sep 7, 2024. Accepted: Dec 5, 2024

Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making

K. Radhika[1], S. Senthil[2] , N. Kavitha[3], R.Jegan[4], M.Anandhkumar[5], A. Bobin[6]

[1]Department of Mathematics, Rajah Serfoji Government College (Autonomous) (Affiliated to Bharathidasan University, Tiruchirappalli), Thanjavur - 613005, Tamilnadu, India.
radhi9791222789@gmail.com

[2]Department of Mathematics, Faculty of Science and Humanities, R.M.K. Engineering College, Chennai - 601 206, Tamilnadu, India
ssl.sh@rmkec.ac.in

[3]Department of Mathematics, Saveetha Engineering College, Thandalam,Chennai,Tamilnadu,India.
sakkavi238@gmail.com

[4]Department of Mathematics, Panimalar Engineering College, Chennai,Tamilnadu, India.
rjch12571@gmail.com

[5]Department of Mathematics, IFET College of Engineering (Autonomous), Villupuram, Tamilnadu, India.
anandhkumarmm@gmail.com

[6]Department of Mathematics, IFET College of Engineering (Autonomous), Villupuram, Tamilnadu, India.
bobinalbert@gmail.com

Abstract

The objective of this study is to establish the results concerning Interval-Valued (IV) Secondary k-Range Symmetric (RS) Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFM). We have applied the RS condition within the neutrosophic environment to explore the relationships between IVQP s−k-RS, s-RS, IVQP k-RS, and IVQP RS matrices. This analysis has yielded significant insights into how these various matrix types interrelate and their structural properties. We have established the necessary and sufficient criteria for IVQP s−k-RS IVQPNFM, along with various generalized inverses of an IVQP s−ks-RS fuzzy matrix to maintain its classification as an IVQP s−k- RS matrix. Furthermore, we have characterized the generalized inverses of an IVQP s−k- RS matrix S corresponding to the sets S = {1,2}, S = {1,2,3}, and S = {1, 2, 4}. This characterization contributes to the foundational understanding of generalized inverses in the context of IVQPNFM. Additionally, a graphical representation of RS, Column symmetric (CS), and kernel symmetric (KS) adjacency and incidence QPNFM is illustrated. It is shown that every adjacency QPNFM is symmetric, RS, CS, and KS, whereas the incidence matrix only satisfies KS conditions. Similarly, every RS adjacency QPNFM is a KS adjacency QPNFM, but a KS adjacency QPNFM does not necessarily imply RS QPNFM. In this paper, we present an application of soft graphs in

decision-making through the use of the adjacency matrix of a soft graph. We have developed an algorithm for this purpose and provide an example to demonstrate its application.

Keywords: IV Neutrosophic Fuzzy matrix, IV RS Neutrosophic fuzzy matrix, s-k- RS IV Neutrosophic fuzzy matrix.

1. Introduction

Interval-Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices are an advanced mathematical framework that extends traditional matrix theory to handle uncertainty, indeterminacy, and inconsistency in complex systems. This model integrates multiple mathematical concepts from neutrosophic logic, interval-valued fuzzy sets, and matrix theory to provide a versatile tool for representing and processing data in uncertain environments. Neutrosophic logic is an extension of fuzzy logic that introduces three membership functions—truth (T), falsity (F), and indeterminacy (I). Unlike traditional fuzzy logic, neutrosophic logic allows for the simultaneous existence of truth, falsity, and indeterminacy in different degrees for a given proposition or element. A neutrosophic fuzzy matrix represents a matrix whose elements are neutrosophic fuzzy sets. Each element in the matrix is characterized by an ordered triple (T,I,F) where:

- T represents the degree of truth,
- I represents the degree of indeterminacy,
- F represents the degree of falsity.

Interval-Valued: In this context, instead of single numerical values for T, I, and F, interval values are used to express the uncertainty or range of possibilities. For example, the truth value might be expressed as an interval [a,b], meaning the actual degree of truth lies somewhere between a and b. Similarly, indeterminacy and falsity values can also be represented as intervals.

Secondary k-Range: This refers to a secondary level of granularity or partitioning in the matrix. The parameter k-range could indicate a specific division or range within the matrix that is used to classify or analyze different sections of data. It helps in categorizing and analyzing more complex scenarios where single-range partitioning is not enough.

Symmetric Matrices: A matrix is called symmetric if it is equal to its transpose, i.e., $S = S^T$. In Interval-Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices, symmetry ensures that the relationship between the elements in the matrix is mirrored, which might be useful in modeling relationships like similarity, mutual influence, or mutual dependence.

Quadri Partitioned: The term "quadri partitioned" refers to the division of the matrix into four quadrants or partitions, where each partition could represent a different aspect or dimension of the problem being analyzed. These partitions allow for more detailed analysis and representation of multi-faceted data.

Matrices play a fundamental role in various scientific and engineering disciplines. However, traditional matrix theory often struggles to address problems that involve diverse forms of uncertainty. To address this limitation, Zadeh [1] introduced fuzzy sets (FS), which are characterized by membership values. However, assigning accurate membership values to a fuzzy set can sometimes pose challenges. To tackle this issue, Atanassov [2] developed intuitionistic fuzzy sets (IFS), which incorporate both

membership and non-membership values to better represent uncertainty. Building on this, Smarandache [3] introduced neutrosophic sets (NSs), a framework designed to handle indeterminate information and address situations involving imprecision, uncertainty, and inconsistency.

Fuzzy matrices have been applied to address specific types of problems, and numerous researchers have expanded on these applications. However, traditional fuzzy matrices only consider membership values and are unable to handle non-membership values. Khan, Shyamal, and Pal [4] were the first to investigate intuitionistic fuzzy matrices (IFMs). Atanassov [5,6] explored both intuitionistic fuzzy sets (IFS) and operations on interval-valued IFS (IV IFS). Hashimoto [7] worked on the canonical form of a transitive matrix, while Kim and Roush [8] examined generalized fuzzy matrices. Lee [9] delved into secondary skew-symmetric and secondary orthogonal matrices, and Hill and Waters [10] focused on k-real and k-Hermitian matrices. Meenakshi [11] contributed to the understanding of fuzzy matrix theory and its applications.

Measuring membership or non-membership as precise values in real-world scenarios often poses challenges. To address these, Anandhkumar [12,13] investigated pseudo-similarity in neutrosophic fuzzy matrices and explored various inverses of neutrosophic fuzzy matrices. Additionally, Punithavalli and Anandhkumar [14] studied reverse sharp and left-Tand right-T partial ordering on intuitionistic fuzzy matrices. Pal and Susanta Kha [15] have explored interval-valued intuitionistic fuzzy matrices, contributing to the understanding of how these matrices can handle uncertainty more effectively. Vidhya and Irene Hepzibah [16] have focused their research on interval-valued neutrosophic fuzzy matrices, further extending the concept by incorporating neutrosophic logic to address indeterminacy along with membership and non-membership values. Anandhkumar et al. [17] have examined reverse sharp and left-T and right-T partial ordering on neutrosophic fuzzy matrices, which enhances the mathematical framework for handling neutrosophic fuzzy relations and their applications.

Anandhkumar et al. [18] have examined reverse tilde T and minus partial ordering on intuitionistic fuzzy matrices, extending the theoretical framework for fuzzy systems. Additionally, Anandhkumar et al. [19] explored partial orderings, characterizations, and generalizations of k-idempotent neutrosophic fuzzy matrices, further enriching the study of neutrosophic fuzzy matrices in handling uncertainties.Building upon these foundational studies, we propose considering membership values as intervals, rather than fixed points, and similarly treating non-membership values as intervals. This approach accommodates a broader range of uncertainty. In this context, we introduce Interval-Valued (IV) Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices (IVSNRNFMs). Furthermore, we define basic operators on these interval-valued neutrosophic fuzzy matrices (IVNFMs), establishing a groundwork for future research and applications in fields that require handling of imprecise and indeterminate information.

The structure of the article is organized as follows:

In section 1, we present introduction,

In section 2 and 3, we introduce notation and some elementary definitions.

In section 4, we provide graphical representations of RS, CS, and KS adjacency matrices.

In section 5, we discuss Algorithm and application of adjacency Neutrosophic fuzzy matrix of a graph in decision making.

In section 6, we introduce Interval valued Secondary k-RS IVQPNFM

In section 7, we discuss IVQP s – k-RS regular Neutrosophic fuzzy matrices.

1.1 Literature Review

Meenakshi and Jaya Shree [20] have explored the concept of **k**-kernel symmetric matrices, while Meenakshi and Krishnamoorthy [21] characterized secondary **k**-Hermitian matrices. Additionally, Meenakshi and Jaya Shree [22] investigated **k**-range symmetric matrices, and Jaya Shree [23] delved into secondary **k-**kernel symmetric fuzzy matrices. Shyamal and Pal [24] contributed to the study of interval-valued fuzzy matrices, while Meenakshi and Kalliraja [25] examined regular interval-valued fuzzy matrices. Anandhkumar [26] explored kernel and **k**-kernel intuitionistic fuzzy matrices, and Jaya Shree [27] discussed secondary k**-**range symmetric fuzzy matrices. Anandhkumar et al. [28] worked on generalized symmetric neutrosophic fuzzy matrices, and Kaliraja and Bhavani [29] researched interval-valued secondary **k**-range symmetric fuzzy matrices. Let S be any fuzzy matrix. If S occurs, it coincides with the transpose of the matrix, denoted S^T. A fuzzy matrix S, belonging to Fn, is referred to as a kernel symmetric matrix if it satisfies $N(S) = N(S^T)$, where N(S) represents the null space of S. However, this does not imply that the range spaces of S and S^T are equal, i.e., $R(S) \neq R(S^T)$. Nevertheless, the converse holds true, meaning that if $R(S) = R(S^T)$, then $N(S) = N(S^T)$, indicating kernel symmetry.

Symmetric matrices play an essential role in the theory of complex matrix entries, particularly in the study of k-Hermitian matrices. This concept has led to the development of k-EP matrices, which generalize both k-Hermitian matrices and EP (equal projection) matrices. Hill and Waters [30] initiated the exploration of k-Hermitian matrices. Following this, Baskett and Katz [31] introduced theorems on the products of EPr matrices, which further expanded the understanding of matrix product theory. Meenakshi and Krishnamoorthy [32] contributed to this area by studying k-EP matrices.In the context of complex matrices, it is well established that the concepts of range symmetry and kernel symmetry are equivalent. However, this equivalence does not hold for interval-valued fuzzy matrices, where the relationship between range and kernel symmetry breaks down due to the inherent uncertainty in interval-valued membership and non-membership functions. This distinction makes the study of interval-valued fuzzy matrices more nuanced and highlights the need for further research into their unique properties.

The concept of interval-valued s−ks -Hermitian and interval-valued kernel symmetric matrices for fuzzy matrices provides a framework for handling uncertainty in complex systems. These matrices extend classical matrix theory by incorporating interval values to represent both membership and non-membership, offering a more flexible approach to manage fuzzy data.Additionally, we have expanded on several fundamental conclusions regarding these two types of matrices. Specifically, an interval-valued secondary s−ks -kernel symmetric fuzzy matrix is characterized by properties that extend traditional kernel symmetry, and suitable criteria for determining generalized inverses (g-inverses) of these matrices are established. These generalized inverses for interval-valued secondary s−k-kernel

symmetric fuzzy matrices serve as essential tools for solving fuzzy matrix equations and other computational problems involving such matrices.

Moreover, we derive the necessary and sufficient conditions for a matrix to qualify as an interval-valued s−k kernel symmetric fuzzy matrix, providing a rigorous foundation for analyzing and utilizing these structures in practical applications. This further enhances the understanding of fuzzy matrix theory in relation to uncertainty and symmetry.Meenakshi, Krishnamoorthy, and Ramesh [33] have also contributed to this area by studying s−k-EP matrices, which play a crucial role in extending matrix theory to more generalized and uncertain environments. These developments highlight the ongoing research in the field of fuzzy matrices and their application in solving real-world problems involving imprecision and indeterminacy.

As a generalization of secondary Hermitian and Hermitian matrices, Meenakshi and Krishnamoorthy [34] introduced the concept of s−k Hermitian matrices, which expand the traditional understanding of matrix symmetry and its applications. Building on this foundation, we extend the notion of s−k Hermitian matrices to include s−k kernel symmetric intuitionistic fuzzy matrices. This extension allows us to incorporate the unique properties of intuitionistic fuzzy sets, such as membership and non-membership values, into the framework of matrix theory.Furthermore, we derive equivalent conditions for various generalized inverses (g-inverses) of secondary k-kernel intuitionistic fuzzy matrices to qualify as secondary k-kernel symmetric intuitionistic fuzzy matrices. This relationship is critical for understanding the interplay between matrix symmetry and the properties of g-inverses in fuzzy systems, thereby enhancing the applicability of these matrices in solving fuzzy matrix equations and related problems.

In addition, Meenakshi and Krishnamoorthy [35] have introduced k-EP matrices, which further generalize existing concepts in fuzzy matrix theory. These developments emphasize the ongoing exploration of advanced matrix structures and their significance in handling uncertainty and indeterminacy in various applications.Shyamal and Pal [36] conducted a study on interval-valued fuzzy matrices, expanding the understanding of how fuzzy data can be represented and manipulated using interval values to express uncertainty.The concept of k-symmetric matrices was introduced by various

authors, including Ann Lec [37], who explored secondary symmetric and skew-symmetric secondary orthogonal matrices. This work contributes to the broader understanding of matrix symmetry and its implications in different mathematical contexts.

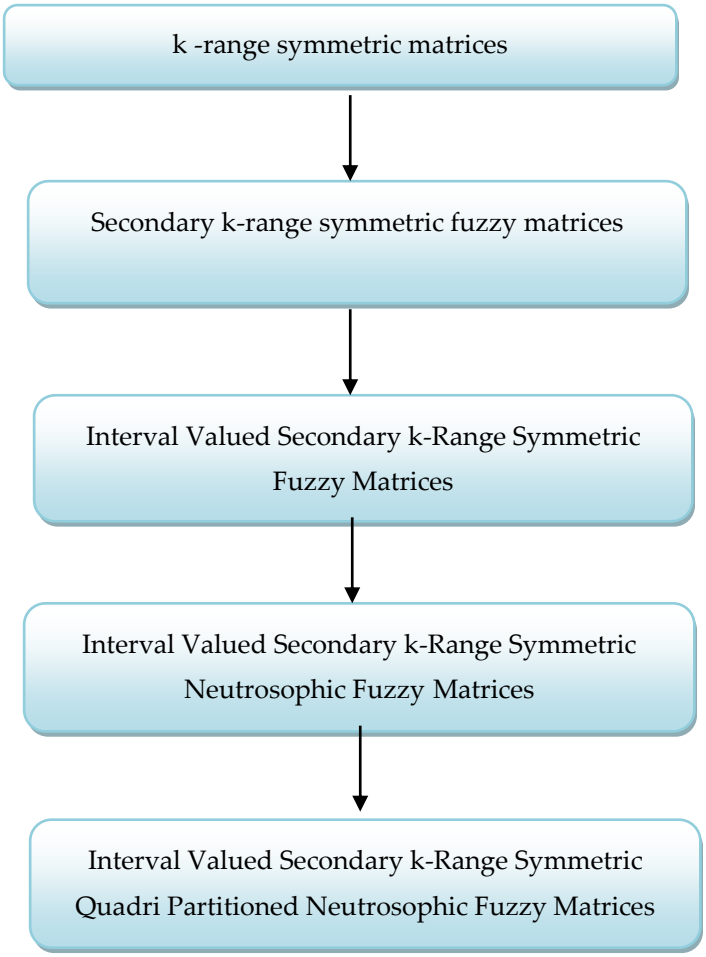
Elumalai and Rajesh Kannan [38] further advanced the study of symmetry in matrices by introducing k-symmetric circulant, sss-symmetric circulant, and s−k -symmetric circulant matrices. These specialized matrices play a vital role in various applications, particularly in fields requiring structured data representations.Additionally, Elumalai and Arthi [39] investigated the properties of k-centrosymmetric and k-skew centrosymmetric matrices, providing valuable insights into how symmetry interacts with the structural properties of matrices.Following this, Gunasekaran and Mohana [40] studied k-symmetric, s-symmetric, and s−k-symmetric double stochastic matrices. Their research

emphasizes the importance of symmetry in the context of stochastic processes and reinforces the growing body of knowledge surrounding symmetric matrices in various mathematical and applied domains. Anandhkumar et al. [41] have studied Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. Anandhkumar et al. introduced the concept of secondary k-range symmetric neutrosophic fuzzy matrices, providing a new approach to enhance the matrix's flexibility in representing uncertain information in diverse applications [42]. Expanding on this foundation, Anandhkumar, Bobin, Chithra, and Kamalakannan proposed generalized symmetric Fermatean neutrosophic fuzzy matrices, which incorporate additional dimensions of indeterminacy to enable more comprehensive modeling [43]. Further developments include secondary k-column symmetric neutrosophic fuzzy matrices, as explored by Anandhkumar, Punithavalli, and Janaki, which focus on enhancing matrix structures for specific applications in decision-making and data analysis [44].

Table 1: Extension of Neutrosophic Fuzzy Matrices Based on Previous Works

References	Extension of Neutrosophic Fuzzy Matrices from Fuzzy Matrices	Year
[20]	On k-kernel symmetric matrices	2009
[22]	On k-range symmetric matrices	2009
[23]	Secondary k-kernel symmetric fuzzy matrices	2014
[27]	Secondary k-range symmetric fuzzy matrices	2018
[29]	Interval Valued Secondary k-Range Symmetric Fuzzy Matrices	2022
[41]	Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices	2023
Proposed	Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making	2024

This table summarizes the extensions of Neutrosophic Fuzzy Matrices as derived from previous works. It highlights the evolution of research in this field, culminating in our proposed extension of Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices in 2024.


k -range symmetric matrices
Secondary k-range symmetric fuzzy matrices
Interval Valued Secondary k-Range Symmetric Fuzzy Matrices
Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices
Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

From Table 1 and the accompanying process flow, it is clear that previous studies have predominantly focused on concepts such as k-kernel, k-range, secondary k-kernel, and secondary k-range matrices within the framework of fuzzy matrices. However, a noticeable research gap exists concerning the exploration of these concepts in a neutrosophic environment, which deals with indeterminate and inconsistent information.

Recognizing this gap, we have made significant strides in establishing results related to kkk-range and secondary k-range matrices specifically within the context of interval-valued neutrosophic fuzzy **matrices**. By addressing this underexplored area, our work contributes to the development of the theoretical foundation necessary for understanding how these matrix types can operate under the complexities of neutrosophic logic. This not only enriches the existing literature but also opens up new avenues for research and applications in dealing with uncertainty and imprecision in various fields.

1.2 Novelties

We have established the concept of Interval-Valued (IV) Secondary k-Range Symmetric (RS) Neutrosophic Fuzzy Matrices. This development is crucial for enhancing the hybrid fuzzy structure,

allowing for more sophisticated modeling of uncertainty. We applied this concept within the framework of Neutrosophic Fuzzy Matrices (NFM) and conducted a detailed study of several results stemming from this application. In addition, we present equivalent characterizations of Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. This section includes various examples that illustrate the properties and applications of these matrices, facilitating a deeper understanding of their structure and significance.

Furthermore, we discuss various generalized inverses associated with regular matrices, providing a comprehensive characterization of the set of all inverses through the lens of Secondary s−k-Range Symmetric Neutrosophic Matrices. This analysis contributes to the broader discourse on matrix theory, particularly in the context of fuzzy and neutrosophic environments, enhancing the tools available for dealing with complex systems characterized by uncertainty.

1.3 Notations:

IV =Interval valued,

IVNFM = Interval valued Neutrosophic Fuzzy Matrix,

RS = Range Symmetric

IVQPNFM = Interval valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$[S_T, S_C, S_I, S_F]_L^T$ = Transpose of the IVQPNFM $[S_T, S_C, S_I, S_F]_L$,

$[S_T, S_C, S_I, S_F]_U^T$ = Transpose of the IVQPNFM $[S_T, S_C, S_I, S_F]_U$,

$[S_T, S_C, S_I, S_F]_L^+$ = Moore-Penrose inverse of IVQPNFM $[S_T, S_C, S_I, S_F]_L$,

$[S_T, S_C, S_I, S_F]_U^+$ = Moore-Penrose inverse of IVQPNFM $[S_T, S_C, S_I, S_F]_U$,

$R([S_T, S_C, S_I, S_F]_L)$ = Row space of IVQPNFM $[S_T, S_C, S_I, S_F]_L$

$R([S_T, S_C, S_I, S_F]_U)$ = Row space of IVQPNFM $[S_T, S_C, S_I, S_F]_U$,

$C([S_T, S_C, S_I, S_F]_L)$ = Column space of IVQPNFM $[S_T, S_C, S_I, S_F]_L$,

$C([S_T, S_C, S_I, S_F]_U)$ = Column space of IVQPNFM $[S_T, S_C, S_I, S_F]_U$

QPNFM : Quadri Partitioned Neutrosophic fuzzy matrices.
PNFM : Permutation Neutrosophic fuzzy matrices.

2. Preliminaries and Definitions

2.1 Preliminary

If $\kappa(x)=(x_{k[1]}, x_{k[2]}, x_{k[3]},\ldots, x_{k[n]}) \in F_{n\times 1}$ for $(x = x_1, x_2,..,x_n) \in F_{[1\times n]}$, where K is involuntary, The corresponding

Permutation matrix is satisfied using the conditions

(P.2.1) $KK^T = K^T K = I_n$, $K = K^T, K^2 = I$

By the definition of V, and $R(x) = Kx$

(P.2.2) $V = V^T, VV^T = V^T V = I_n$ and $V^2 = I$

(P.2.3) $R\left([S_T, S_C, S_I, S_F]_L\right) = R\left([S_T, S_C, S_I, S_F]_L\right)V,$

$$R\left([S_T, S_C, S_I, S_F]_L\right) = R\left([S_T, S_C, S_I, S_F]_L\right)K$$

$$R\left([S_T, S_C, S_I, S_F]_U\right) = R\left([S_T, S_C, S_I, S_F]_U\right)V,$$

$$R\left([S_T, S_C, S_I, S_F]_U\right) = R\left([S_T, S_C, S_I, S_F]_U\right)K$$

(P.2.4) $R\left([S_T, S_C, S_I, S_F]_L V\right)^T = R\left(V[S_T, S_C, S_I, S_F]_L^T\right)$,

$$R\left(V[S_T, S_C, S_I, S_F]_L\right)^T = R\left([S_T, S_C, S_I, S_F]_L^T V\right)$$

$$R\left([S_T, S_C, S_I, S_F]_U V\right)^T = R\left(V[S_T, S_C, S_I, S_F]_U^T\right),$$

$$R\left(V[S_T, S_C, S_I, S_F]_U\right)^T = R\left([S_T, S_C, S_I, S_F]_U^T V\right)$$

3.Quadri Partitioned Neutrosophic Soft Set (QPNSS)

Definition 3.1. Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all QPNSS of X. The collection (F, A) is termed the (QPNSS) over X, where F is a mapping given by F : A⟶ P(X).Here,

$A = \{< x, T_A(x), C_A(x), U_A(x), FA(x) >: x \in U\}$ with $T_A, F_A, C_A, U_A : X \longrightarrow [0,1]$ and $0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$. In this context

- $T_A(x)$ is the truth membership,
- $C_A(x)$ is contradiction membership,
- $U_A(x)$ is ignorance membership
- $F_A(x)$ is the false membership.

Definition:3.2 A QPNSS (F, A) over the universe X is considered to be an empty QPNSS with respect to the parameter A if $T = 0$, $C = 0$, $U = 1$, $F = 1$.

Definition: 3.3 A QPNSS (F, A) over the universe X is considered to be universe QPNSS with respect to the parameter A if $T = 1$, $C = 1$, $U = 0$, $F = 0$.

Definition: 3.4 Let $S = <[S_T, S_C, S_I, S_F]>$ be a QPNFM, if R $[[S_T, S_C, S_I, S_F]]$ =

R $[[S_T, S_C, S_I, S_F]^T]$ then $S = <[S_T, S_C, S_I, S_F]>$ is called as RS.

Example: 3.1 Consider an QPNFM

$$[S_T, S_C, S_I, S_F] = \begin{bmatrix} <0.4,0,1,0.6> & <1,0,1,1> & <0.2,0.3,0.4,0.7> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.7> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

Here, R $[S_T, S_C, S_I, S_F]$ = R $[[S_T, S_C, S_I, S_F]^T]$

The following matrices does not satisfy the range symmetric condition

$$[S_T, S_C, S_I, S_F] = \begin{bmatrix} <0.4,0,1,0.7> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.4,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$$[S_T, S_C, S_I, S_F]^T = \begin{bmatrix} <0.4,0,1,0.7> & <1,0,1,1> & <0.2,0.4,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$\big[(0.4,0,1,0.7)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)\big] \in R([S_T, S_C, S_I, S_F])$,

$\big[(0.4,0,1,0.7)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)\big] \notin R([S_T, S_C, S_I, S_F]^T)$,

$\big[(1,0,1,1)\ (1,0,1,1)\ (1,0,1,1)\big] \in R[S_T, S_C, S_I, S_F]$,

$\big[(1,0,1,1)\ (1,0,1,1)\ (1,0,1,1)\big] \in R([S_T, S_C, S_I, S_F]^T)$,

$\big[(0.2,0.4,0.4,0.6)\ (1,0,1,1)\ (1,0,1,1)\big] \in R([S_T, S_C, S_I, S_F])$,

$\big[(0.2,0.4,0.4,0.6)\ (1,0,1,1)\ (1,0,1,1)\big] \notin R([S_T, S_C, S_I, S_F]^T)$,

$R([S_T, S_C, S_I, S_F]) \notin R([S_T, S_C, S_I, S_F]^T)$.

Note:3.1 For QPNFM R with det $[S_T, S_C, S_I, S_F]$ > <0,0,1,1> has non- zero rows and non-columns, hereafter N($[S_T, S_C, S_I, S_F]$) = <0,0,1,1> = N($[S_T, S_C, S_I, S_F]^T$) . Furthermore, a symmetric matrix $[S_T, S_C, S_I, S_F]$ = $[S_T, S_C, S_I, S_F]^T$ that is N($[S_T, S_C, S_I, S_F]$)= N($[S_T, S_C, S_I, S_F]^T$).

Definition :3.5 Let $S = <[S_T, S_C, S_I, S_F]> \in (QPNFM)_n$, if N($[S_T, S_C, S_I, S_F]$) = N($[S_T, S_C, S_I, S_F]^T$) and then R is called KS- QPNFM where N($[S_T, S_C, S_I, S_F]$)= {x/x $[S_T, S_C, S_I, S_F]$} = (0,0,1,1) and x ∈ $F_{1\times n}$}.

Example: 3.2 Consider an QPNFM

$$[S_T, S_C, S_I, S_F] = \begin{bmatrix} <0.3,0.3,0.4,0.7> & <0.5,0.3,0.6,0.3> & <0.2,0.3,0.4,0.5> \\ <0.4,0.7,0.6,0.3> & <0.4,0.8,0.1,0.2> & <0.2,0.5,0.1,0.4> \\ <0.2,0.4,0.4,0.1> & <0.4,0.4,0.5,0.2> & <0.4,0.5,0.6,0.1> \end{bmatrix}$$

$$N([S_T, S_C, S_I, S_F]) = N([S_T, S_C, S_I, S_F]^T) = (0,0,1,1).$$

Definition 3.6. Symmetric QPNFM. If $S = <[S_T, S_C, S_I, S_F]> \in (QPNFM)_n$ is said to be symmetric QPNFM if $r_{ij} = r_{ji}$

Example: 3.3 Consider an QPNFM

$$[S_T, S_C, S_I, S_F] = \begin{bmatrix} <0.4,0,1,0.2> & <1,0,1,1> & <0.2,0.3,0.4,0.7> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.7> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

Here, $[S_T, S_C, S_I, S_F] = [S_T, S_C, S_I, S_F]^T$

Every row single (1,1,0,0) with (0,0,1,1) 's everywhere else is called QPPNFM.

Example: 3.4 Consider a QPNFPM, $K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$

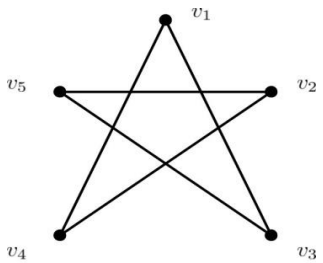
4. Graphical Representation of KS Adjacency IVINFM.

Definition 4.1. Adjacency IVINFM

An QPNM is a square matrix representing a finite graph. The elements of this matrix indicate whether pairs of vertices in the graph are connected. In the case of a finite simple graph, the IVINFM can be represented as a binary matrix, typically denoted as [1,1,0,0] and [0,0,1,1] where the diagonal elements are consistently set to [0,0,1,1]. Let G(V, E) denote a graph with n vertices. The adjacency matrix S = $[s_{ij}]$ is a symmetric matrix defined by $S = [s_{ij}] = \begin{cases} [1,1,0,0] & \text{when } v_i \text{ is adjacent to } v_j \\ [0,0,1,1] & \text{otherwise} \end{cases}$, denoted by S(G) or

Example: 4.1 Consider an IVINM and a equivalent graph

$$S = \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_2 \\ v_5 \end{array} \begin{pmatrix} v_1 & v_3 & v_4 & v_2 & v_5 \\ [0,0,1,1] & [1,1,0,0] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] \\ [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [1,1,0,0] \\ [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \\ [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] \\ [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \end{pmatrix}$$



H

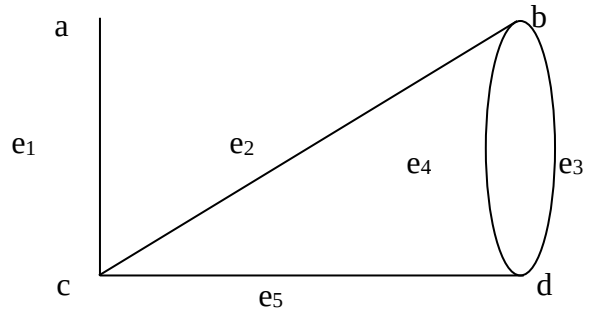
Definition 4.2. Incidence IVINFM

Let G(V, E) represent a simple graph with n vertices. Let V = {V_1, V_2, …, V_n} and E = {e_1, e_2, ..., e_m}. Then, the incidence IVINFM I = [m_{ij}] is a $n \times m$ matrix defined by

$$I = [m_{ij}] = \begin{cases} [1,1,0,0] & \text{when } v_i \text{ is incident to } e_j \\ [0,0,1,1] & \text{otherwise} \end{cases}$$, denoted by S(G) or S_G.

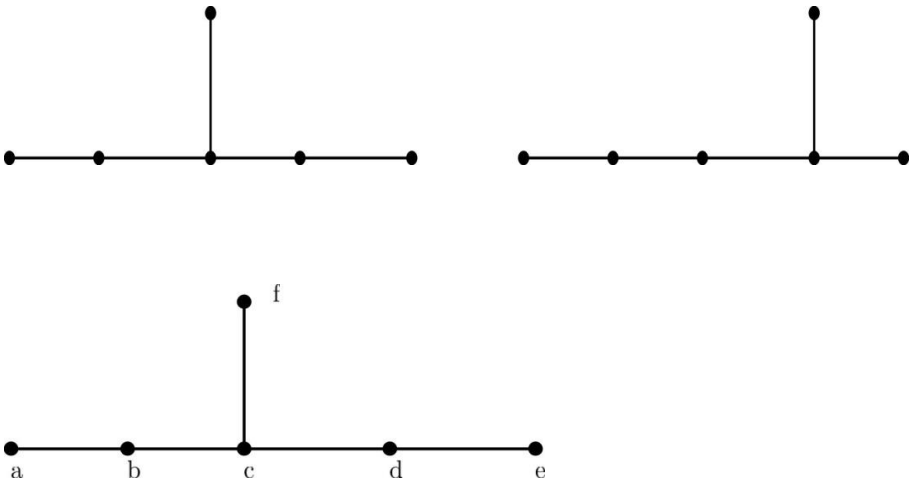
Example:4.2 Consider an incidence IVINFM and a equivalent graph. The incidence IVINFM is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] \\ [0,0,1,1] & [1,1,0,0] & [1,1,0,0] & [1,1,0,0] & [0,0,1,1] \\ [1,1,0,0] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [1,1,0,0] \\ [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [1,1,0,0] & [1,1,0,0] \end{bmatrix}$$



Definition 4.3. **Isomorphic of Graph:** Two graphs are said to be isomorphic if number of vertices, edges, degree sequence and adjacency IVINFM are equal.

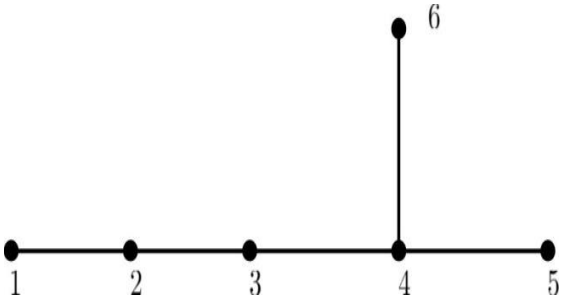
Relation between isomorphism, non-isomorphism and KS



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} a & b & c & d & e & f \\ [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] \\ [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] \\ [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] \\ [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \\ [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] \\ [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] \end{bmatrix}$$

The adjacency IVINFM of the graph is given by

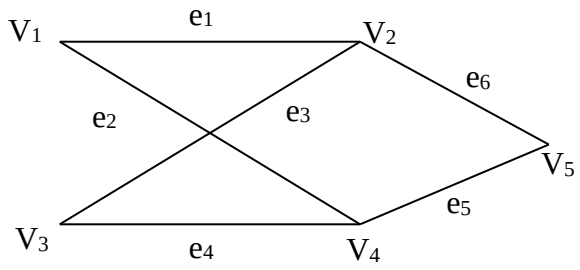
Consider the graph H and name as follows

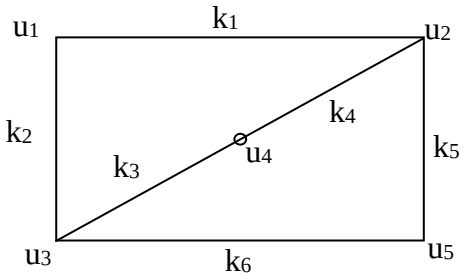


The adjacency IVINM of the graph is given by

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] \\ [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] & [0,0,1,1] \\ [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] \\ [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [1,1,0,0] \\ [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] \\ [0,0,1,1] & [0,0,1,1] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [0,0,1,1] \end{bmatrix}$$

The two graphs presented have identical numbers of vertices, edges, and degree sequences, yet their adjacency IVINFMs differ. Therefore the given Graph is not isomorphic but KS.





Let us form the adjacency IVINFM AG and AH

$$A_G = \begin{bmatrix} & v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \\ v_2 & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] \\ v_3 & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \\ v_4 & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] \\ v_5 & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \end{bmatrix}$$

$$A_H = \begin{bmatrix} & u_1 & u_2 & u_3 & u_4 & u_5 \\ u_1 & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \\ u_2 & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] \\ u_3 & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \\ u_4 & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] \\ u_5 & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] & [1,1,0,0] & [0,0,1,1] \end{bmatrix}$$

The two graphs provided have the same number of vertices, edges, and degree sequences, and their adjacency IVINFMs are also identical. Therefore, the graphs are isomorphic and also KS IVINFM.

Every isomorphic and non-isomorphic graph is KS adjacency IVINM but converse need not be true.

In this section, we introduce an algorithm designed to reduce parameters through the use of an adjacency matrix associated with a soft graph. We then apply this algorithm to a decision-making problem.

5. Algorithm

Consider the product set { M_1, M_2, M_3, …..M_n} with parameters {S_1,S_2,…,S_k}. To select the best product based on these parameters, we propose the following algorithm, which utilizes the adjacency matrix of a soft graph.

• Form a bipartite graph G=(V,E) for the given problem. In this graph:

V represents the set of vertices, which includes the products { M_1, M_2, M_3, …..M_n} and the parameters {S_1,S_2,…,S_k}. If E represents the set of edges, which connect each product M_i to the relevant parameters S_i

This bipartite graph effectively illustrates the relationships between the products and their associated parameters.

• To construct a soft graph (F,A) with A={M1,M2,M3,…,Mn} follow these steps:

1. **Define the Set S(x)**: For a given vertex x in the graph, define S(x)={z∈V:d(x,z)≤1} where d(x,z) denotes the distance between vertices x and z.
2. **Define the Set T(x)**: Define T(x)={xu∈E: u∈S(x)} where E represents the set of edges and u is a vertex connected to x.
3. **Construct F(x)**: The soft graph F(x) is then represented as F(x)=(S(x),T(x)), where S(x) is the set of vertices within a distance of (1,1,0,0) from x and T(x) is the set of edges connecting x to these vertices.

• Construct the adjacency matrix of the given soft graph (F,A). In this matrix:

- The rows correspond to the products Mi.
- The columns correspond to the parameters Pj.

Each entry (Mi,Sj) in the matrix represents the relationship or connection between product Mi and parameter Sj. If there is a connection, the entry is (1,1,0,0); otherwise, it is (0,0,1,1).

• If any entry (Mi,Sj) in the adjacency matrix is either (1,1,0,0) or (0,0,1,1) for all i=1,2,…,n, then the parameter Sj should be removed. This indicates that the parameter Sj does not contribute to distinguishing between products and can therefore be excluded from consideration.

• To determine the row totals in the modified adjacency matrix of the soft graph:

Modify the Adjacency Matrix: Make any necessary modifications to the original adjacency matrix based on the specific requirements or criteria provided.

Calculate Row Totals: For each row in the modified adjacency matrix, sum the entries. This sum represents the total number of connections or relationships for each vertex in the graph.

Identify the Last Column: The last column of the matrix should contain these row totals.

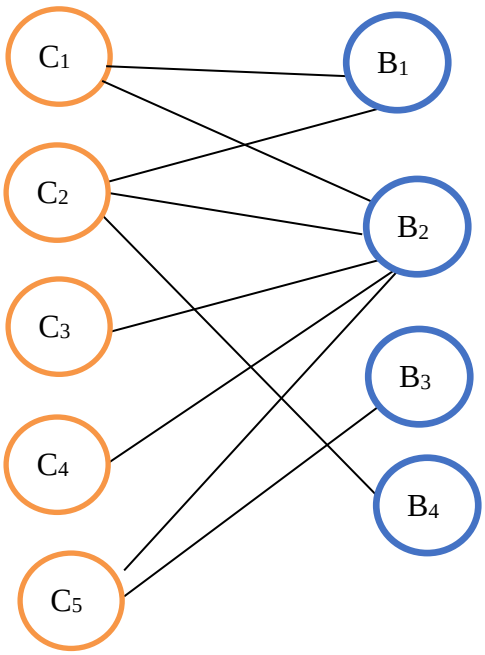
• Determine the product Mi that has the highest row total..

• The product with the highest row total will be the most favorable option.

5.1 Application:

The candidate is selecting a new course to enrol in from five available options: C1, C2, C3, C4, and C5. Each course offers various benefits (B1, B2, B3, and B4).Course C1 provides benefits B1 and B3, Course C2 includes benefits B1, B2, and B4, Course C3 has only benefit B2., Course C4 also offers benefit B2 and

Course C5 features benefits B2 and B3. Which course should the candidate choose to get the most value based on the benefits that align with their learning goals? First, generate a graph based on the given problem as outlined in the description.



Let V={C_1,C_2,C_3,C_4,C_5,B_1,B_2,B_3,B_4}.

Select A={C_1,C_2,C_3,C_4,C_5}.

Define S(x)={ z∈V: d(x,z) ≤ 1},

T(x)={ xu∈E: :u∈S(x)}. Now, let F(x)=(S(x),T(x)), where: F(C1)={C_1,B_1,B_3}, F(C2)={C_2,B_1,B_2,B_4}, F(C3)={C_3,B_2}, F(C4)={C_4,B_2} and F(C5)={C_5,B_2,B_3}.

Thus, (F, A) is a soft graph.

The adjacency matrix for the graph described above is provided as follows.

	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	C_5	Row total =Degree of vertex
C_1	(1,1,0,0)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	2

C_2	(1,1,0,0)	(1,1,0,0)	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	3
C_3	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	1
C_4	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	1
C_5	(0,0,1,1)	(1,1,0,0)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	2
B_1	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	2
B_2	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	5
B_3	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	1
B_4	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	1

The row total in the final column of the adjacency matrix represents the vertex degree in the soft graph (F,A). Given that the entry (M_i,S_2) is (1,1,0,0) for i=1,2,3,4,5 we should remove the second column from the matrix.

	B_1	B_3	B_4	C_1	C_2	C_3	C_4	C_5	Row total = Degree of vertex
C_1	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	1
C_2	(1,1,0,0)	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	2
C_3	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	0
C_4	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	0
C_5	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	1
B_1	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	2
B_2	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	5
B_3	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	1
B_4	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,1)	(1,1,0,0)	(0,0,1,1)	(0,0,1,1)	1

In the final column of the matrix, examine the row totals for new course C_1, C_2, C3, C4, and C_5. The row total for C_2 is the highest among them. Therefore, C_2 is the optimal choice.

6. Interval valued Secondary k-RS IVQPNFM

Definition 6.1. For an IVQPNFM $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \text{IVQPNFM}_{nn}$ is an IVQP

s - symmetric IVQPNFM iff $[S_T, S_C, S_I, S_F]_L = V\left([S_T, S_C, S_I, S_F]_L^T\right)V$ and $[S_T, S_C, S_I, S_F]_U$

$= V\left([S_T, S_C, S_I, S_F]_U^T\right)V$.

Definition 6.2 For an IVQPNFM $S =< [S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \text{IVQPNFM}_{nn}$ is an IVQP s-

RS IVQPNFM iff $R\left([S_T, S_C, S_I, S_F]_L\right) = R\left(V[S_T, S_C, S_I, S_F]_L^T V\right)$,

$R\left([S_T, S_C, S_I, S_F]_U\right) = R\left(V[S_T, S_C, S_I, S_F]_U^T V\right)$.

Definition 6.3. For an IVQPNFM $S =< [S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U >$ is an IV s-k- RS

IVQPNFM iff $R\left([S_T, S_C, S_I, S_F]_L\right) = R\left(KV[S_T, S_C, S_I, S_F]_L^T VK\right)$, $R\left([S_T, S_C, S_I, S_F]_U\right) =$

$R\left(KV[S_T, S_C, S_I, S_F]_U^T VK\right)$.

Lemma 6.1. For an IVQPNFM $S =< [S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \text{IVQPNFM}_{nn}$ is an IVQP

s- RS IVQPNFM $\Leftrightarrow$ $\text{V}\,S =< V[S_T, S_C, S_I, S_F]_L, \text{V}[S_T, S_C, S_I, S_F]_U >$ s- RS IVQPNFM.

$\Leftrightarrow$ $S\,\text{V} =< [S_T, S_C, S_I, S_F]_L\,\text{V}, [S_T, S_C, S_I, S_F]_U\,\text{V} >$ is RS IVQPNFM.

Proof. Let an IVQPNFM $S =< [S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \text{IVQPNFM}_{nn}$ is s-RS IVQPNFM

$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_L\right) = R\left(V[S_T, S_C, S_I, S_F]_L^T V\right)$

$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_L\,\text{V}\right) = R\left([S_T, S_C, S_I, S_F]_L V\right)^T$

$\Leftrightarrow [S_T, S_C, S_I, S_F]_L\,\text{V}$ is RS. [By P.2.2]

$\Leftrightarrow R\left(\text{V}[S_T, S_C, S_I, S_F]_L\,\text{V}\text{V}^T\right) = R\left(VV[S_T, S_C, S_I, S_F]_L^T V\right)$

$\Leftrightarrow R\left(\text{V}[S_T, S_C, S_I, S_F]_L\right) = R\left(\text{V}[S_T, S_C, S_I, S_F]_L\right)^T$

$\Leftrightarrow \text{V}[S_T, S_C, S_I, S_F]_L$ is RS.

Similar manner

$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_U\right) = R\left(V[S_T, S_C, S_I, S_F]_U^T V\right)$

$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_U\,\text{V}\right) = R\left([S_T, S_C, S_I, S_F]_U V\right)^T$

$\Leftrightarrow [S_T, S_C, S_I, S_F]_U \text{ V}$ is RS.

$\Leftrightarrow R\left(\text{V}[S_T, S_C, S_I, S_F]_U \text{ VV}^T\right) = R\left(VV[S_T, S_C, S_I, S_F]_U{}^T V\right)$

$\Leftrightarrow R\left(\text{V}[S_T, S_C, S_I, S_F]_U\right) = R\left(\text{V}[S_T, S_C, S_I, S_F]_U\right)^T$

$\Leftrightarrow \text{V}[S_T, S_C, S_I, S_F]_U$ is RS.

Therefore, $\text{V}\,S = < V[S_T, S_C, S_I, S_F]_L, \text{V}[S_T, S_C, S_I, S_F]_U >$ is an IV symmetric.

Example 6.1 Let us consider an IVQPNFM

$$S = \begin{bmatrix} <[1,1],[1,1],[0,0],[0,0]> & <[0.1,0.3],[0.2,0.4],[0.2,0.5],[0.3,0.5]> \\ <[0.1,0.3],[0.2,0.4],[0.2,0.5],[0.3,0.5]> & <[1,1],[1,1],[0,0],[0,0]> \end{bmatrix}$$

Lower Limit QPNFM, $[S_T, S_C, S_I, S_F]_L = \begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix}$,

Upper Limit QPNFM, $[S_T, S_C, S_I, S_F]_U = \begin{bmatrix} <1,1,0,0> & <0.3,0.4,0.5,0.5> \\ <0.3,0.4,0.5,0.5> & <1,1,0,0> \end{bmatrix}$

$V = \begin{bmatrix} <0,0,1,1> & <1,1,0,0> \\ <1,1,0,0> & <0,0,1,1> \end{bmatrix}$, $\text{K} = \begin{bmatrix} <1,1,0,0> & <0,0,1,1> \\ <0,0,1,1> & <1,1,0,0> \end{bmatrix}$,

$$KV[S_T, S_C, S_I, S_F]_L{}^T VK = \begin{bmatrix} <1,1,0,0> & <0,0,1,1> \\ <0,0,1,1> & <1,1,0,0> \end{bmatrix} \begin{bmatrix} <0,0,1,1> & <1,1,0,0> \\ <1,1,0,0> & <0,0,1,1> \end{bmatrix}$$

$$\begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix} \begin{bmatrix} <0,0,1,1> & <1,1,0,0> \\ <1,1,0,0> & <0,0,1,1> \end{bmatrix} \begin{bmatrix} <1,1,0,0> & <0,0,1,1> \\ <0,0,1,1> & <1,1,0,0> \end{bmatrix}$$

$$KV[S_T, S_C, S_I, S_F]_L{}^T VK = \begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix}$$

$$KV[S_T, S_C, S_I, S_F]_L{}^T VK = [S_T, S_C, S_I, S_F]_L$$

Similarly, $KV[S_T, S_C, S_I, S_F]_U{}^T VK = [S_T, S_C, S_I, S_F]_U$

$$[S_T, S_C, S_I, S_F]_L = K[S_T, S_C, S_I, S_F]_L K$$

$$K[S_T, S_C, S_I, S_F]_L K = \begin{bmatrix} <1,1,0,0> & <0,0,1,1> \\ <0,0,1,1> & <1,1,0,0> \end{bmatrix} \begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix}$$

$$\begin{bmatrix} <1,1,0,0> & <0,0,1,1> \\ <0,0,1,1> & <1,1,0,0> \end{bmatrix}$$

$$K[S_T, S_C, S_I, S_F]_L K = \begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix} = [S_T, S_C, S_I, S_F]_L$$

Similarly, $[S_T, S_C, S_I, S_F]_U = K[S_T, S_C, S_I, S_F]_U K$

Therefore S is symmetric NFM and its satisfies the condition

$[S_T, S_C, S_I, S_F]_L = [S_T, S_C, S_I, S_F]_L^T$, $[S_T, S_C, S_I, S_F]_U = [S_T, S_C, S_I, S_F]_U^T$

Therefore S is range symmetric QPNFM and its satisfies the condition

$R[S_T, S_C, S_I, S_F]_L = R[S_T, S_C, S_I, S_F]_L^T$, $R[S_T, S_C, S_I, S_F]_U = R[S_T, S_C, S_I, S_F]_U^T$

Therefore S is kernel symmetric QPNFM and its satisfies the condition

$N[S_T, S_C, S_I, S_F]_L = N[S_T, S_C, S_I, S_F]_L^T$, $N[S_T, S_C, S_I, S_F]_U = N[S_T, S_C, S_I, S_F]_U^T$

Therefore S is s-k symmetric QPNFM and its satisfies the condition

$$R([S_T, S_C, S_I, S_F]_L) = R\left(KV[S_T, S_C, S_I, S_F]_L^T VK\right),$$

$$R([S_T, S_C, S_I, S_F]_U) = R\left(KV[S_T, S_C, S_I, S_F]_U^T VK\right).$$

not both k –symmetric and s- k - symmetric NFM.
Therefore S is both k –symmetric and and s- k - symmetric QPNFM and also s-k- RS QPNFM.

Theorem 6.1. The following conditions are equivalent for $S \in \text{IVQPNFM}_n$

(i) $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \text{IVQPNFM}_{nn}$ is an IVQP s – k RS.

(ii) $\text{KV}\, S = <KV[S_T, S_C, S_I, S_F]_L, \text{KV}[S_T, S_C, S_I, S_F]_U >$ is an IVQP RS.

(iii) $S\,\text{KV} = <[S_T, S_C, S_I, S_F]_L KV, [S_T, S_C, S_I, S_F]_U KV >$ is an IVQP RS.

(iv) $\text{V}\, S = <V[S_T, S_C, S_I, S_F]_L, \text{V}[S_T, S_C, S_I, S_F]_U >$ is an IVQP k- RS.

(v) $S\,\mathrm{K} = <[S_T, S_C, S_I, S_F]_L\ \mathrm{K}, [S_T, S_C, S_I, S_F]_U\ \mathrm{K}>$ is an IVQP s- RS.

(vi) S^T is an IVQP s-k RS.

(vii) $\mathrm{R}([S_T, S_C, S_I, S_F]_L) = \left(\mathrm{R}[S_T, S_C, S_I, S_F]_L^{\,T}\ \mathrm{VK}\right)$,

$\mathrm{R}([S_T, S_C, S_I, S_F]_U) = \left(\mathrm{R}[S_T, S_C, S_I, S_F]_U^{\,T}\ \mathrm{VK}\right)$

(viii) $\mathrm{R}([S_T, S_C, S_I, S_F]_L^{\,T}) = \left(\mathrm{R}[S_T, S_C, S_I, S_F]_L\ \mathrm{VK}\right)$,

$\mathrm{R}([S_T, S_C, S_I, S_F]_U^{\,T}) = \left(\mathrm{R}[S_T, S_C, S_I, S_F]_U\ \mathrm{VK}\right)$

(ix) $\mathrm{C}(\mathrm{KV}[S_T, S_C, S_I, S_F]_L) = C\left(\mathrm{KV}[S_T, S_C, S_I, S_F]_L^{\,T}\right)^T$,

$\mathrm{C}(\mathrm{KV}[S_T, S_C, S_I, S_F]_U) = C\left(\mathrm{KV}[S_T, S_C, S_I, S_F]_U^{\,T}\right)^T$

(x) $[S_T, S_C, S_I, S_F]_L = \mathrm{VK}[S_T, S_C, S_I, S_F]_L^{\,T}\ \mathrm{VKH}_1$,

$[S_T, S_C, S_I, S_F]_U = \mathrm{VK}[S_T, S_C, S_I, S_F]_U^{\,T}\ \mathrm{VKH}_1\ \text{for}\ \mathrm{H}_1 \in IVQPNFM$

(xi) $[S_T, S_C, S_I, S_F]_L = \mathrm{H}_1\mathrm{KV}[S_T, S_C, S_I, S_F]_L^{\,T}\ \mathrm{KV}$,

$[S_T, S_C, S_I, S_F]_U = \mathrm{H}_1\mathrm{KV}[S_T, S_C, S_I, S_F]_U^{\,T}\ \mathrm{VK}\ \text{for}\ \mathrm{H}_1 \in IVQPNFM$

(xii) $[S_T, S_C, S_I, S_F]_L^{\,T} = \mathrm{KV}[S_T, S_C, S_I, S_F]_L\ \mathrm{VKH}_1$,

$[S_T, S_C, S_I, S_F]_U^{\,T} = \mathrm{KV}[S_T, S_C, S_I, S_F]_U\ \mathrm{VKH}_1\ \text{for}\ \mathrm{H}_1 \in IVQPNFM$

(xiii) $[S_T, S_C, S_I, S_F]_L^{\,T} = \mathrm{H}_1\mathrm{KV}[S_T, S_C, S_I, S_F]_L\ \mathrm{KV}$,

$[S_T, S_C, S_I, S_F]_U = \mathrm{H}_1\mathrm{KV}[S_T, S_C, S_I, S_F]_U\ \mathrm{VK}\ \text{for}\ \mathrm{H}_1 \in IVQPNFM$

Proof: (i) iff (ii) iff (iv)

Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U> \in \mathrm{IVQPNFM}_{mn}$ is an IVQP s – κ RS

Let $[S_T, S_C, S_I, S_F]_L$ is an IVQP s – κ RS.

$\Leftrightarrow R([S_T, S_C, S_I, S_F]_L) = R(KV[S_T, S_C, S_I, S_F]_L^{\,T} VK)$,

$\mathrm{R}([S_T, S_C, S_I, S_F]_U) = R(KV[S_T, S_C, S_I, S_F]_U^{\,T} VK)$,

$\Leftrightarrow R(KV[S_T, S_C, S_I, S_F]_L) = R(KV[S_T, S_C, S_I, S_F]_L)^T$,

$\mathrm{R}([S_T, S_C, S_I, S_F]_U) = R(K[S_T, S_C, S_I, S_F]_U)^T$ By (P.2.3)

$\Leftrightarrow$ KV $S =< KV[S_T, S_C, S_I, S_F]_L, \mathrm{KV}[S_T, S_C, S_I, S_F]_U >$ is an IVQP RS

$\Leftrightarrow$ V $S =< V[S_T, S_C, S_I, S_F]_L, \mathrm{V}[S_T, S_C, S_I, S_F]_U >$ is an IVQP k- RS

As a conclusion (i) iff (ii) iff (iv) is true

(i) iff (ii) iff (v)

Let $S =< [S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP s – κ RS

$\Leftrightarrow R(KV[S_T, S_C, S_I, S_F]_L) = R(KV[S_T, S_C, S_I, S_F]_L)^T$,

$\mathrm{R}(KV[S_T, S_C, S_I, S_F]_U) = R(KV[S_T, S_C, S_I, S_F]_U)^T$, ,

$\Leftrightarrow R(VK(KV[S_T, S_C, S_I, S_F]_L)) = R((VK)[S_T, S_C, S_I, S_F]_L{}^T VK(VK)^T)$

$\mathrm{R}(VK(KV[S_T, S_C, S_I, S_F]_U)) = R((VK)[S_T, S_C, S_I, S_F]_U{}^T VK(VK)^T)$

$\Leftrightarrow SKV = \left[[S_T, S_C, S_I, S_F]_L KV, [S_T, S_C, S_I, S_F]_U KV\right]$ is an IVQP RS

$\Leftrightarrow SK = \left[[S_T, S_C, S_I, S_F]_L K, [S_T, S_C, S_I, S_F]_U K\right]$ is an IVQP s- RS

As a conclusion (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) is true. (ii) $\Leftrightarrow$ (ix)

$KVS = \left[KV[S_T, S_C, S_I, S_F]_L, KV[S_T, S_C, S_I, S_F]_U\right]$ is an IVQP RS

$\Leftrightarrow R\left(KV[S_T, S_C, S_I, S_F]_L\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right)$,

$\mathrm{R}\left(KV[S_T, S_C, S_I, S_F]_U\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right)$

(ii) $\Leftrightarrow$ (ix) is true. (ii) $\Leftrightarrow$ (vii)

$KVS = \left[KV[S_T, S_C, S_I, S_F]_L, KV[S_T, S_C, S_I, S_F]_U\right]$ is an IVQP RS.

$\Leftrightarrow R\left(KV[S_T, S_C, S_I, S_F]_L\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right)$,

$\mathrm{R}\left(KV[S_T, S_C, S_I, S_F]_U\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right)$

$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_L\right) = R\left([S_T, S_C, S_I, S_F]_L{}^T VK\right)$,

$$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right) = R\left([S_T, S_C, S_I, S_F]_U{}^T VK\right)$$

As a conclusion (ii) $\Leftrightarrow$ (vii) is true. (iii) $\Leftrightarrow$ (viii)

$$SVK = \left[[S_T, S_C, S_I, S_F]_L VK, [S_T, S_C, S_I, S_F]_U VK\right]$$

$$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_L VK\right) = R\left(\left([S_T, S_C, S_I, S_F]_L VK\right)^T\right),$$

$$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U VK\right) = R\left(\left([S_T, S_C, S_I, S_F]_U VK\right)^T\right)$$

$$\Leftrightarrow R\left([S_T, S_C, S_I, S_F]_L VK\right) = R\left([S_T, S_C, S_I, S_F]_L\right)^T,$$

$$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U VK\right) = R\left([S_T, S_C, S_I, S_F]_U\right)^T$$

As a conclusion (iii) $\Leftrightarrow$ (viii) is true.

(i) $\Leftrightarrow$ (vi)

Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP s – κ RS

$$\Leftrightarrow R([S_T, S_C, S_I, S_F]_L) = R(KV[S_T, S_C, S_I, S_F]_L{}^T VK),$$

$$\mathrm{R}([S_T, S_C, S_I, S_F]_U) = R(KV[S_T, S_C, S_I, S_F]_U{}^T VK),$$

$$\Leftrightarrow \ (KVS)^T = (KV[S_T, S_C, S_I, S_F]_L, KV[S_T, S_C, S_I, S_F]_U)^T \text{ is an IVQP RS}$$

$$\Leftrightarrow \ S^T VK = ([S_T, S_C, S_I, S_F]_L VK, [S_T, S_C, S_I, S_F]_U VK) \text{ is an IVQP RS}$$

$$\Leftrightarrow \ S^T = \left([S_T, S_C, S_I, S_F]_L{}^T, [S_T, S_C, S_I, S_F]_U{}^T\right) \text{ is an IVQP s – κ RS}$$

As a conclusion (i) $\Leftrightarrow$ (vi) is true

(i) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (xi)

Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP s – κ RS

Consider $[S_T, S_C, S_I, S_F]_L$ is a s – κ RS

$$\Leftrightarrow C([S_T, S_C, S_I, S_F]_L{}^T) = C(\mathrm{KV}[S_T, S_C, S_I, S_F]_L VK),$$

$$\mathrm{C}([S_T, S_C, S_I, S_F]_U{}^T) = C(\mathrm{KV}[S_T, S_C, S_I, S_F]_U VK) \qquad \text{By (P.2.3)}$$

$$\Leftrightarrow [S_T, S_C, S_I, S_F]_L = H_1 \mathrm{KV}[S_T, S_C, S_I, S_F]_L{}^T VK,$$

$$[S_T, S_C, S_I, S_F]_U{}^T = H_1 \mathrm{KV}[S_T, S_C, S_I, S_F]_U VK \text{ for } \mathrm{H}_1 \in IVNFM.$$

As a result (i) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (xi) true.

(ii) $\Leftrightarrow$ (xiii) $\Leftrightarrow$ (x)

$\Leftrightarrow SVK = \left[[S_T, S_C, S_I, S_F]_L VK, [S_T, S_C, S_I, S_F]_U VK\right]$ is an IVQP RS

$\Leftrightarrow SV = \left[[S_T, S_C, S_I, S_F]_L V, [S_T, S_C, S_I, S_F]_U V\right]$ is an IVQP κ- RS

$\Leftrightarrow R(\mathrm{V}[S_T, S_C, S_I, S_F]_L) = R(\mathrm{K}\left(\mathrm{V}[S_T, S_C, S_I, S_F]_L\right)^T K)$,

$\mathrm{R}(\mathrm{V}[S_T, S_C, S_I, S_F]_U) = R(K\left(V[S_T, S_C, S_I, S_F]_U\right)^T K)$,

$\Leftrightarrow R([S_T, S_C, S_I, S_F]_L) = R\left([S_T, S_C, S_I, S_F]_L\right)^T VK)$,

$\mathrm{R}([S_T, S_C, S_I, S_F]_U) = R\left([S_T, S_C, S_I, S_F]_U\right)^T VK)$,

$\Leftrightarrow C([S_T, S_C, S_I, S_F]_L{}^T) = C\left(\mathrm{KV}[S_T, S_C, S_I, S_F]_L\right)^T K)$,

$\mathrm{C}([S_T, S_C, S_I, S_F]_U{}^T) = C\left(KV[S_T, S_C, S_I, S_F]_U\right)$

$[S_T, S_C, S_I, S_F]_L = VK[S_T, S_C, S_I, S_F]_L{}^T \ \mathrm{VK}\, H_1$,

$[S_T, S_C, S_I, S_F]_U = VK[S_T, S_C, S_I, S_F]_U \ \mathrm{VK}\ for\ H_1 \in IVNFM$

As a conclusion (ii) $\Leftrightarrow$ (xiii) $\Leftrightarrow$ (x) is true

The above statement can be reduced to the equivalent requirement that a matrix be an IVQP s- RS for K = I in particular.

Corollary:6.1 The following statements are equivalent for $S \in \mathrm{IVQPNFM}_{nn}$

(i) $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IV s-RS.

(ii) $VS = <V[S_T, S_C, S_I, S_F]_L, \mathrm{V}[S_T, S_C, S_I, S_F]_U >$ is an IVQP RS.

(iii) $SV = <[S_T, S_C, S_I, S_F]_L \ \mathrm{V}, [S_T, S_C, S_I, S_F]_U \ \mathrm{V} >$ is an IVQP RS.

(iv) $S^T = <[S_T, S_C, S_I, S_F]_L{}^T, [S_T, S_C, S_I, S_F]_U{}^T >$ is an IVQP s – RS.

(v) $R\left([S_T, S_C, S_I, S_F]_L\right) = R\left([S_T, S_C, S_I, S_F]_L{}^T V\right)$,

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right) = R\left([S_T, S_C, S_I, S_F]_U{}^T V\right)$

(vi) $R\left([S_T, S_C, S_I, S_F]_L^{\,T}\right) = R\left([S_T, S_C, S_I, S_F]_L V\right)$,

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U^{\,T}\right) = R\left([S_T, S_C, S_I, S_F]_U \mathrm{V}\right)$

(vii) $C\left(\mathrm{KV}[S_T, S_C, S_I, S_F]_L\right) = C\left(\mathrm{V}[S_T, S_C, S_I, S_F]_L\right)^T$,

$\mathrm{C}\left(\mathrm{KV}[S_T, S_C, S_I, S_F]_U\right) = C\left(\mathrm{V}[S_T, S_C, S_I, S_F]_U\right)^T$

(viii) $[S_T, S_C, S_I, S_F]_L = \mathrm{V}[S_T, S_C, S_I, S_F]_L^{\,T}\ \mathrm{VH}_1$,

$[S_T, S_C, S_I, S_F]_U = \mathrm{V}[S_T, S_C, S_I, S_F]_U^{\,T}\ \mathrm{VH}_1$ for $\mathrm{H}_1 \in IVQPNFM$

(ix) $[S_T, S_C, S_I, S_F]_L = \mathrm{H}_1\mathrm{V}[S_T, S_C, S_I, S_F]_L^{\,T}\ \mathrm{V}$,

$[S_T, S_C, S_I, S_F]_U = \mathrm{H}_1\mathrm{V}[S_T, S_C, S_I, S_F]_U^{\,T}\ \mathrm{V}$ for $\mathrm{H}_1 \in IVQPNFM$

(x) $[S_T, S_C, S_I, S_F]_L^{\,T} = \mathrm{V}[S_T, S_C, S_I, S_F]_L\ \mathrm{VH}_1$,

$[S_T, S_C, S_I, S_F]_U^{\,T} = \mathrm{V}[S_T, S_C, S_I, S_F]_U\ \mathrm{VH}_1$ for $\mathrm{H}_1 \in IVQPNFM$

(xi) $[S_T, S_C, S_I, S_F]_L^{\,T} = \mathrm{H}_1\mathrm{V}[S_T, S_C, S_I, S_F]_L\ \mathrm{V}$,

$[S_T, S_C, S_I, S_F]_U = \mathrm{H}_1\mathrm{V}[S_T, S_C, S_I, S_F]_U\ \mathrm{V}$ for $\mathrm{H}_1 \in IVQPNFM$

Theorem 6.2. For $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ then any two of the conditions below imply the other

(i) $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP κ- RS.

(ii) $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP s-κ- RS.

(iii) $R\left([S_T, S_C, S_I, S_F]_L\right)^T = R\left(\mathrm{VK}[S_T, S_C, S_I, S_F]_L\right)^T$,

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right)^T = R\left(\mathrm{VK}[S_T, S_C, S_I, S_F]_U^{\,T}\right)$

Proof: (i) and (ii) implies (iii)

Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP s – k RS

$\Rightarrow R\left([S_T, S_C, S_I, S_F]_L\right) = R\left([S_T, S_C, S_I, S_F]_L{}^T VK\right),$

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right) = R\left([S_T, S_C, S_I, S_F]_U{}^T VK\right)$

$\Rightarrow R\left(K[S_T, S_C, S_I, S_F]_L K\right) = R\left(K[S_T, S_C, S_I, S_F]_L{}^T K\right),$

$\mathrm{R}\left(K[S_T, S_C, S_I, S_F]_U K\right) = R\left(K[S_T, S_C, S_I, S_F]_U{}^T K\right)$

$\Rightarrow R[S_T, S_C, S_I, S_F]_L{}^T = R\left(\left(VK[S_T, S_C, S_I, S_F]_L\right)^T\right),$

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right)^T = R\left(\left(VK[S_T, S_C, S_I, S_F]_U\right)^T\right)$

(i) & (ii) implies (iii) is true

(i)& (iii) implies (ii)

$S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP κ- RS

$\Rightarrow R\left(K[S_T, S_C, S_I, S_F]_L K\right) = R\left(\left([S_T, S_C, S_I, S_F]_L\right)^T\right),$

$\mathrm{R}\left(K[S_T, S_C, S_I, S_F]_U K\right) = R\left(\left([S_T, S_C, S_I, S_F]_U\right)^T\right)$

Therefore, (i) & (iii)

$\Rightarrow R\left([S_T, S_C, S_I, S_F]\right) = R\left([S_T, S_C, S_I, S_F]_L{}^T VK\right),$

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right) = R\left([S_T, S_C, S_I, S_F]_U{}^T VK\right)$

$\Rightarrow R\left([S_T, S_C, S_I, S_F]_L\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right),$

$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right)$

$S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IV s-k-RS

$\Rightarrow$ (ii) is true

(ii) & (iii) implies (i)

$S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP s- κ- RS

$\Rightarrow R\left(K[S_T, S_C, S_I, S_F]_L K\right) = R\left(K[S_T, S_C, S_I, S_F]_L{}^T K\right),$

$\mathrm{R}\left(K[S_T, S_C, S_I, S_F]_U K\right) = R\left(K[S_T, S_C, S_I, S_F]_U{}^T K\right)$

Therefore,(ii) and (iii) $\Rightarrow R\left([S_T, S_C, S_I, S_F]_L\right) = R\left(K[S_T, S_C, S_I, S_F]_L{}^T K\right),$

$$\mathrm{R}\left([S_T, S_C, S_I, S_F]_U\right) = R\left(K[S_T, S_C, S_I, S_F]_U{}^T K\right)$$

$S = <[S_T, S_C, S_I, S_F], [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ is an IVQP κ- RS .

Therefore, (i) is true, Hence the theorem.

7. IVQP s – k RS regular Neutrosophic fuzzy matrices

In this section, we explore the various generalized inverses of matrices in the context of Interval-Valued Quasi-Periodic Neutrosophic Fuzzy Matrices (IVQPNFM). We establish the necessary criteria for different generalized inverses of an IVQP s−ks-ks−k Range Symmetric (RS) Neutrosophic Fuzzy Matrix to also qualify as IVQP s−ks-ks−k RS matrices. Additionally, we characterize the generalized inverses of an IVQP s−k RS matrix corresponding to the sets S={1,2}, S={1,2,3}, and S={1,2,4}. This characterization enhances our understanding of the structural properties of these matrices and their inverses.

Theorem 7.1: Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$, Z belongs to S{1,2} and SZ, ZS are an IV s- κ- RS. Then S is an IVQP s- κ - RS iff $Z = <[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L, [\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U >$ is an IVQP s- κ – RS.

Proof: Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$

$$R\left(KV[S_T, S_C, S_I, S_F]_L\right) = R\left(KV[S_T, S_C, S_I, S_F]_L \, \mathrm{Z}[S_T, S_C, S_I, S_F]\right) \subseteq R\left(\mathrm{Z}[S_T, S_C, S_I, S_F]_L\right)$$

$$= R\left(ZVV[S_T, S_C, S_I, S_F]_L\right) \subseteq R\left(ZVKKV[S_T, S_C, S_I, S_F]_L\right) \subseteq R\left(KV[S_T, S_C, S_I, S_F]_L\right)$$

$$\text{Hence, } \mathrm{R}\left(KV[S_T, S_C, S_I, S_F]_L\right) = R\left(\mathrm{Z}[S_T, S_C, S_I, S_F]_L\right)$$

$$= \mathrm{R}\left(KV\left(\mathrm{Z}[S_T, S_C, S_I, S_F]_L\right)^T VK\right)$$

$$= \mathrm{R}\left([S_T, S_C, S_I, S_F]_L{}^T [\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L{}^T VK\right)$$

$$= \mathrm{R}\left([\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L{}^T VK\right) = \mathrm{R}\left(\left(KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right)^T\right)$$

$$R\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right) = R\left([S_T, S_C, S_I, S_F]_L{}^T VK\right)$$

$$= \mathrm{R}\left(KV[S_T, S_C, S_I, S_F]_L [\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right)$$

$$= \mathrm{R}\left(KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right)$$

Similarly,

$$\text{Hence, } \mathrm{R}\left(KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right) = R\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right)$$

$\Leftrightarrow R\left(KV[S_T,S_C,S_I,S_F]_L\right)=R\left(\left(KV[S_T,S_C,S_I,S_F]\right)^T\right),$

$R\left(KV[S_T,S_C,S_I,S_F]_U\right)=R\left(\left(KV[S_T,S_C,S_I,S_F]_U\right)^T\right)$

$\Leftrightarrow R\left(KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)=R\left(\left(KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)^T\right),$

$R\left(KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_U\right)=R\left(\left(KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_U\right)^T\right)$

$\Leftrightarrow KVX=\left[KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L,KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_U\right]$ is an IVQP RS

$Z=<[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L,[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]>$ is an IVQP RS.

Theorem 7.2:, Let $S=<[S_T,S_C,S_I,S_F]_L,[S_T,S_C,S_I,S_F]_U>\in \mathrm{IVQPNFM}_{nn}$, $Z=<[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L$

$,[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_U>\in \mathrm{S}\{1,2,3\,\mathrm{R}(\mathrm{KV}[S_T,S_C,S_I,S_F]_L)=R\left(\mathrm{KV}[\mathrm{X}_{\mu L},\mathrm{X}_{\lambda L},\mathrm{X}_{\nu L}]\right)^T$

$,\mathrm{R}(\mathrm{KV}[S_T,S_C,S_I,S_F]_U)=R\left(\mathrm{KV}[\mathrm{Z}_{\mu U},\mathrm{Z}_{\lambda U},\mathrm{Z}_{\nu U}]\right)^T$ .Then $S=<[S_T,S_C,S_I,S_F]_L,$

$[S_T,S_C,S_I,S_F]_U>\in \mathrm{IVQPNFM}_{nn}$ is IVQP s-κ- RS $\Leftrightarrow$ $Z=<[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L,[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_U>$ is IVQP s- k – RS.

Proof: Given P{1,2,3}, Hence ,

$[S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L[S_T,S_C,S_I,S_F]_L=[S_T,S_C,S_I,S_F]_L,$

$[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L[S_T,S_C,S_I,S_F][\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L=[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L,$

$\left([S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)^T=[S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L$

Consider, $\mathrm{R}\left(\left(KV[S_T,S_C,S_I,S_F]_L\right)^T\right)=\mathrm{R}\left([\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L{}^T[S_T,S_C,S_I,S_F]_L{}^T VK\right)$

$=\mathrm{R}\left(KV\left([S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)^T\right)$

$=\mathrm{R}\left(\left([S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)^T\right)$ $\quad[By\ P_{\cdot 2.3}]$

$=\mathrm{R}\left([S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)$

$=\mathrm{R}\left([\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)$

$\left[\text{By using } [\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L=[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L[S_T,S_C,S_I,S_F]_L[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right]$

$=\mathrm{R}\left(KV[\mathrm{Z}_T,\mathrm{Z}_C,\mathrm{Z}_I,\mathrm{Z}_F]_L\right)$ $\quad[By\ P_{2.3}]$

Similarly, we can consider, $\mathrm{R}\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right) = \mathrm{R}\left([\mathrm{Z}_\mu, \mathrm{Z}_\lambda, \mathrm{Z}_\nu]_U{}^T [S_T, S_C, S_I, S_F]_U{}^T VK\right)$

$= \mathrm{R}\left(KV\left([S_T, S_C, S_I, S_F]_U[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right)^T\right) = \mathrm{R}\left(\left([S_T, S_C, S_I, S_F]_U[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right)^T\right)$

$= \mathrm{R}\left([S_T, S_C, S_I, S_F]_U[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right) \qquad \left[(PZ)^T = \mathrm{PZ}\right]$

$= \mathrm{R}\left([\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right) \qquad [\text{By using } \mathrm{Z} = \mathrm{Z}PZ]$

$= \mathrm{R}\left(KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right) \qquad [\text{By } P_{2.3}]$

If KVA is an IVQP RS

$\Leftrightarrow R\left(KV[S_T, S_C, S_I, S_F]_L\right) = \mathrm{R}\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right),$

$R\left(KV[S_T, S_C, S_I, S_F]_U\right) = \mathrm{R}\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right)$

$\Leftrightarrow R\left(KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right) = \mathrm{R}\left(\left(KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right)^T\right),$

$KVX = [K[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L, KV[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U]$ is an IVQP RS.

$Z = <[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L, [\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U >$ is an IVQP s-k RS.

Theorem 7.3: Let $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$, Z ∈ S {1, 2, 4},

$\mathrm{R}(\mathrm{KV}[S_T, S_C, S_I, S_F]_L)^T = R\left(\mathrm{KV}[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right), \mathrm{R}(\mathrm{KV}[S_T, S_C, S_I, S_F]_U)^T$

$= R\left(\mathrm{KV}[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U\right)$. Then KVP is an IV s- κ-Ks iff $Z = <[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L,$

$[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_U >$ is an IVQP s- κ- RS.

Proof: Given, S {1, 2, 4}, Hence $[S_T, S_C, S_I, S_F]_L[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L[S_T, S_C, S_I, S_F]_L = [S_T, S_C, S_I, S_F]_L,$

$[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_\nu]_L[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L = [\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L,$

$\left([\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L[S_T, S_C, S_I, S_F]_L\right)^T = [\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L[S_T, S_C, S_I, S_F]_L$

Consider, $\mathrm{R}\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right) = \mathrm{R}\left([\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L{}^T [S_T, S_C, S_I, S_F]_L{}^T VK\right)$

$= R\left(KV\left([S_T, S_C, S_I, S_F]_L[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right)^T\right)$

$= R\left(\left([S_T, S_C, S_I, S_F]_L[\mathrm{Z}_T, \mathrm{Z}_C, \mathrm{Z}_I, \mathrm{Z}_F]_L\right)^T\right) \qquad [By\ P_{2.3}]$

$$= R\left([S_T, S_C, S_I, S_F]_L [Z_T, Z_C, Z_I, Z_F]_L\right) = R\left([Z_T, Z_C, Z_I, Z_F]_L\right)$$

$$= R\left(\mathrm{KV}[Z_T, Z_C, Z_I, Z_F]_L\right) \qquad [By\ P_{2.3}]$$

$$R\left(\left(KV[S_T, S_C, S_I, S_F]_U\right)^T\right) = \mathrm{R}\left([Z_T, Z_C, Z_I, Z_F]_U{}^T [S_T, S_C, S_I, S_F]_U{}^T VK\right)$$

$$= R\left(KV\left([S_T, S_C, S_I, S_F]_U [Z_T, Z_C, Z_I, Z_F]_U\right)^T\right)$$

$$= R\left(\left([S_T, S_C, S_I, S_F]_U [Z_T, Z_C, Z_I, Z_F]_U\right)^T\right) \qquad [By\ P_{2.3}]$$

$$= R\left([S_T, S_C, S_I, S_F]_U [Z_T, Z_C, Z_I, Z_F]_U\right) \qquad [\ (S\,Z)^T = SZ\ ]$$

$$= N\left([X_{\mu U}, X_{\lambda U}, X_{\nu U}]\right) = R\left(\mathrm{KV}[Z_T, Z_C, Z_I, Z_F]_U\right) \qquad [By\ P_{2.3}]$$

If KVP is an IVQP RS

$$\Leftrightarrow R\left(KV[S_T, S_C, S_I, S_F]_L\right) = \mathrm{R}\left(\left(KV[S_T, S_C, S_I, S_F]_L\right)^T\right),$$

$$R\left(KV[S_T, S_C, S_I, S_F]\right) = \mathrm{R}\left(\left(KV[S_T, S_C, S_I, S_F]\right)^T\right)$$

$$\Leftrightarrow R\left(KV[Z_T, Z_C, Z_I, Z_F]_L\right) = \mathrm{R}\left(\left(KV[Z_T, Z_C, Z_I, Z_F]_L\right)^T\right),$$

$KVX = [KV[Z_T, Z_C, Z_I, Z_F]_L, KV[Z_T, Z_C, Z_I, Z_F]_U]$ is an IVQP RS.

$Z = <[Z_T, Z_C, Z_I, Z_F], [Z_T, Z_C, Z_I, Z_F]>$ is an IVQP s-k RS.

The aforementioned Theorems reduce to comparable criteria, in particular for K = I, for different g-inverses of interval valued s- RS to be IV secondary RS.

Corollary 7.1: For $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$, $Z \in$ P {1, 2} and

$SZ = <[S_T, S_C, S_I, S_F]_L [Z_T, Z_C, Z_I, Z_F], [S_T, S_C, S_I, S_F]_U [Z_T, Z_C, Z_I, Z_F]_U >,$

$ZS = <[Z_T, Z_C, Z_I, Z_F]_L [S_T, S_C, S_I, S_F]_L, [Z_T, Z_C, Z_I, Z_F]_U [S_T, S_C, S_I, S_F]_U >,$ are is an IVQP s-RS. Then S is an IVQP s- RS iff $Z = <[Z_T, Z_C, Z_I, Z_F]_L, [Z_T, Z_C, Z_I, Z_F]_U >$ is an IVQP s- RS.

Corollary 7.2: For $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \mathrm{IVQPNFM}_{nn}$ $Z \in$ S {1, 2, 3},

$$\mathrm{R}(\mathrm{KV}[S_T, S_C, S_I, S_F]_L) = R\left(\mathrm{V}[Z_T, Z_C, Z_I, Z_F]_L\right)^T, \mathrm{R}(\mathrm{KV}[S_T, S_C, S_I, S_F])$$

$= R\left(\mathrm{V}[Z_T, Z_C, Z_I, Z_F]_U\right)^T$. Then S is an IVQP s- RS iff $Z = <[Z_T, Z_C, Z_I, Z_F]_L, [Z_T, Z_C, Z_I, Z_F]_U >$ is an IVQP s- RS.

Corollary 7.3: For $S = <[S_T, S_C, S_I, S_F]_L, [S_T, S_C, S_I, S_F]_U > \in \text{IVQPNFM}_{nn}$, $Z \in$ P {1, 2, 4}

, $\text{R}(\text{V}[S_T, S_C, S_I, S_F]_L)^T = R\left(\text{V}[\text{Z}_T, \text{Z}_C, \text{Z}_I, \text{Z}_F]_L\right)^T$, $\text{R}(\text{V}[S_T, S_C, S_I, S_F]_U)^T$

$= R\left(\text{V}[\text{Z}_T, \text{Z}_C, \text{Z}_I, \text{Z}_F]_U\right)$. Then S is an IVQP s- RS iff Z is an IVQP s-RS.

8. Conclusion:

Firstly, we present equivalent characterizations for IVQP k-Range Symmetric (RS), IVQP RS, Interval-Valued s-Range Symmetric, and IVQP s−k Range Symmetric Neutrosophic Fuzzy Matrices. Additionally, we provide an example demonstrating that while an s−k-symmetric fuzzy matrix can be considered an s−k-RS Neutrosophic fuzzy matrix, the reverse does not necessarily hold true. Secondly, we examine various generalized inverses associated with regular matrices and establish a characterization of the complete set of all inverses. This exploration aids in understanding the intricate relationships between different types of matrix inverses. Thirdly, we determine equivalent conditions for various generalized inverses of IVQPNFM s−k-Range Symmetric and s-Range Symmetric Neutrosophic Fuzzy Matrices. This analysis provides crucial insights into the structural properties and potential applications of these matrices. Additionally, we provide a graphical representation of RS, CS, and KS adjacency and incidence QPNFM. While every adjacency QPNFM is symmetric, RS, CS, and KS, the incidence matrix satisfies only KS conditions. Every RS adjacency QPNFM is a KS adjacency QPNFM, but a KS adjacency QPNFM does not necessarily imply RS QPNFM. Soft graphs represent a novel area of research in mathematics. In this paper, we explore their application in decision-making by utilizing the adjacency matrix of a soft graph and developing a corresponding algorithm. Finally, we conclude by introducing the concept of IVQP Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. Looking ahead, our future work will focus on investigating IVQP Secondary k-Kernel Symmetric Neutrosophic Fuzzy Matrices, further enriching the field of neutrosophic fuzzy matrix theory.

REFERENCES

[1]. Zadeh L.A., Fuzzy Sets, Information and control.,(1965),8, pp. 338-353.

[2]. K.Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Physica-Verlag, 1999.

[3]. Smarandache,F, Neutrosophic set, a generalization of the intuitionistic fuzzy set. Int J Pure Appl Math.; .,(2005),.24(3):287–297.

[4]. M.Pal, S.K.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets, 8(2) (2002), 51-62.

[5]. K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 ,(1986), 87-96.

[6]. K.Atanassov, Operations over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64, (1994) ,159-174.

[7]. H.Hashimoto, Canonical form of a transitive matrix, Fuzzy Sets and Systems, 11 (1983),157-162.

[8]. K.H.Kim and F.W.Roush, Generalised fuzzy matrices, Fuzzy Sets and Systems, 4, (1980) ,293-315.

[9]. A.Lee, Secondary Symmetric, Secondary Skew Symmetric, Secondary Orthogonal

Matrices, Period Math, Hungary, 7, (1976) ,63-76.

[10]. R.D. Hill and S.R.Waters, On k-Real and k-Hermitian matrices, Linear Algebra and its Applications, 169, (1992), 17-29.

[11]. AR.Meenakshi, Fuzzy Matrix: Theory and Applications, MJP Publishers, Chennai, 2008.

[12]. M. Anandhkumar. V.Kamalakannan,S.M.Chitra, and Said Broumi, Pseudo Similarity of Neutrosophic Fuzzy matrices, International Journal of Neutrosophic Science, Vol. 20, No. 04, PP. 191-196, 2023 .

[13]. M.Anandhkumar.B.Kanimozhi,V.Kamalakannan,S.M.Chitra, and Said Broumi, On various Inverse of Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023.

[14]. G. Punithavalli , M. Anandhkumar, Reverse Sharp And Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices,TWMS J. App. and Eng. Math. 14(4),1772-1783, (2024).

[15]. M Pal and Susanta K. Khan Interval-Valued Intuitionistic Fuzzy Matrices, NIFS 11 (2005), 1, 16-27.

[16]. R. Vidhya And R. Irene Hepzibah On Interval Valued Neutrosophic Fuzzy Matrices, Advances and Applications in Mathematical Sciences Volume 20, Issue 4, February 2021, Pages 561-57.

[17]. M. Anandhkumar ,T. Harikrishnan, S. M. Chithra , V. Kamalakannan , B. Kanimozhi , Broumi Said ,Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, Vol. 21, No. 04, PP. 135-145, 2023.

[18]. M.Anandhkumar, B.Kanimozhi, S.M. Chithra, V.Kamalakannan, Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic Fuzzy Matrices, Mathematical Modelling of Engineering Problems, 2023, 10(4), pp. 1427–1432.

[19]. M. Anandhkumar ,T. Harikrishnan,S. M. Chithra,V. Kamalakannan,B. Kanimozhi. "Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices." International Journal of Neutrosophic Science, Vol. 23, No. 2, 2024 ,PP. 286-295.

[20]. AR.Meenakshi and D.Jaya Shree, On k-kernel symmetric matrices, International Journal of Mathematics and Mathematical Sciences, 2009, Article ID 926217, 8 Pages.

[21]. AR.Meenakshi and S.Krishanmoorthy, On Secondary k-Hermitian matrices, Journal of Modern Science, 1 (2009) 70-78.

[22]. AR.Meenakshi and D.Jaya Shree, On K -range symmetric matrices, Proceedings of the National conference on Algebra and Graph Theory, MS University, (2009), 58- 67.

[23]. D.Jaya shree , Secondary κ-Kernel Symmetric Fuzzy Matrices, Intern. J. Fuzzy Mathematical Archive Vol. 5, No. 2, 2014, 89-94 ISSN: 2320 –3242 (P), 2320 –3250 , Published on 20 December 2014.

[24]. A. K. Shyamal and M. Pal, Interval valued Fuzzy matrices, Journal of Fuzzy Mathematics 14(3) (2006), 582-592.

[25]. A. R. Meenakshi and M. Kalliraja, Regular Interval valued Fuzzy matrices, Advance

in Fuzzy Mathematics 5(1) (2010), 7-15.

[26]. G. Punithavalli, M. Anandhkumar, Kernel and K-Kernel Symmetric Intuitionistic Fuzzy Matrices,TWMS J. App. and Eng. Math,14(3),1231-1240,(2024).

[27]. D. Jaya Shree, Secondary κ-range symmetric fuzzy matrices, Journal of Discrete Mathematical Sciences and Cryptography 21(1):1-11,2018.

[28]. M. Anandhkumar; G.Punithavalli; T.Soupramanien; Said Broumi, Generalized Symmetric Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 57,2023, 57, pp. 114–12.

[29]. M. Kaliraja And T. Bhavani, Interval Valued Secondary κ-Range Symmetric Fuzzy Matrices, Advances and Applications in Mathematical Sciences Volume 21, Issue 10, August 2022, Pages 5555-5574.

[30]. Hill R. D., and Waters S. R, on κ-real and κ-Hermitian matrices, Linear Algebra and its Applications,169, 17–29, .,(1992).

[31]. Baskett T. S., and Katz I. J, Theorems on products of EPr matrices," Linear Algebra and its Applications, 2, 87–103, (1969).

[32]. Meenakshi AR., and Krishanmoorthy S.,(2009), on Secondary k-Hermitian matrices, Journal of Modern Science, 1, 70-78.

[33]. Meenakshi AR., Krishnamoorthy S., and Ramesh G., (2008) on s-k-EP matrices", Journal of Intelligent System Research, 2, 93-100.

[34]. Meenakshi AR., and Krishanmoorthy S.,(2009), on Secondary k-Hermitian matrices, Journal of Modern Science, 1, 70-78.

[35]. Meenakshi AR., and Krishnamoorthy S.(1998), on κ-EP matrices, Linear Algebra and its Applications, 269, 219–232.

[36]. Shyamal A. K., and Pal. M., (2006) , Interval valued Fuzzy matrices, Journal of Fuzzy Mathematics 14(3), 582-592.

[37]. Ann Lec.,(1976), Secondary symmetric and skew symmetric secondary orthogonal matrices (i) Period, Math Hungary, 7, 63-70.

[38]. Elumalai N., and Rajesh kannan K.,(2016), k - Symmetric Circulant, s - Symmetric Circulant and s – k Symmetric Circulant Matrices, Journal of ultra scientist of physical sciences, 28 (6), 322-327.

[39]. Elumalai. N., and Arthi B., (2017), Properties of k - CentroSymmetric and k – Skew CentroSymmetric Matrices, International Journal of Pure and Applied Mathematics , 10, 99-106.

[40]. Gunasekaran K., and Mohana N., (2014), k-symmetric Double stochastic, s-symmetric Double stochastic,s-k-symmetric Double stochastic Matrices, International Journal of Engineering Science Invention, 3 (8).

[41]. Anandhkumar, M.; G. Punithavalli; R. Jegan; and Said Broumi. "Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 61, 1 (2024).

[42]. M. Anandhkumar, H. Prathab, S. M. Chithra, A. S. Prakaash, A. Bobin, Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, 23(4) , 23-28, (2024).

[43]. M. Anandhkumar, A. Bobin; S. M. Chithra; and V. Kamalakannan. "Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 70, 1 (2024).

[44]. M.Anandhkumar, G. Punithavalli; and E. Janaki. "Secondary k-column symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 64, 1, (2024).

Received: Sep 2, 2024. Accepted: Dec 3, 2024

NSS
Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

A Methodological Approach to Decision-Making Using Interval-Valued Neutrosophic Framework for Enhancing and Evaluating English Blended Teaching Quality

Ying Du*
Shanghai Technical Institute of Electronics and Information, Shanghai, 201411, China
*Corresponding author, E-mail:19101750983@163.com

Abstract: The future of blended English instruction in universities is promising, driven by ongoing technological advancements and the increasing diversity of educational needs. This teaching model is expected to better integrate the strengths of both online and in-person learning, providing students with more flexible and personalized learning experiences. By combining these two modalities, blended instruction can improve student motivation and engagement while allowing educators to continuously refine their teaching methods and materials based on real-time data and feedback. As a result, blended learning is projected to become a more widespread and preferred method of instruction in higher education. In terms of evaluating the quality of blended English teaching, multiple-attribute decision-making (MADM) techniques play an important role. Interval-valued neutrosophic sets (IVNSs) have been extensively used and studied in MADM for their ability to handle uncertainty and imprecision in decision-making processes. In this context, the ORESTE is applied under the IVNS framework for ranking the alternatives. This method is particularly useful for evaluating the quality of blended English instruction. A numerical example is provided in the paper to demonstrate its application in assessing the quality of blended teaching. The study concludes that the IVN-ORESTE method not only offers stability but also provides a degree of flexibility, which is crucial for adapting to the varying needs and conditions of higher education environments.

Keywords: Multiple attribute decision-making (MADM); IVNSs; IVN-ORESTE; Effect evaluation

1. Introduction

In the rapidly evolving landscape of higher education, blended learning has emerged as a significant trend, particularly because of its distinct educational advantages. In the context of university English instruction, this model merges traditional face-to-face classroom teaching with modern online platforms, creating a dynamic and flexible learning environment for students[1]. Often referred to as hybrid learning, blended learning integrates in-person teaching with digital

experiences, optimizing educational resources and fusing learning activities across both online and offline environments. The primary goal of this approach is to enhance the effectiveness of teaching while improving the efficiency of student learning. In university English courses, blended learning typically involves a combination of video lectures, online forums, and interactive tasks, alongside conventional classroom interactions and discussions[2]. The widespread availability of internet technologies and mobile devices has facilitated the rapid adoption and expansion of blended learning across the globe. Many universities now implement this model as a strategic approach to improve both teaching quality and the overall learning experience. For instance, online platforms allow instructors to share instructional videos and supplementary materials, enabling students to access these resources anytime and anywhere, which promotes autonomous learning. Meanwhile, traditional classroom settings continue to offer opportunities for more direct face-to-face communication and in-depth discussions[3]. However, successfully implementing blended learning requires careful consideration of several key factors, including curriculum design, technological support, teacher training, and student acceptance. Effective curriculum design is crucial; it must ensure that online and offline activities are integrative and complementary, rather than simply existing side-by-side. Additionally, educational institutions need to provide reliable technical support, such as video conferencing tools and online discussion platforms, to facilitate smooth and uninterrupted teaching activities [4]. Blended learning has demonstrated significant advantages in enhancing student motivation, improving learning efficiency, and deepening learning engagement. Students can use online resources to engage in pre-learning or to review course materials, ensuring continuity and depth in their education. Simultaneously, face-to-face interactions in the classroom allow for timely feedback from instructors, helping students to better understand and master the course content. This balance between self-directed online learning and structured classroom interaction provides a more holistic educational experience[5]. Despite its benefits, blended learning also presents several challenges. A major obstacle is the seamless integration of online and offline instructional strategies, which requires careful planning to avoid fragmentation of the learning experience. Sustaining student interest and participation in both modes of instruction can also be difficult, especially when students face distractions in online environments[6]. Additionally, providing effective evaluation and feedback for student progress in a blended format requires new approaches that account for both digital and in-person learning activities. Furthermore, technical issues such as inconsistent internet access or software compatibility problems can diminish the quality of the teaching and learning experience[7]. As educational technologies continue to evolve, and as pedagogical philosophies shift, blended learning in university English instruction is poised for significant transformation. Future developments will likely incorporate cutting-edge technologies such as artificial intelligence and big data analytics, which can enhance the personalization and overall efficiency of teaching. These advancements will enable instructors to tailor learning experiences to individual student needs while utilizing data-driven insights to

improve instructional strategies. Simultaneously, educators and researchers will continue to refine and optimize blended learning models to accommodate the diverse learning preferences of students, ensuring that this approach remains relevant in an ever-changing educational landscape. In conclusion, blended learning is increasingly becoming a central method for improving the quality and efficiency of university English education[8]. Through ongoing experimentation and scholarly research, this educational model is expected to reach its full potential, significantly enriching students' learning experiences in the years to come.

Smarandache [9] introduced neutrosophic sets (NSs), later refined by Wang et al. [10] into single-valued NSs (SVNSs), and further by Wang et al. [11] into interval-valued NSs (IVNSs) for MADM. IVNSs offer significant advantages in handling uncertainty and imprecision. First, IVNSs represent uncertainty through three independent parameters: truth-membership, falsity-membership, and indeterminacy-membership. This allows for a more flexible approach to dealing with complex and ambiguous decision-making environments. Unlike traditional fuzzy sets, IVNSs allow each parameter to be expressed as an interval, which enhances their ability to represent imprecise information. Second, IVNSs can simultaneously handle uncertainty, vagueness, and inconsistency, making them highly adaptable in areas such as multi-attribute decision-making, risk assessment, and complex system analysis. Additionally, by incorporating various neutrosophic logic operations, IVNSs provide flexibility in addressing different levels of subjective factors and objective conditions in the decision-making process. In summary, the primary advantages of IVNSs lie in their ability to comprehensively manage uncertain information and offer stable and reliable decision support in scenarios requiring precise evaluation of complex issues. This makes them particularly valuable in multi-attribute decision-making processes, especially when dealing with fuzziness and uncertainty. The English blended teaching quality evaluation is looked as MADM.

Roubens first presented the ORESTE approach at a conference in 1980, and it was later developed in an essay that same year. When a decision-maker gives an analyst a preliminary ranking of the traits for decision-making, ORESTE is utilized. Additionally, the best option—which has a variety of qualitative and quantitative characteristics—is chosen from the group of options. This method is frequently employed. Features of the ORESTE include It's among the compensatory techniques, Features ought to be autonomous, the qualitative attributes do not have to be transformed into quantitative attributes [12].

This study aims to propose the IVN-ORESTE method under IVNSs and apply it to practical MADM processes. The proposed method enhances decision-making by accommodating uncertainty and providing a more adaptable framework for evaluating alternatives in complex scenarios.

The structure of the remainder of this paper is organized as follows: Section 2 provides an in-depth discussion of IVNSs, explaining their fundamental concepts and applications. Section 3 focuses on the construction of the IVN-ORESTE method specifically for MADM scenarios, highlighting its flexibility and adaptability in complex decision environments. In Section 4, we

present a numerical example that applies the IVN-ORESTE method to the evaluation of English blended teaching quality. This section includes a comprehensive case study. Finally, Section 5 concludes the paper, summarizing the key findings and contributions of the study, while also suggesting potential areas for future research and improvement in the application of IVNS-based decision-making techniques.

2. Preliminary

Definitions of NSs for additional computational analysis are presented in this section. In fact, Smarandache (1999) extended the IFS standard interval [0,1] by introducing neturosophic logic and NSs for the first time in literature [13-15].

2.1 Definition 1.

Let X be a space of points and $x \in X$, An NS A in X can be defined as truth $T_A(x)$, an indeterminacy $I_A(x)$, and falsity $F_A(x)$ membership function.

$$0-\leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3+ \tag{1}$$

2.2 Definition 2.

We can define some operations of IVNN as:

$$\mu A = \begin{Bmatrix} \left[1-\left(1-T_A^L(x)\right)^{\mu}, 1-\left(1-T_A^U(x)\right)^{\mu}\right], \\ \left[\left(\left(I_A^L(x)\right)^{\mu}, I_A^U(x)\right)^{\mu}\right], \\ \left[\left(\left(F_A^L(x)\right)^{\mu}, F_A^U(x)\right)^{\mu}\right] \end{Bmatrix} \tag{2}$$

$$A^{\mu} = \begin{Bmatrix} \left[\left(T_A^L(x)\right)^{\mu}\right], \\ \left[1-\left(1-I_A^L(x)\right)^{\mu}, 1-\left(1-I_A^U(x)\right)^{\mu}\right], \\ \left[1-\left(1-F_A^L(x)\right)^{\mu}, 1-\left(1-F_A^U(x)\right)^{\mu}\right] \end{Bmatrix} \tag{3}$$

$$A+B = \begin{Bmatrix} [T_A^L(x)+T_B^L(x)-T_A^L(x)T_B^L(x), T_A^U(x)+T_B^U(x)-T_A^U(x)T_B^U(x)], \\ [I_A^L(x)I_B^L(x), I_A^U(x)I_B^U(x)], \\ [F_A^L(x)F_B^L(x), F_A^U(x)F_B^U(x)] \end{Bmatrix} \tag{4}$$

$$A-B = \begin{Bmatrix} [T_A^L(x)-T_B^L(x), T_A^U(x)-T_B^U(x)], \\ \left[\max\left(I_A^L(x), I_B^L(x)\right), \max\left(I_A^U(x), I_B^U(x)\right)\right], \\ \left[\max\left(F_A^L(x), F_B^L(x)\right), \max\left(F_A^U(x), F_B^U(x)\right)\right] \end{Bmatrix} \tag{5}$$

$$A+B = \begin{Bmatrix} [T_A^L(x)T_B^L(x), T_A^U(x)T_B^U(x)], \\ [I_A^L(x)+I_B^L(x)-I_A^L(x)I_B^L(x), I_A^U(x)+I_B^U(x)-I_A^U(x)I_B^U(x)], \\ [F_A^L(x)+F_B^L(x)-F_A^L(x)F_B^L(x), F_A^U(x)+F_B^U(x)-F_A^U(x)F_B^U(x)] \end{Bmatrix} \tag{6}$$

$$\frac{A}{B} = \begin{Bmatrix} \left[\min\left\{\frac{T_A^L(x)}{T_B^L(x)}, \frac{T_A^L(x)}{T_B^U(x)}, \frac{T_A^U(x)}{T_B^L(x)}, \frac{T_A^U(x)}{T_B^U(x)}\right\}, \max\left\{\frac{T_A^L(x)}{T_B^L(x)}, \frac{T_A^L(x)}{T_B^U(x)}, \frac{T_A^U(x)}{T_B^L(x)}, \frac{T_A^U(x)}{T_B^U(x)}\right\}\right], \\ \min\left\{\frac{I_A^L(x)}{I_B^L(x)}, \frac{I_A^L(x)}{I_B^U(x)}, \frac{I_A^U(x)}{I_B^L(x)}, \frac{I_A^U(x)}{I_B^U(x)}\right\}, \max\left\{\frac{I_A^L(x)}{I_B^L(x)}, \frac{I_A^L(x)}{I_B^U(x)}, \frac{I_A^U(x)}{I_B^L(x)}, \frac{I_A^U(x)}{I_B^U(x)}\right\}, \\ \min\left\{\frac{F_A^L(x)}{F_B^L(x)}, \frac{F_A^L(x)}{F_B^U(x)}, \frac{F_A^U(x)}{F_B^L(x)}, \frac{F_A^U(x)}{F_B^U(x)}\right\}, \max\left\{\frac{F_A^L(x)}{F_B^L(x)}, \frac{F_A^L(x)}{F_B^U(x)}, \frac{F_A^U(x)}{F_B^L(x)}, \frac{F_A^U(x)}{F_B^U(x)}\right\} \end{Bmatrix} \tag{7}$$

$$A^{-1}=\left\{\begin{matrix}\left[\left(T_A^L(x)\right)^{-1},\left(T_A^U(x)\right)^{-1}\right],\\ \left[\left(I_A^L(x)\right)^{-1},\left(I_A^U(x)\right)^{-1}\right],\\ \left[\left(F_A^L(x)\right)^{-1},\left(F_A^U(x)\right)^{-1}\right]\end{matrix}\right\} \tag{8}$$

$$\frac{A}{\mu}=\left\{\begin{matrix}\left[\min\left(\frac{T_A^L(x)}{\mu},1\right),\min\left(\frac{T_A^U(x)}{\mu},1\right)\right],\\ \left[\min\left(\frac{I_A^L(x)}{\mu},1\right),\min\left(\frac{I_A^U(x)}{\mu},1\right)\right],\\ \left[\min\left(\frac{F_A^L(x)}{\mu},1\right),\min\left(\frac{F_A^U(x)}{\mu},1\right)\right]\end{matrix}\right\} \tag{9}$$

3. MADM approach

This section shows the steps of the IVNN-ORESTE for ranking the alternatives [16-17].

Step 1. Build the decision matrix

The decision matrix is built based on the opinions of experts and decision makers.

$$T=\begin{pmatrix}t_{11} & \cdots & t_{1n}\\ \vdots & \ddots & \vdots\\ t_{m1} & \cdots & t_{mn}\end{pmatrix} \tag{10}$$

Step 2. We used the IVNN to evaluate the criteria and alternatives.

Step 3. Apply the score function.

The score function is applied to obtain crisp values.

Step 4. Combine the decision matrix.

Step 5. Compute the criteria weights.

The criteria weights are computed using the average method.

Step 6. Obtain the position matrix.

The position matrix is obtained by ranking the alternatives based on the decision matrix and criteria.

Step 7. Obtain the block distance of each alternative.

$$d\left(0,A_{ij}\right)=\gamma t_{ij}(\gamma)+(1-\gamma)t_j \tag{11}$$

Where $0<\gamma<1$

Step 8. Compute the block distance of each alternative in the position matrix is computed to build the block distance matrix.

Step 9. Rank the alternatives.

The alternatives are ranked based on:

$$d(0, A_{ij}) \leq d(0, A_{i'j}), R(A_{ij}) \leq R(A_{ij}) \leq R(A_{i'j}) \quad (12)$$

The total rank of alternatives is:

$$R(A_i) = \sum_{j=1}^{n} R(A_{ij}) \quad (13)$$

4. Case and comparative study

The evaluation of blended English language teaching quality in universities spans several key dimensions, including course content, instructional methods, student engagement, and the integration of technology. At the heart of this assessment lies the development of course content, which plays a critical role in the effectiveness of blended learning environments. An ideal blended teaching approach seamlessly merges online and in-person components, ensuring that both complement each other rather than replicate content. Typically, online components involve video lectures, discussion forums, and digital resources that support self-paced learning outside the traditional classroom. Educators must carefully design these online materials, tailoring them to the specific needs of the digital environment rather than simply duplicating classroom content. For example, video lectures should be concise, information-rich, and crafted to engage learners with shorter attention spans typically found in online settings. Innovative teaching methodologies are also essential for enhancing the quality of blended learning. Instructors are encouraged to adopt dynamic, participatory activities that foster deeper student engagement. Virtual group discussions, live Q&A sessions, and other interactive elements can significantly boost student motivation and participation. Additionally, the use of technology to provide immediate feedback on student performance is crucial. Timely feedback not only helps students gauge their progress but also enables them to adjust their learning strategies in real-time, thus enhancing educational outcomes. Student engagement, another vital indicator of blended learning quality, is often a reflection of teaching effectiveness. High levels of engagement suggest that students are actively involved and able to learn autonomously. To promote engagement, educators can design tasks that are both relevant and suitably challenging, as well as offer personalized learning paths that cater to individual student needs. Continuous adjustment of teaching methods, based on student feedback and learning data, is also essential for maintaining high engagement levels and improving overall teaching quality. The application of technology is another cornerstone of successful blended learning. Learning Management Systems (LMS), interactive tools, and a variety of online resources can significantly improve teaching quality and student outcomes. To maximize the benefits of these technologies, schools should ensure that they are user-friendly and accessible. Furthermore, teachers should receive adequate technical training to effectively incorporate these tools into their teaching practices. Finally, structured assessment and feedback mechanisms are crucial for maintaining and improving the quality of blended teaching. A systematic approach to assessment should be established to monitor both the teaching process and learning outcomes. Regular feedback from students and educators should be gathered, focusing not only on academic performance but also on the overall

learning experience. This feedback helps refine teaching content, course design, and instructional methods. In conclusion, the quality of blended English teaching in universities is a multi-dimensional evaluation system. It encompasses course content, teaching methods, student engagement, the effective use of technology, and ongoing assessment and feedback. By continuously optimizing these aspects, educators can significantly improve teaching effectiveness, ensuring students achieve their best possible learning outcomes. The evaluation of blended teaching quality constitutes a MADM process. Ten possible English Colleges are evaluated through 14 attributes:

1. Teaching Effectiveness

- The ability of the blended teaching approach to improve student learning outcomes, comprehension, and language proficiency.

2. Student Engagement

- The extent to which students are actively involved and participate in both online and in-class activities.

3. Curriculum Design

- The alignment of course content, learning objectives, and assessment methods with the blended learning model.

4. Technology Integration

- The seamless and effective use of digital tools and platforms to support teaching and learning.

5. Assessment of Quality

- The ability of assessments (formative and summative) to measure learning outcomes accurately and provide meaningful feedback.

6. Teacher Competence

- The skills and preparedness of instructors to deliver blended teaching effectively, including their ability to use technology and manage hybrid classrooms.

7. Resource Availability

- Accessibility and quality of instructional materials, such as videos, slides, e-books, and online exercises.

8. Student Satisfaction

- Perceptions of students regarding the quality of teaching, course structure, and overall learning experience.

9. Flexibility

- The degree to which the blended teaching model accommodates different learning paces, styles, and schedules.

10. Interaction Quality

- The effectiveness of interactions between students and teachers, as well as peer-to-

peer communication in online and offline settings.

11. Feedback Mechanism

- Timeliness, quality, and relevance of feedback provided to students on their performance.

12. Technological Accessibility

- Availability and ease of use of technological tools and platforms for students and teachers, including access to reliable internet.

13. Innovation in Pedagogy

- Implementation of innovative teaching methods such as gamification, flipped classrooms, and collaborative projects.

14. Monitoring and Evaluation

- Systems in place to regularly monitor and evaluate the effectiveness of the blended teaching approach and make data-driven improvements.

The IVNN-ORESTE method is illustrated for evaluation of blended teaching quality under IVNSs.

Step 1. Eq. (10) is used to build the decision matrix. Three experts are building the decision matrix using the IVNN as shown in Table 1. The decision matrix is built between 14 criteria and ten alternatives.

Step 2. Set of IVNN are used for building the decision matrix.

Step 3. The score function was applied to convert the IVNN into single number.

Step 4. The mean method was used to combine three decision matrices into one.

Step 5. The criteria weights are computed as shown in Figure 1.

We show the Teaching Effectiveness criterion has the highest weight equal to 0.07762, followed by the Student Engagement with weight equal to 0.07609, and Curriculum Design criterion with weight equal to 0.0749. we show the Monitoring and Evaluation has the lowest weight.

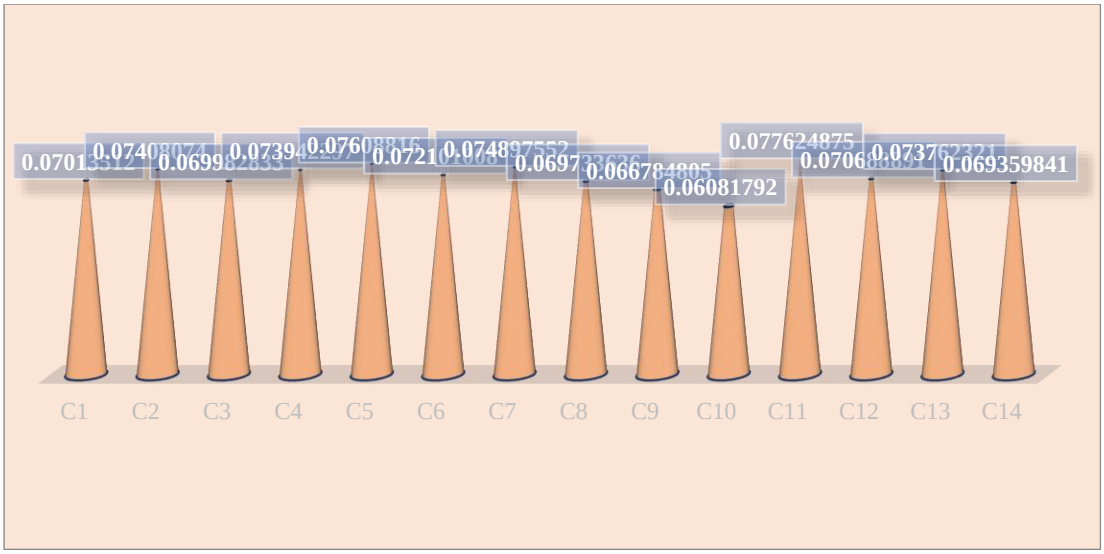


Figure 1. The criteria weights.

Table 1. The decision matrix.

Step 6. We obtained the position matrix.

Step 7. Then we obtain the block distance of each alternative using Eq. (11).

Step 8. Then we computed the block distance of each alternative in the position matrix is computed to build the block distance matrix.

Step 9. Then we ranked the alternatives using Eq. (13) as shown in Figure 2. We show the alternative 5 is the best and alternative 8 is the worst.

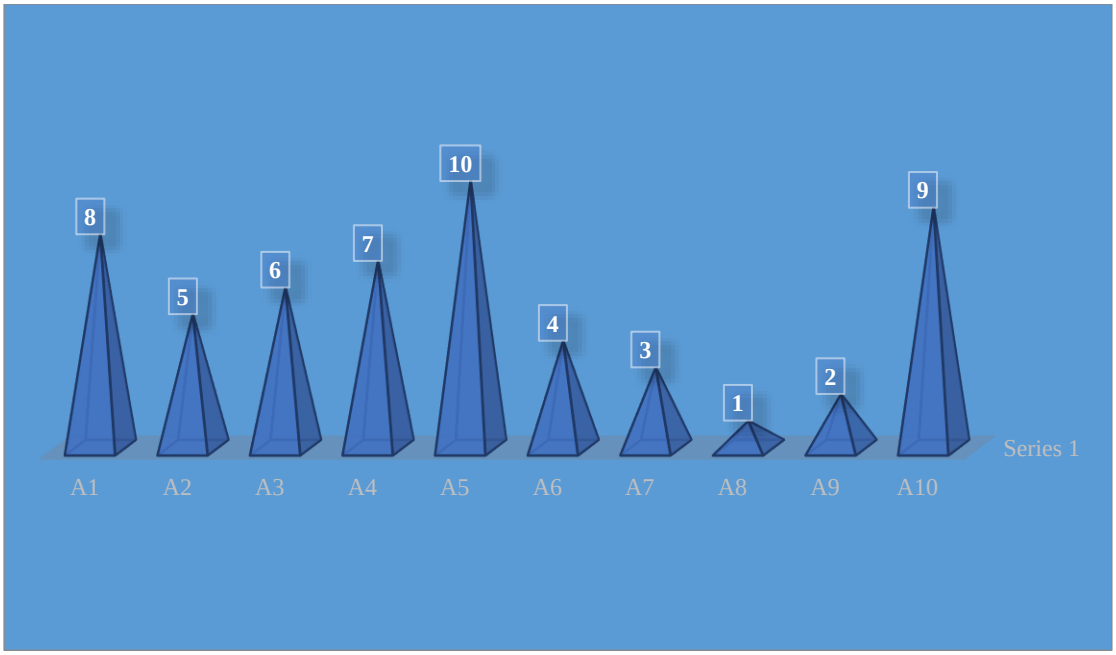


Figure 2. The rank of alternatives.

4.1. Comparative study

Then, the IVN- ORESTE approach is compared with INN-VIKOR approach [18], INN-CODAS approach[19], INN-EDAS approach [20] and INN-TODIM approach [21]. The comparative results are conducted in Figure 3.

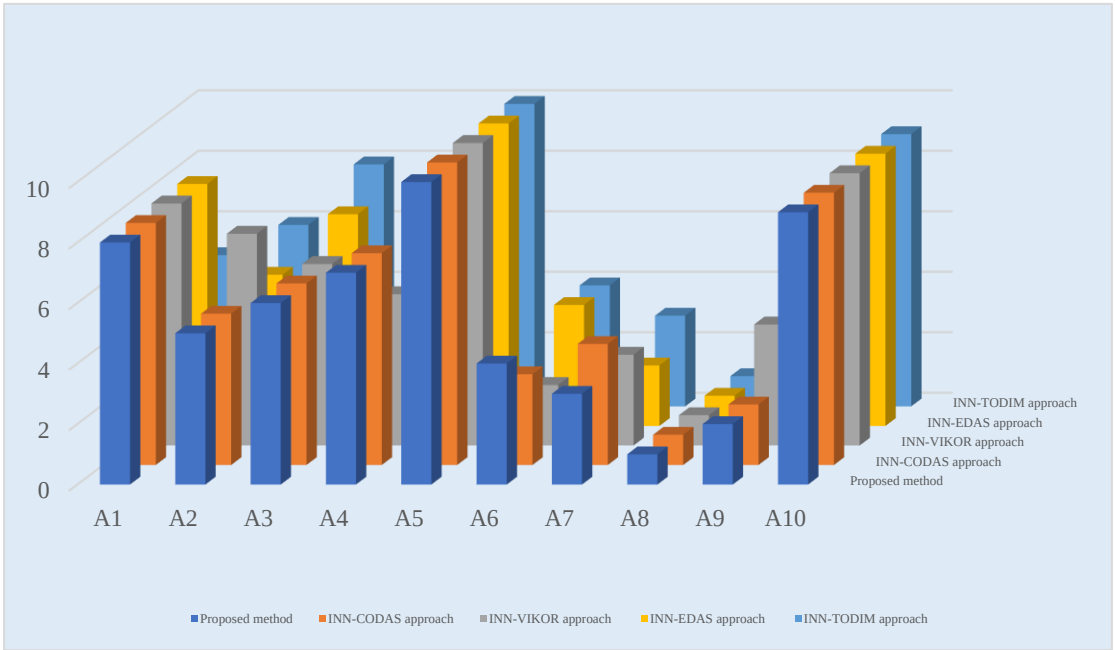


Figure 3. The comparative analysis.

In accordance with correlation coefficients information between the INN-VIKOR approach

[18], INN-CODAS approach[19], INN-EDAS approach [20], INN-TODIM approach [21] is 0.90303, 987879, 975758, 915152,, respectively. Compared the final results of IVN- ORESTE approach with INN-VIKOR approach [18], INN-CODAS approach[19], INN-EDAS approach [20], INN-TODIM approach [21], the illustrated results are slightly different and the illustrated best college and worst college is same. This validation confirms that the IVN-ORESTE method is both reasonable and effective. The primary advantages of the proposed IVN- ORESTE approach are: (1) criteria are independent; (2) It is no need to change the qualitative criteria into quantitative criteria. Moreover, the IVN-ORESTE allows decision-makers the flexibility to adjust parameters based on personal preferences, risk tolerance, and other subjective considerations. This adaptability enables decision-makers to tailor the decision-making process to better align with their specific needs and satisfaction. As a result, the method provides a more personalized and context-sensitive approach to decision-making, ensuring that the final ranking reflects the unique priorities of the decision maker. Overall, the IVN-ORESTE method offers a robust framework for decision-making, combining analytical rigor with flexibility, making it highly suitable for complex MADM situations where both objectivity and subjective preferences need to be considered.

5. Conclusion

Blended English teaching at the university level integrates traditional classroom instruction with digital learning platforms, creating a dynamic and flexible educational model. This approach leverages a variety of educational resources and technological tools to enhance both the efficiency and quality of instruction. Through online learning, students can access course materials at any time and from any location, which significantly increases the flexibility of the learning process. This not only fosters independent learning but also allows students to pace their studies according to individual needs and schedules. At the same time, offline classroom teaching remains essential for fostering direct teacher-student interaction. In-person discussions and activities help deepen students' understanding of complex concepts, reinforce knowledge, and provide immediate feedback. The blended teaching model is particularly well-suited for language learning, as it combines the benefits of autonomous learning with those of interactive communication. For example, students can engage in language exercises, quizzes, and simulation tests via online platforms, while face-to-face sessions enhance their speaking and listening skills through active participation and real-time interaction. Moreover, this hybrid model also helps students develop digital literacy and adaptability, essential skills in today's technology-driven work environments. By navigating between online and offline modes of learning, students are better prepared for the demands of modern professional settings, where digital competence is increasingly important. Blended learning thus not only improves academic outcomes but also equips students with practical skills for their future careers. In evaluating the quality of blended English teaching, MADM techniques are employed. In this study, we ORESTE method to IVNSs and introduce the IVN-ORESTE, specifically designed for MADM applications. An illustrative example is provided to

demonstrate the evaluation of English blended teaching quality using this method.

Reference

[1] Q. Cai, O. Ouyang, Fuzzy Comprehensive Evaluation Research on College English Course Teaching Quality, Educational Materials, (2019) 208-210.

[2] Y. Guo, Discussion on How to Improve the Quality of College English Teaching, Curriculum Education Research, (2018) 108-109.

[3] X. He, J. Wu, Attach Great Importance to College English Teaching Reform and Strive to Improve Teaching Quality, Campus English, (2014) 17-17.

[4] Y. Li, Research on Measures to Improve Intensive Reading Teaching Quality under College English Teaching Reform, Journal of Jiamusi Vocational College, (2015) 270-271.

[5] X. Zheng, An Empirical Study on the Effectiveness of the Chunk-Based Situational Teaching Model in Improving College English Oral Teaching Quality, Journal of Weifang University, 19 (2019) 91-94.

[6] X. Wang, C. Zhao, Y. Zuo, Exploration and Practice of Improving College English Teaching Quality in Local Applied Universities, English Square, (2024) 114-117.

[7] Y. Song, Research on the Construction of a Classroom Teaching Quality Evaluation System for College English Courses, Overseas English, (2023) 143-146.

[8] D. Li, Research and Practice on the Construction of College English Teaching Quality Monitoring and Assurance System under the Background of Engineering Education Accreditation, Campus English, (2024) 85-87.

[9] F. Smarandache, A unifying field in logics: Neutrosophic logic, Multiple-Valued Logic, 8 (1999).

[10] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single valued neutrosophic sets, Infinite study, 2010.

[11] H. Wang, F. Smarandache, R. Sunderraman, Y.-Q. Zhang, interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing, Infinite Study, 2005.

[12] M. Madić and G. Petrović, “Application of the ORESTE method for solving decision making problems in transportation and logistics,” *UPB Sci. Bull. Ser. D Mech. Eng.*, vol. 78, no. 4, pp. 83–94, 2016..

[13 H. Zhang, J. Wang, and X. Chen, "An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets," *Neural Comput. Appl.*, vol. 27, pp. 615–627, 2016.

[14] M. Yazdani, A. E. Torkayesh, Ž. Stević, P. Chatterjee, S. A. Ahari, and V. D. Hernandez, "An interval valued neutrosophic decision-making structure for sustainable supplier selection," *Expert Syst. Appl.*, vol. 183, p. 115354, 2021.

[15] I. Deli, "Interval-valued neutrosophic soft sets and its decision making," *Int. J. Mach. Learn. Cybern.*, vol. 8, pp. 665–676, 2017.

Cybernetics, 10 (2019) 347-355.

[16] H. Jafari, "Identification and prioritization of grain discharging operations risks by using ORESTE method," *J. Asian Bus. Strateg.*, vol. 3, no. 9, p. 233, 2013.

[17] A. Alinezhad, J. Khalili, A. Alinezhad, and J. Khalili, "ORESTE method," *New methods Appl. Mult. Attrib. Decis. Mak.*, pp. 17–21, 2019.

[18] R. Bausys, E.K. Zavadskas, Multicriteria decision making approach by VIKOR under interval neutrosophic set environment, Econ. Comput. Econ. Cybern. Stud., 49 (2015) 33-48.

[19] E. Bolturk, A. Karasan, Interval valued neutrosophic CODAS method for renewable energy selection, in: 13th International Conference on Fuzzy Logic and Intelligent Technologies in Nuclear Science (FLINS), World Scientific Publ Co Pte Ltd, Belfast, IRELAND, 2018, pp. 1026-1033.

[20] A. Karasan, C. Kahraman, A novel interval-valued neutrosophic EDAS method: prioritization of the United Nations national sustainable development goals, Soft Comput., 22 (2018) 4891-4906.

[21] Y.R. Hong, D.S. Xu, K.L. Xiang, H. Qiao, X.X. Cui, H.X. Xian, Multi-Attribute Decision-Making Based on Preference Perspective with Interval Neutrosophic Sets in Venture Capital, Mathematics, 7 (2019) 13.

Received: Sep 5, 2024. Accepted: Dec 6, 2024

NSS

Neutrosophic Sets and Systems, Vol. 78, 2025

University of New Mexico

A Neutrosophic Fixed Charge Multi-Objective 4 Dimensional Multi-item Transportation Problem with Carbon Emission and Budget Constraints

A.N. Revathi[1] and S. Mohanaselvi[1,*]

[1]Department of Mathematics, SRM Valliammai Engineering College,

[1,*] Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chennai– 603 203, Tamil Nadu, India.

anrevathi 77@gmail.com

*Correspondence: mohanass@srmist.edu.in; Tel.:+91 − 9940087667

Abstract. This paper seeks to address 4-dimensional transportation challenges by integrating budget constraints, vehicle speed, carbon emissions, and optimizing for maximum profit and minimum time. The rising number of vehicles, varying road conditions, and diverse vehicle types used to meet daily needs have exacerbated issues like global warming and greenhouse gas (GHG) emissions. Our goal is to reduce carbon emissions for a cleaner environment. Single-objective, single-item transportation systems are often insufficient for scenarios with multiple criteria. Thus, we adopt a multi-objective, multi-item decision-making approach to tackle real-life transportation problems. Practical challenges frequently involve uncertainties stemming from time pressures, insufficient data, information gaps, or inaccuracies in measurement. Recognizing this, we incorporate indeterminacy in our problem design. To address the uncertainties and ambiguities found in fuzzy and intuitionistic fuzzy settings, we utilize a neutrosophic set, which is well-suited to handle such complexities. Consequently, we employ neutrosophic linear programming to derive a compromise solution for the proposed neutrosophic fixed charge multi-objective, four-dimensional, multi-item transportation problem, considering carbon emissions and budget constraints. The model's effectiveness is showcased using a numerical example, with the results displayed and analyzed. Finally, sensitivity analysis and conclusions, along with future research directions, are presented.

Keywords: Neutrosophic linear programming; Carbon emission;Fixed charge transportation problem; Decision making on multi-objective.

1. Introduction

In recent decades, the traditional transportation system has primarily focused on a single objective function. However, transportation problems in the real world are inherently multi-dimensional, involving criteria such as transit cost, average delivery time, and fixed charges (FC) for routes. These factors become critical when transporting a homogeneous product to various destinations within a competitive economic landscape. Due to these complexities, real-world decision-making cannot be solved effectively by single-objective transportation problem (TP) models.

To address this, traditional TPs have evolved to incorporate multi-objective decision-making models, which handle competing objectives such as cost, time, and carbon emissions. Hitchcock [1] first defined the TP, combining it with linear programming (Hitchcock-Koopmans). Numerous studies on multi-objective TP (MOTP) in both precise and imprecise environments have emerged since then. Maity et al. [2] investigated the application of artificial intelligence to multi-modal TP, while Ebrahimnejad [3] introduced an approach for solving TP using generalized trapezoidal fuzzy numbers. Li and Lai [4] explored MOTP in a fuzzy environment, and Maity et al. [5] designed a framework for uncertain environments, emphasizing cost reliability. These studies underline the increasing complexity of transportation issues and the need for sophisticated models.

Motivation for this research arises from the limitations of existing methods in handling real-world complexities, particularly the combination of fixed costs, multiple objectives, carbon emissions, and budget constraints. The fixed-charge transportation problem (FCTP), for example, extends the basic TP by introducing a 0-1 decision variable and fixed charges, including vehicle rental, highway tolls, and machine establishment costs. First introduced by Hirsch and Dantzig [6] , FCTP plays a crucial role in optimizing transportation costs by accounting for both fixed and direct costs. Recent research by Midya and Roy [7, 8] applied interval and rough interval programming to FCTP in stochastic environments, while Roy et al. [9] addressed random rough variables.

Tools such as interval programming and neutrosophic logic have emerged as powerful methods for addressing uncertainty in transportation problems. The Solid Transportation Problem (STP), originally defined by Haley [10], adds conveyance constraints along with source and destination constraints, while Roy and Midya [11] tackled the Fixed Charge Solid Transportation Problem (FCSTP) using intuitionistic fuzzy logic. Once a single-type conveyance is utilized

in FCTP, TP is redefined and renamed as the fixed charge solid transportation problem (FCSTP). FCSTP was resolved in intuitionistic fuzzy (IF) by Roy and Midya [11] using concepts of product blending and multi-objective setting. The STP and facility location with carbon policies were represented by Das et al. [12]. A MOSTP with IF-type variables and parameters was solved by Ghosh et al. [13]. Using two-fold uncertainty, Roy et al. [14] examined FCSTP with multi-item and multi-objective.

In the context of an uncertain interval programming environment, Sifaouil and Aider [15] examined safety precautions and budgetary restrictions in a FCSTP with multiple objectives and items. The transportation industry primarily uses trucks, cars, ships, planes, buses, trains, and other modes of transportation to move goods and people. The transportation system is a major contributor to CO2 and other greenhouse gas emissions due to the combustion processes in engines. Light-duty vehicles, including passenger cars and minibuses, account for half of these emissions, while heavy-duty vehicles such as trucks, ships, and freight transport are responsible for the remaining half.

Air and environmental pollution are major risks associated with greenhouse gas emissions. The type of fuel, engine, traffic laws, road conditions, driving regulations, etc. all affect the carbon emission(CE). A range of researchers have explored CE across various contexts. Here are some key studies that focus on the impact of CE within transportation-related challenges These studies demonstrate the growing complexity of transportation problems, particularly when considering environmental factors like carbon emissions (CE). The transportation industry is a major source of CE, with light-duty vehicles and freight transport contributing heavily to greenhouse gas emissions.

Given the growing environmental concerns, many researchers have focused on the impact of CE in transportation. Ding et al. [16] examined potential reductions in China's transport sector, while Sengupta et al. [17] used the Gamma type-2 defuzzification approach to address CE within STPs. Song and Leng [18] explored CE policies for single-period problems. Strategies for reducing CO2 emissions in smart transport were studied by Tarulescu et al. [19]. FCSTP with multi-stage, multi-objective in a green supply chain was presented by Midya et al. [20]. These studies highlight the need for transportation models that effectively balance cost, time, and environmental impact. Ali and Muthuswamy [21] emphasize the need for sustainable evaluation of power production systems to achieve a cleaner energy future, incorporating environmental impacts, resource efficiency, and economic viability through a multi-criteria decision-making (MCDM) approach using the VIKOR method integrated with

neutrosophic sets.

Novelty in this research lies in the introduction of a Neutrosophic Fixed Charge Multi-Objective 4-Dimensional Multi-Item Transportation Problem (NFCMO4DMITP) model, which integrates fixed charges, carbon emissions, and budget constraints. The neutrosophic theory provides a powerful framework to handle uncertainty, allowing for the formulation of models that account for both objective and subjective uncertainties in transportation scenarios. This model is particularly useful in multi-dimensional problems where multiple products, routes, and vehicles are involved.

The environment in which the model is applied involves uncertainty in both transportation costs and emissions, as well as the presence of budget constraints, making it suitable for real-world scenarios with complex logistics challenges. The contribution of this research is the development of a comprehensive model that addresses not only the traditional transportation objectives but also the emerging concerns related to carbon emissions and environmental sustainability.

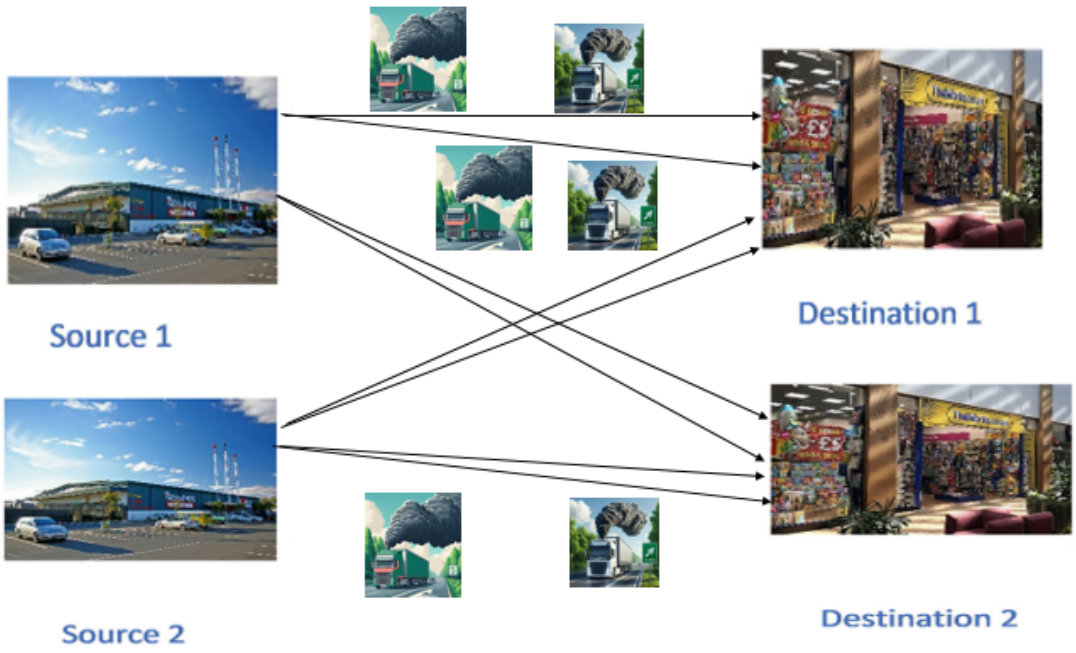


FIGURE 1. Pictorial representation of 4DTP

The rest of the paper is structured as follows: Section 2 provides fundamental definitions and theorems related to neutrosophic theory. Section 3 outlines the mathematical formulation

TABLE 1. Existing work on TP

References	Dimension	Objective	FC	TC	CE	Time	Environment	Solution method
Thamaraiselvi and Santhi [22]	2	S	-	✓	-	-	N	NLGP
Sengupta et al. [23]	3	M	-	✓	-	✓	T2Z	IFP
Roy et al. [24]	2	M	-	✓	✓	-	IF	Interval-valued IFP
Gessesse et al. [25]	2	M	-	✓	✓	-	Burr distribution	GA based FP
Paul et al. [26]	3	S	-	✓	-	-	N	LINGO 17.0 (ranking function)
Midya et al. [27]	3	M	✓	✓	✓	✓	IF	Min-max GP
Aktar et al. [28]	4	S	-	✓	_		T2F	GRGT
Adhami and Ahmad [29]	2	M	-	✓	-	-	PFN	PHFP
Malik and Gupta [30]	2	M	-	✓	-	-	Interval-valued IF	GP
Samanta et al. [31]	4	M	-	-	✓	-	LR-type IF	Convex combination method
Mohammed and Wang [32]	3	M	-	✓	✓	✓	F	LP-matrix, GP
Rizk-Allah et al. [33]	2	M	-	✓	✓	-	C	NCPA
Kumar et al. [34]	2	S	-	✓	-	-	PF	PF
Pratihar et al. [35]	2	S	✓	-	-	-	T2F	Modified VAM
Khalifa et al. [36]	2	S	-	✓	-	-	N	KKM approach
This paper	4	M	✓	✓	✓	✓	N	Interval-valued NP

S-Single, M-Multi, F-Fuzzy, N-Neutrosophic IF-Intuitionistic fuzzy, T2F-Type 2 Fuzzy, FP-Fuzzy Programming, PF-Pythagorean Fuzzy, PHFP-Pythagorean hesitant Fuzzy programming, T2Z-Type 2 zigzag, GP- Goal Programming, NP-Neutrosophic Programming, GA-Genetic Algorithm, GRGT- generalised reduced gradient technique, PFN- parabolic Fuzzy Number C-Crisp

of the proposed NFCMO4DMITP model. Section 4 details the solution methodology, while Section 5 discusses the benefits of the study. Finally, Sections 6, 7, and 8 present a numerical example, sensitivity analysis, and conclusions with future research directions.

2. **Preliminaries**

In this section, we begin by presenting some fundamental definitions of neutrosophic set theory, followed by a discussion on the neutrosophic compromise programming approach.

Definition 2.1. A neutrosophic set(NS) ${}^{n}\tilde{N}$ defined in the universal set Y is characterized by truth membership function (TMF) $T_{{}^{n}\tilde{N}}(y)$ indeterminacy membership function(IMF) $I_{{}^{n}\tilde{N}}(y)$

and a falsity membership function (FMF) $F_{{}^n\tilde{N}}(y)$ and is denoted by

$$ {}^n\tilde{N} = \{\langle y, T_{{}^n\tilde{N}}(y), I_{{}^n\tilde{N}}(y), F_{{}^n\tilde{N}}(y)\rangle \,|y \in Y\} \tag{1}$$

where $T_{{}^n\tilde{N}}(y), I_{{}^n\tilde{N}}(y), F_{{}^n\tilde{N}}(y)$ are real standard or non standard subsets belonging to]0- ,1+[. Also TMF, IMF and FMF are the functions from Y to]0- ,1+[. Also we have $0^- \leq \sup T_{{}^n\tilde{N}}(y) + \sup I_{{}^n\tilde{N}}(y) + \sup F_{{}^n\tilde{N}}(y) \leq 3^+$.

Wang et al. [37] introduced Single valued Neutrosophic set (SVNS) in engineering problem as it is computationally more comfortable.

Definition 2.2. A SVNS ${}^n\tilde{N}$ defined on Y is expressed as
${}^n\tilde{N} = \{\langle y, T_{{}^n\tilde{N}}(y), I_{{}^n\tilde{N}}(y), F_{{}^n\tilde{N}}(y)\rangle \,|y \in Y\}$ where
$T_{{}^n\tilde{N}}(y), I_{{}^n\tilde{N}}(y), F_{{}^n\tilde{N}}(y) \in [0,1], \forall y \in Y$ and $0 \leq T_{{}^n\tilde{N}}(y), I_{{}^n\tilde{N}}(y), F_{{}^n\tilde{N}}(y) \leq 3$.
Clearly, SVNS is subset of NS.

Definition 2.3. The complement of NS ${}^n\tilde{N}$ is denoted by $c^n\tilde{N}$ and is defined by $T_{c^n\tilde{N}}(y) = F_{{}^n\tilde{N}}(y), I_{c^n\tilde{N}}(y) = 1 - I_{{}^n\tilde{N}}(y),\ F_{c^n\tilde{N}}(y) = T_{{}^n\tilde{N}},\ \forall y \in Y$

Definition 2.4. The standard formulation of an optimization problem with i objectives, i variables and K constraints is as follows
Minimize $Z(Y) = (Z_1(y), Z_2(y), Z_3(y), \cdots, Z_i(y))$
subject to
$h_k(y) \leq 0, (k = 1, 2, \cdots, K)$
$y_u \geq 0, (u = 1, 2, \cdots, i), y_u \in Y \subset R^n$.

By individually solving the n objective functions subject to the constraints, we obtained i solutions $Y_1, Y_2, \cdots, Y_i$.
To find the bounds for each objective function, we obtain the values of each objective at each of these n solutions.

$$\begin{aligned} &(i.e.)\, U_i = \max\{Z_i(Y_1), Z_i(Y_2), \cdots, Z_i(Y_i)\} \\ &\text{and } L_i = \min\{Z_i(Y_1), Z_i(Y_2), \cdots, Z_i(Y_i)\} \end{aligned} \tag{2}$$

The TMF $T_{{}^n\tilde{N}}(y)$, IMF $I_{{}^n\tilde{N}}(y)$ and FMF $F_{{}^n\tilde{N}}(y)$ functions for the considered objective functions $Z_i(y)$ are respectively defined. Hence, the upper and lower bounds for TMF, IMF and FMF are given by

$$\left.\begin{aligned} &U_i^T = U_i, L_i^T = L_i \\ &U_i^F = U_i^T, L_i^F = L_i^T + t_i(U_i^T - L_i^T) \\ &U_i^I = L_i^T + s_i(U_i^T - L_i^T), L_i^I = L_i^T \end{aligned}\right\} \tag{3}$$

where t_i, s_i are predetermined real numbers in (0,1).

Using the above upper and lower bounds, TMF, IMF and FMF are interpreted as follows:

$$T_i(Z_i^*(y)) = \begin{cases} 1 & \text{if } Z_i^*(y) < L_i^T \\ \dfrac{U_i^T - Z_i^*(y)}{U_i^T - L_i^T} & \text{if } L_i^T \le Z_i^*(y) \le U_i^T \\ 0 & \text{if } Z_i^*(y) > U_i^T \end{cases} \tag{4}$$

$$I_i(Z_i^*(y)) = \begin{cases} 1 & \text{if } Z_i^*(y) < L_i^I \\ \dfrac{U_i^I - Z_i^*(y)}{U_i^I - L_i^I} & \text{if } L_i^I \le Z_i^*(y) \le U_i^I \\ 0 & \text{if } Z_i^*(y) > U_i^I \end{cases} \tag{5}$$

$$F_i(Z_i^*(y)) = \begin{cases} 1 & \text{if } Z_i^*(y) > U_i^F \\ \dfrac{Z_i^*(y) - L_i^F}{U_i^F - L_i^F} & \text{if } L_i^F \le Z_i^*(y) \le U_i^F \\ 0 & \text{if } Z_i^*(y) < L_i^F \end{cases} \tag{6}$$

where $U_i^{(.)} \neq L_i^{(.)}$ for all objectives.

Definition 2.5. Interval valued neutrosophic number: Samarandache [38] An neutrosophic number(NN) is represented by ${}^n\tilde{b} = c + dI$, where c,d $\in R$ and I $\subset [0,1]$ is the indeterminacy. Now $I = [I^L, I^U]$, and therefore the interval form of ${}^n\tilde{b} = [c + dI^L, c + dI^U] = [b^L, b^U]$.

Definition 2.6. Neutrosophic linear programming problem(NLPP): Ye [39] A NLPP is defined as the general optimization problem if the following conditions are met;

(1) The objective function in the neutrosophic framework is linear.
(2) Every decision variable is non-negative.
(3) All structural constraints are either " $\le$ " or " $\ge$ " types.

Definition 2.7. Ya [39] In general, a constrained optimization problem in n decision variable with neutrosophic number(NN) is formulated as below:

$$\text{minimize/maximize}^n Z(y, I)$$

subject to

$$f_i(y, I) \le 0, i = 1, 2, \cdots, m$$
$$p_k(y, I) = 0, k = 1, 2, \cdots, K$$

$y \in Y$ When indeterminacy I is treated as a potential interval range, the optimal solutions across all feasible intervals define the feasible region or feasible set for y and $I = [I^L, I^U]$. In this scenario, the value of the NN objective function represents an optimal possible interval NN for ${}^n Z(y, I)$. For instance, consider the following optimization problem with $I = [0, 1]$.

$$\text{maximize } {}^n Z(y, I) = (4 + 5I)y_1 + (6 + I)y_2$$

$$= [4y_1 + 6y_2, 9y_1 + 7y_2]$$

subject to

$$(2 + 2I)y_1 + (3 + I)y_2 \leq (8 + 2I)$$

$$= [2y_1 + 3y_2, 4y_1 + 4y_2] \leq [8, 10]$$

$$(5 + 2I)y_1 + (8 + I)y_2 \leq (20 + 2I)$$

$$= [5y_1 + 8y_2, 7y_1 + 9y_2] \leq (20 + 2I)$$

This problem can now be novelly transformed into two equivalent crisp sub-problems to determine the worst (lower bound) and best (upper bound) solutions as follows:

Sub-Problem(i)

maximize $4y_1 + 6y_2$

subject to

$2y_1 + 3y_2 \leq 8$

$4y_1 + 4y_2 \leq 10$

Sub-Problem(ii)

maximize $9y_1 + 7y_2$

subject to

$2y_1 + 3y_2 \leq 8$

$4y_1 + 4y_2 \leq 10$

By addressing the two subproblems, we derive the optimal solution for the original problem, which is represented as the NN interval: $Z = [15, 22.5]$.

Nomenclature

m	index for origins
t	index for destinations
v	indexed by the type of transport
w	index for the routes
g	index for the goods
${}^{n}(\tilde{Z}_i)$	objective function with NN (i=1,2)
${}^{\mathrm{In}}\widetilde{\mathbf{Z}}_{\mathbf{i}}$	the interval valued objective function (i=1,2)
${}^{\mathrm{L}}\mathbf{Z}_{\mathbf{i}}$	deterministic form of the lower-level objective functions(i=1,2)
${}^{\mathrm{U}}\mathbf{Z}_{\mathbf{i}}$	deterministic form of the upper-level objective functions(i=1,2)
${}^{\mathrm{n}}\widetilde{\mathbf{P}}_{\mathrm{mg}}$	unit purchasing price of the g^{th} product at m^{th} origin

Nomenclature

$^{\mathbf{n}}\widetilde{\mathbf{S}}_{\mathrm{tg}}$	unit selling price of g^{th} product at t^{th} destination
$\mathbf{y}_{\mathrm{mtvwg}}$	quantity of g^{th} goods that to be transit from m^{th} origin to t^{th} destination with v^{th} vehicle via w^{th} route
$^{\mathbf{n}}\widetilde{\mathbf{C}}_{\mathrm{mtvwg}}$	the unit transportation cost of the product from m^{th} origin to t^{th} destination via v^{th} mode of conveyance per unit distance
$^{\mathrm{n}}\widetilde{\mathbf{f}}_{\mathrm{mtvwg}}$	the FC, during the transit activity happens from m^{th} origin to t^{th} destination by v^{th} conveyance via w^{th} route
$^{\mathbf{n}}\widetilde{\mathbf{d}}_{\mathrm{mtvwg}}$	distance from from m^{th} origin to t^{th} destination via w^{th} route
V_v	speed of the v^{th} transport
γ_{mtw}	disturbance rate of the speed due to w^{th} route from m^{th} origin to the t^{th} destination
$^{\mathrm{n}}\widetilde{\mathbf{L}}_{\mathrm{mtvg}}$	time for loading and unloading the g^{th} goods during transit from the m^{th} origin to the t^{th} destination using the v^{th} mode of transport
$^{\mathrm{n}}\widetilde{\mathbf{a}}_{\mathrm{mg}}$	g^{th} good's availability at m^{th} origin
$^{\mathrm{n}}\widetilde{\mathbf{b}}_{\mathrm{tg}}$	g^{th} good's demand at the t^{th} destination
$^{\mathbf{n}}\widetilde{\mathbf{e}}_{\mathbf{v}}$	v^{th} transport's capacity
$h(\mathbf{y}_{\mathrm{mtvwg}})$	binary variable takes the value 1 if $\mathbf{y}_{\mathrm{mtvwg}} \neq \mathbf{0}$ **and 0 if** $\mathbf{y}_{\mathrm{mtvwg}} = \mathbf{0}$
$^{\mathrm{n}}\widetilde{\mathrm{Bud}}_{\mathbf{t}}$	total budget at the t^{th} destination
$^{\mathrm{n}}\widetilde{\alpha}$	carbon tax at per unit of its CE
$^{\mathbf{n}}\widetilde{\mathbf{e}}_{\mathrm{mtv}}$	charge of CE per unit

3. Mathematical Formation

This section contains the mathematical formulation of the proposed NFCMO4DMITPCEBC.

An NFCMO4DMITPCEBC with vehicle speed is formulated as follows. Let there be M origins O_m $(m = 1, 2, \ldots M)$, T-demands D_t $(t = 1, 2, \ldots, T)$, W-roads Q_w $(w = 1, 2 \ldots W)$, G- goods P_g $(g = 1, 2, \ldots G)$, V-conveyances E_v $(v = 1, 2, \ldots, V)$. The transportation model seeks to maximize profits and minimize transit times, but accurately estimating parameters is challenging due to uncertainties in practical applications. Decision-makers face unknowns such as raw material availability, fluctuating shipping costs, and demand variations. The novelty of this research is its approach to managing these uncertainties, improving decision-making in complex transportation systems.

Maximizing the total profit is the first objective and minimizing the transportation time is the second objective of the model which are given below.

$$\text{Maximum profit} = \text{Total revenue} - \text{Total purchase price}$$

$$- \text{Total transportation cost}$$

$$\text{Minimum time} = \text{Transporting time} + \text{Loading/Unloading time}$$

The transporting time will be taken into consideration if the goods are transported from m^{th} origin to t^{th} destination by v^{th} transport via w^{th} road. First constraint require that all goods available (a_m) at each origin m be delivered,

Second constraint force, delivery (b_t) units of goods to each destination,

Third constraint force capacity of the vehicle (e_v),

Fourth constraint force, budget of the destination, Fifth constraints represent the fixed carbon capacity of the v^{th} transport mode.

Assume that H is a function of y_{mtvwg} takes values 0 and 1 to describe the transportation activity from source m to destination t through v^{th} transport mode. Finally, sixth and seventh constraints set the ranges of y_{umv} and H(x_{umv}) respectively.

The mathematical formulation of NFCMO4DMITPCEBC is given as below Model 1:

Model 1:

$$\text{Max } {}^{n}\widetilde{\mathbf{Z}}_{\mathbf{1}} = \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}[{}^{n}\widetilde{S_{\text{tg}}} - {}^{n}\widetilde{P}_{\text{mg}} - {}^{n}\widetilde{\alpha}\ {}^{n}\widetilde{e_{\text{mtv}}} - {}^{n}\widetilde{C}_{\text{mtvwg}} d_{\text{mtw}}] y_{\text{mtvwg}}$$
$$- \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} {}^{n}\widetilde{f}_{\text{mtvg}}\ h_{\text{mtvwg}})$$

$$\text{Min } {}^{n}\widetilde{\mathbf{Z}}_{\mathbf{2}} = \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\frac{d_{\text{mtw}} \text{x}_{\text{mtvw}}}{V_{\text{v}}\ (1-\gamma_{\text{mtw}})} + \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}({}^{n}\widetilde{\delta}_{\text{mtvg}}\ y_{\text{mtvwg}})$$

where

$$\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}} \ \leq {}^{n}\widetilde{a}_{\text{mg}}\ ,\ m = 1, \cdots, M,\ g = 1, \cdots, G$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}} \ \geq {}^{n}\widetilde{b}_{\text{tg}}\ ,\ t = 1, \cdots, T,\ g = 1, \cdots, G$$

$$\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{w=1}^{W}\sum_{g=1}^{G} y_{\text{mtvwg}} \leq {}^{n}\widetilde{e}_{v}\ \ ,\ v = 1, \cdots, V$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}\left({}^{n}\widetilde{P}_{\text{mg}} + {}^{n}\widetilde{C}_{\text{mtvwg}} + {}^{n}\widetilde{\alpha}\ \ {}^{n}\widetilde{e_{\text{mtv}}}\right) y_{\text{mtvwg}}$$
$$+ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} {}^{n}\widetilde{f}_{\text{mtvw}} \text{x}_{\text{mtvw}} \leq {}^{n}\widetilde{\text{Bud}}_{t}\ ,\ \forall\ t = 1, \cdots, T.$$

$$y_{\text{mtvwg}} \ \geq 0,\ \forall\ m, t, v, w, g.$$

$$\sum_{\mathbf{m=1}}^{\mathbf{M}}\sum_{\mathbf{t=1}}^{\mathbf{N}}\sum_{w=1}^{W}\sum_{g=1}^{G} {}^{n}\widetilde{e}_{\text{mtv}} y_{\text{mtvwg}} \leq {}^{n}\widetilde{E}_{\text{v}}\ \ , v = 1, \cdots, V.$$

$$x_{\text{mtvw}} = \begin{cases} 1 \ if \ \sum_{g} y_{\text{mtvwg}} > 0 \\ 0 \ if \sum_{g} y_{\text{mtvwg}} \ = 0 \end{cases}$$

$$h_{\text{mtvwg}} = \begin{cases} 1 \ if \sum_{g} y_{\text{mtvwg}} > 0 \\ 0 \ if \sum_{g} y_{\text{mtvwg}} = 0 \end{cases}$$

The condition for feasibility of the TP is as below

$$\sum_{m=1}^{M}\sum_{g=1}^{G} {}^{n}\tilde{a}_{mg} \geq \sum_{t=1}^{T}\sum_{g=1}^{G} {}^{n}\tilde{b}_{tg}; \sum_{v=1}^{V} {}^{n}\tilde{e}_{v} \geq \sum_{t=1}^{T}\sum_{g=1}^{G} {}^{n}\tilde{b}_{tg}. \tag{7}$$

The parameters considered in the above model are NN, leading to Model 1 being identified as NFCMO4DMITPCEBC. To derive the Pareto optimal solution, we treat minimizing ${}^{In}\tilde{Z}_1$ is equivalent to maximizing$(-{}^{n}\tilde{Z}_1\ )$ and minimizing ${}^{In}\tilde{Z}_2$ is equivalent to minimizing ${}^{n}\tilde{Z}_2$.

Without losing generality, we specify the NN in Model 1 as ${}^{n}\tilde{a} = q + rI$, with I representing indeterminacy. As a result, Model 1 is reformulated into Model 2, as outlined below:

Model 2:

$$\begin{aligned} \text{Min } {}^{In}\widetilde{\mathbf{Z}}_{\mathbf{1}} = & \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} [-(S^{1}_{\text{tg}} + I_{\text{s}} S^{2}_{\text{tg}}) + (P^{1}{}_{\text{mg}} + I_{\text{p}} P^{2}{}_{\text{mg}}) + (\alpha^{\ 1} \\ & + I_{\ \alpha} \alpha^{\ 2})(e_{\text{mtv}}{}^{1} + I_{\text{e}} e_{\text{mtv}}{}^{2}) + (C^{1}{}_{\text{mtvwg}} + I_{\text{c}} C^{2}{}_{\text{mtvwg}}) d_{\text{mtw}}] y_{\text{mtvwg}} \\ & - \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} (f^{1}{}_{\text{mtvg}} + (I_{\text{f}} f^{2}_{\text{mtvg}}\ h_{\text{mtvwg}}) \end{aligned}$$

$$\text{Min } {}^{In}\widetilde{\mathbf{Z}}_{\mathbf{2}} = \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W} \frac{d_{\text{mtw}} \text{x}_{\text{mtvw}}}{V_{\text{v}}\ (1 - \gamma_{\text{mtw}})} + \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} (\delta^{1}{}_{\text{mtvg}}\ + I_{\text{mtvg}}\ \delta^{2}{}_{\text{mtvg}}\ ) y_{\text{mtvwg}}$$

where

$$x_{\text{mtvw}} = \begin{cases} 1 \ if \ \sum_{g} y_{\text{mtvwg}} > 0 \\ 0 \ if \sum_{g} y_{\text{mtvwg}} \ = 0 \end{cases}$$

$$h_{\text{mtvwg}} = \begin{cases} 1 \ if \sum_{g} y_{\text{mtvwg}} > 0 \\ 0 \ if \sum_{g} y_{\text{mtvwg}} = 0 \end{cases}$$

$$\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}}\ \leq (a^{1}{}_{\text{mg}} + I_{\text{a}} a^{2}{}_{\text{mg}})\ ,\ m = 1, \cdots, M,\ g = 1, \cdots, G$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}} \ \geq (b^{1}{}_{\text{tg}} + I_{\text{b}} b^{2}{}_{\text{tg}}) \ , \ t = 1, \cdots, T, \ g = 1, \cdots, G$$

$$\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{w=1}^{W}\sum_{g=1}^{G} y_{\text{mtvwg}} \leq (e^{1}{}_{v} \ \ + I_{v} \ \ e^{2}{}_{v} \ \ ,) \ v = 1, \cdots, V$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} \left((P^{1}{}_{\text{mg}} + I_{\text{p}} P^{2}{}_{\text{mg}}) + (C^{1}{}_{\text{mtvwg}} + I_{\text{C}} C^{2}{}_{\text{mtvwg}}) + (\alpha^{\ 1} \ + I^{1} \alpha^{\ 2}) \ (e^{1}_{\text{mtv}} + I_{\text{e}} e^{2}_{\text{mtv}}) \right) y_{\text{mtvwg}}$$

$$+ \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} f^{1}{}_{\text{mtvw}} + I_{\text{f}} f^{2}{}_{\text{mtvw}} \text{x}_{\text{mtvw}} \leq (\text{Bud}^{1}{}_{t} + \text{I}_{\text{Bud}} \text{Bud}^{2}{}_{t}) \ , \ \forall \ t = 1, \cdots, T.$$

$$y_{\text{mtvwg}} \ \geq 0, \ \forall \ m, t, v, w, g.$$

$$\sum_{\mathbf{m=1}}^{\mathbf{M}}\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{w=1}^{W}\sum_{g=1}^{G} (e^{1}{}_{\text{mtv}} + I_{\text{e}} e^{2}{}_{\text{mtv}} y_{\text{mtvwg}}) \leq (E^{1}{}_{\text{v}} \ \ + I_{\text{E}} \ \ E^{2}{}_{\text{v}} \ \ , v = 1, \cdots, V.)$$

Now a NN can be represented as ${}^{n}\tilde{a} = [q + rI^{L}, q + rI^{U}] = [a^{L}, a^{U}]$, which constitutes an interval number with defined lower and upper bounds.This leads to the need to decompose Model 2 into two distinct crisp problems, namely Model 3 and Model 4. Here, Model 3 is recognized as the lower-level problem, whereas Model 4 is identified as the upper-level problem. By applying inequalities, the interval constraints are converted into a deterministic format. Therefore, the deterministic versions of these models are as follows:

Model 3:

$$\text{Min } {}^{IL}\mathbf{Z_1} = \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} [-S^{L}{}_{\text{tg}} + P^{L}{}_{\text{mg}} + \alpha^{L} \ e^{L}_{\text{mtv}} + C^{L}{}_{\text{mtvwg}} d_{\text{mtw}}] y_{\text{mtvwg}}$$

$$+ \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} f^{L}{}_{\text{mtvg}} \ h_{\text{mtvwg}})$$

$$\text{Min } {}^{IL}\mathbf{Z_2} = \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W} \frac{d_{\text{mtw}} \text{x}_{\text{mtvw}}}{V_{\text{v}} \ (1 - \gamma_{\text{mtw}})} + \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} (\delta^{L}{}_{\text{mtvg}} \ y_{\text{mtvwg}})$$

where

$$x_{\text{mtvw}} = \begin{cases} 1 \ if \ \sum\limits_{g} y_{\text{mtvwg}} > 0 \\ 0 \ if \sum\limits_{g} y_{\text{mtvwg}} \ = 0 \end{cases}$$

$$h_{\text{mtvwg}} = \begin{cases} 1 \ if \sum\limits_{g} y_{\text{mtvwg}} > 0 \\ 0 \ if \sum\limits_{g} y_{\text{mtvwg}} = 0 \end{cases}$$

$$\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}} \ \leq a^{U}{}_{\text{mg}} \ , \ m = 1, \cdots M, \ g = 1, \cdots, G$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mnvwg}} \ \geq b^{L}{}_{\text{ng}}\ ,\ t=1,\cdots,T,\ g=1,\cdots,G$$

$$\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{w=1}^{W}\sum_{g=1}^{G} y_{\text{mtvwg}} \leq e^{U}{}_{v}\ \ ,\ v=1,\cdots,V$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}\left(P^{L}{}_{\text{mg}}+C^{L}{}_{\text{mtvwg}}+\alpha^{\ L}\ e^{L}_{\text{mtv}}\right)y_{\text{mtvwg}}$$
$$+\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} f^{L}{}_{\text{mtvw}}\text{x}_{\text{mtvw}} \leq \text{Bud}^{L}{}_{t}\ ,\ \forall\ t=1,\cdots,T.$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}\left(P^{U}{}_{\text{mg}}+C^{U}{}_{\text{mtvwg}}+\alpha^{\ U}\ e^{U}_{\text{mtv}}\right)y_{\text{mtvwg}}$$
$$+\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} f^{U}{}_{\text{mtvw}}\text{x}_{\text{mtvw}} \leq \text{Bud}^{U}{}_{t}\ ,\ \forall\ t=1 \text{ to T}.$$

$$y_{\text{mtvwg}}\ \geq 0,\ \forall\ m,t,v,w,g.$$
$$\sum_{\mathbf{m=1}}^{\mathbf{M}}\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{w=1}^{W}\sum_{g=1}^{G} e^{L}{}_{\text{mtv}}y_{\text{mtvwg}} \leq E^{L}{}_{\text{v}}\ \ ,v=1,\cdots,V.$$
$$y_{\text{mtvwg}}\ \geq 0,\ \forall\ m,t,v,w,g.$$
$$\sum_{\mathbf{m=1}}^{\mathbf{M}}\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{w=1}^{W}\sum_{g=1}^{G} e^{U}{}_{\text{mtv}}y_{\text{mtvwg}} \leq E^{U}{}_{\text{v}}\ \ ,v=1,\cdots,V.$$

Model 4:

$$\text{Min }{}^{IU}\mathbf{Z_1}=\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}[-S^{U}{}_{\text{tg}}+P^{U}{}_{\text{mg}}+\alpha^{U}\ e^{U}_{\text{mtv}}+C^{U}{}_{\text{mtvwg}}d_{\text{mtw}}]y_{\text{mtvwg}}$$
$$+\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} f^{U}{}_{\text{mtvg}}\ h_{\text{mtvwg}})$$
$$\text{Min }{}^{IU}\mathbf{Z_2}=\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\frac{d_{\text{mtw}}\text{x}_{\text{mtvw}}}{V_{\text{v}}\ (1-\gamma_{\text{mtw}})}+\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G}(\delta^{U}{}_{\text{mtvg}}\ y_{\text{mtvwg}})$$

where

$$x_{\text{mtvw}}=\begin{cases}1\ if\ \sum\limits_{g} y_{\text{mtvwg}}>0\\ 0\ if \sum\limits_{g} y_{\text{mtvwg}}\ =0\end{cases}$$

$$h_{\text{mtvwg}}=\begin{cases}1\ if \sum\limits_{g} y_{\text{mtvwg}}>0\\ 0\ if \sum\limits_{g} y_{\text{mtvwg}}=0\end{cases}$$

$$\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}} \ \leq {a^U}_{\text{mg}} \ , \ m=1,\cdots,M, \ g=1,\cdots,G$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W} y_{\text{mtvwg}} \ \geq {b^L}_{\text{tg}} \ , \ t=1\ldots\ldots T, \ g=1,\cdots,G$$

$$\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{w=1}^{W}\sum_{g=1}^{G} y_{\text{mtvwg}} \leq {e^U}_{v} \ , \ v=1,\cdots,V.$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} \left({P^L}_{\text{mg}} + {C^L}_{\text{mtvwg}} + \alpha^{\ L}\ e^L_{\text{mtv}}\right) y_{\text{mtvwg}}$$
$$+\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} {f^L}_{\text{mtvw}}\text{x}_{\text{mtvw}} \leq {\text{Bud}^L}_{t} \ , \ \forall \ t=1,\cdots,T.$$

$$\sum_{m=1}^{M}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} \left({P^U}_{\text{mg}} + {C^U}_{\text{mtvwg}} + \alpha^{\ U}\ e^U_{\text{mtv}}\right) y_{\text{mtvwg}}$$
$$+\sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{v=1}^{V}\sum_{w=1}^{W}\sum_{g=1}^{G} {f^U}_{\text{mtvw}}\text{x}_{\text{mtvw}} \leq {\text{Bud}^U}_{t} \ , \ \forall \ t=1,\cdots,T.$$

$$y_{\text{mtvwg}} \ \geq 0, \ \forall \ m,t,v,w,g.$$
$$\sum_{\mathbf{m=1}}^{\mathbf{M}}\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{w=1}^{W}\sum_{g=1}^{G} {e^L}_{\text{mtv}} y_{\text{mtvwg}} \leq {E^L}_{\text{v}} \ , v=1 \text{ to V}$$
$$y_{\text{mtvwg}} \ \geq 0, \ \forall \ m,t,v,w,g.$$
$$\sum_{\mathbf{m=1}}^{\mathbf{M}}\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{w=1}^{W}\sum_{g=1}^{G} {e^U}_{\text{mtv}} y_{\text{mtvwg}} \leq {E^U}_{\text{v}} \ , v=1 \text{ to V}$$

Definition 3.1. A Pareto-optimal solution, also known as a compromise solution, for Model 3 and Model 4 is a feasible solution $y^* = (y^*_{mtvwg})$ such that there does not exist any other feasible solution $y = (y_{mtvwg})$ with ${}^{In}Z_i(y) \leq^{In} Z_i(y)$ for $i = 1, 2$ and ${}^{In}Z_i(y) \leq^{In} Z_i(y^*)$ for at least one i.

Theorem 3.2. *The union of the Pareto-optimal solutions from Model 3 and Model 4 constitutes the Pareto-optimal solution for Model 2, and by extension, for Model 1 when expressed as an interval NN.*

Proof: *Theorem 3.2 follows directly from Definition 2.5.*

4. Solution Methodology

Models 3 and 4 offer the lower and upper bounds for the objective functions, respectively; however, they do not present a comprehensive compromise solution for Model 2. To address

TABLE 2. Pay-off matrix

	Z_1	Z_2
Y_1^*	$Z_1(Y_1^*)$	$Z_2(Y_1^*)$
Y_2^*	$Z_1(Y_2^*)$	$Z_2(Y_2^*)$

this, we employ NLPP techniques to derive an optimal compromise solution for the multi-objective decision-making problem. The steps to solve the proposed model using NLPP are outlined below.

(1) Convert the neutrosophic optimization problem into a crisp format, resulting in two distinct sub-problems.
(2) Address each sub-problem separately while adhering to all constraints.
(3) Designate the upper bound as the Positive Ideal Solution (PIS) and the lower bound as the Negative Ideal Solution (NIS) for each objective function within the pay-off matrix, as illustrated in Table 2. The PIS and NIS are defined as follows:
PIS=Z_i^* =Max $(Z_i(Y_1^*), Z_i(Y_2^*)$)(i=1,2) and
NIS=Z_i =Min $(Z_i(Y_1^*), Z_i^*(Y_2^*)$)(i=1,2) respectively.
(4) Construct the TMF and the IMF with the highest degree, and the FMF with the lowest degree.
(5) Define the tolerance levels and create the membership functions in accordance with the limits specified in Equations (4), (5), and (6). Here,

$$\left.\begin{aligned} &U_i^T = U_i, L_i^T = L_i \\ &U_i^F = U_i^T, L_i^F = L_i^T + r_i(U_i^T - L_i^T) \\ &U_i^I = L_i^T + s_i(U_i^T - L_i^T), L_i^I = L_i^T \end{aligned}\right\} \tag{8}$$

where r_i, s_i are predetermined real numbers in (0,1).

(6) Choose the values of $\alpha_T, \alpha_I,$ andα_F within the interval $[0, 1]$ for each NN as the truth, indeterminacy and falsity degrees respectively.

(7) Construct NLPP that represents in Model 5

Model 5

$$\text{Maximize}\{T_i({}^{In}Z_i^*(Y))\} : i = 1, 2.$$
$$\text{Minimize}\{F_i({}^{In}Z_i^*(Y))\} : i = 1, 2.$$
$$\text{Maximize}\{I_i({}^{In}Z_i^*(Y))\} : i = 1, 2.$$

subject to the constraints given in model 3.

Model 5 can be transformed into Model 6 as outlined below:

Model 6

$$\begin{aligned}
&\text{Max}\, \alpha_T \\
&\text{Max}\, \alpha_I \\
&\text{Min}\, \alpha_F \\
&\text{subject to} \\
&T_i({}^{In}Z_i(y)) \geq \alpha_T,\ I_i({}^{In}Z_i)(y) \geq \alpha_I,\ F_i({}^{In}z_i)(y) \leq \alpha_F \\
&\alpha_T \geq \alpha_I, \alpha_T \geq \alpha_F, \alpha_T + \alpha_I + \alpha_F \leq 3, \alpha_T + \alpha_I + \alpha_F \geq 0, \alpha_T, \alpha_I, \alpha_F \in [0,1]
\end{aligned} \tag{9}$$

also with constraints given in model 3. The simplified NLPP model (Model 6) that yields the compromise solution for the MOTP (i.e., Model 7) is as follows:

Model 7

$$\begin{aligned}
&\text{Max}\, \alpha_T - \alpha_F + \alpha_I \\
&\text{subject to} \\
&{}^{In}Z_i^*(y) + (U_i^T - L_i^T)\alpha_T \leq U_i^T \\
&{}^{In}Z_i^*(y) + (U_i^I - L_i^I)\alpha_I \leq U_i^I \\
&{}^{In}Z_k^*(y) - (U_i^F - L_i^F)\alpha_F \leq L_i^F \\
&\alpha_T \geq \alpha_I, \alpha_T \geq \alpha_F, \alpha_T + \alpha_I + \alpha_F \leq 3, \\
&\alpha_T, \alpha_I, \alpha_F \in [0,1], i = 1,2.
\end{aligned} \tag{10}$$

with the constraints given in model 3.

5. **Advantages of the Proposed Model**

(1) NS stand out due to their three independent membership degrees: TMF (T), IMF (I), and F (F), enabling them to effectively handle imprecise parameters. Unlike Intuitionistic Fuzzy Sets (IFS), which are restricted to managing incomplete information, NS can cope with both incomplete and indeterminate information. Consequently, NS are more proficient at addressing complete uncertainty compared to IFSs.

(2) In a NS, decision makers aim to maximize the TMF and the IMF while minimizing the FMF, reflecting more realistic scenarios in real-life problems.

(3) In the context of transportation systems, CE contribute significantly to air pollution, increasing CE charges and subsequently reducing the system's profit. Therefore, reducing CE is crucial for maximizing profit, a factor addressed in this paper. Additionally,

the inclusion of vehicle speed and disturbance rate enhances system reliability and effectively reduces transportation time.

(4) To enhance profit maximization, budget constraints and carbon capacity are also factored into this problem. These extra limitations contribute to reducing the overall time required to solve the proposed problem.

Despite these advantages, it is essential to consider the following potential drawbacks of the model to ensure well-rounded decision-making.

5.1. *Limitations of the Model*

While the model proves highly effective for real-world decision-making, it also presents certain limitations, as outlined below:

- Sensitivity to Input Variability: Minor fluctuations in key input data, such as changes in fuel prices or emission rates, can significantly influence the results. This heightened sensitivity can diminish the reliability of the model's recommendations.

- Regulatory and Compliance Challenges: Carbon emission regulations differ across regions, making it difficult to adapt the transportation model uniformly. This can lead to reduced flexibility and operational constraints in certain regions.

6. **Numerical Example**

To demonstrate the applicability of the suggested approach with maximum profit and minimum time in the transportation system, we include a real-world example in this section. Taking into account the two Orissa export hubs for two different kinds of sea fish (like Jharkhand and Chhattisgarh) in the transit network. CE charges for heavy-duty vehicles in transportation systems are in addition to other expenses like selling price, purchasing cost, transportation cost, and FC. In addition, the time factor needs to be included when sending data over long distances. NN are defined as supply, demand, and conveyance based on a number of complex factors. All NN, which are provided in Tables 3, 4, 5, 6 , 7 and 8, are the transportation cost per unit item from source to destination, FC, distance between the origin and destination, rate of disturbances of different vehicles, loading and unloading timings and the rate of CE. The cost of transport and CE are expressed in dollars. each unit and time, measured in hours. Additionally, NN are selected here, along with the selling price, purchasing cost, carbon tax,

TABLE 3. Unit transportation cost

m	t	v	$\tilde{C}_{mtv11}$	$\tilde{C}_{mtv12}$	$\tilde{C}_{mtv21}$	$\tilde{C}_{mtv22}$
1	1	1	4+3I	3+2I	3+2I	3+3I
		2	4+4I	5-I	8+I	8-5I
	2	1	2+I	6+3I	7+I	2+5I
		2	5+I	3+I	2+2I	2+3I
2	1	1	2+3I	4+2I	4+2I	5+3I
		2	2+3I	5+2I	2+2I	5+3I
	2	1	1+2I	1+2I	2+2I	2+2I
		2	1+2I	1+2I	6+2I	2+2I

source, demand, conveyance, budget, loading and unloading schedules, and carbon capacity. The goal is to maximize selling price while minimizing fixed costs, transportation costs, CE charges, and purchasing costs to achieve maximum profit. Additionally, time is limited, necessitating a boundary for the budget and carbon capacity. As a result, the issue is now NFCMO4DMITPCEBC due to the description of such conditions with the objective functions that contradict each other. The problem can be expressed mathematically by using the models 3 and 4. The following provides an illustration of the problem's solution. Furthermore, the following presumptions apply to the planned study.

Selling price $=^{\mathrm{n}}\tilde{S}_{\mathrm{tg}}$:{ $^{\mathrm{n}}\tilde{S}_{11} = 100 + 175I$; $^{\mathrm{n}}\tilde{S}_{12} = 119 + 135I$,$^{n}\tilde{S}_{21} = 221 + 52I$; $^{\mathrm{n}}\tilde{S}_{22} = 203 + 50I$ }

Purchase price $=^{\mathrm{n}}\tilde{P}_{\mathrm{mg}}$:{ $^{\mathrm{n}}\tilde{P}_{11} = 6 + 12I$; $^{\mathrm{n}}\tilde{P}_{12} = 11 + 2I$,$^{n}\tilde{P}_{21} = 12 + 2I$; $^{\mathrm{n}}\tilde{P}_{22} = 11 + I$}

Carbon tax= $^{\mathrm{n}}\tilde{\alpha}$:{ α=2+I}

Source $=(^{\mathrm{n}}\tilde{a}_{\mathrm{mg}})$: { $^{\mathrm{n}}\tilde{a}_{11} = 380 + 8I$; $^{\mathrm{n}}\tilde{a}_{12} = 290 + 10I$;$^{n}\tilde{a}_{21} = 380 + 8I$; ;$^{n}\tilde{a}_{22} = 380 + 8I$}

Demand $=(^{\mathrm{n}}\tilde{b}_{\mathrm{tg}})$: { $^{\mathrm{n}}\tilde{b}_{11} = 150 + 7I$; $^{\mathrm{n}}\tilde{b}_{12} = 130 + 10I$; $^{n}\tilde{b}_{21} = 90 + 7I$;$^{n}\tilde{b}_{22} = 120 + 8I$}

Conveyance$=(^{\mathrm{n}}\tilde{e}_{v})$: { $^{\mathrm{n}}\tilde{e}_{1} = 425 + 12I$; $^{\mathrm{n}}\tilde{e}_{2} = 430 + 19I$ }

Budget $=(^{\mathrm{n}}\tilde{\mathrm{Bud}}_{t})$: { $^{\mathrm{n}}\tilde{B}_{1} = 10800 + 1200I$; $^{\mathrm{n}}\tilde{B}_{2} = 7400 + 1800I$}

Carbon Capacity $=(^{\mathrm{n}}\tilde{c}_{v})$: { $^{\mathrm{n}}\tilde{c}_{1} = 2900 + 100I$; $^{\mathrm{n}}\tilde{c}_{2} = 3400 + 100I$}

Here, we define indeterminacy as I=[0 1].

Vehicle Speed $=(V_v)$: $(V_1 = 35; V_2 = 25)$

Solve the models which are obtained from model 3 and model 4 individually By employing the LINGO18 iterative scheme to solve the models which are obtained from model 3 and model 4 individually , we derive their respective solutions and compute the values of the objective functions, as detailed in Tables 9 and 10. Due to the contradictory nature of these solutions, we seek an overall compromise solution by solving Model 7 using the proposed NLPP. The Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS), denoted as IU and IL respectively, are

Table 4. Fixed charge

m	t	v	$\tilde{f}_{mtv11}$	$\tilde{f}_{mtv12}$	$\tilde{f}_{mtv21}$	$\tilde{f}_{mtv22}$
1	1	1	12+2I	10+2I	14+2I	12+2I
		2	20+5I	5-I	25+I	5-2I
	2	1	4+2I	1+I	4+I	20+2I
		2	4+2I	12+2I	5+2I	10+5I
2	1	1	10+2I	8+2I	7+7I	6+6I
		2	4+20I	2+3I	1+24I	4+2I
	2	1	4+I	18+2I	3+1I	17+4I
		2	4+I	12+I	4+2I	12+2I

Table 5. Distance between the origins and destinations

m	D_{m11}	D_{m12}	D_{m21}	D_{m22}
1	43	45	45	35
2	43	40	56	45

Table 6. Rate of disturbances of different vehicles

m	γ_{m11}	γ_{m12}	γ_{m21}	γ_{m22}
1	0.03	0.04	0.02	0.03
2	0.04	0.034	0.022	0.05

Table 7. Time for loading and unloading the goods

m	t	$\tilde{\delta}_{mt11}$	$\tilde{\delta}_{mt12}$	$\tilde{\delta}_{mt21}$	$\tilde{\delta}_{mt22}$
1	1	6+13I	4+10I	3+6I	5+I
	2	5+4I	3+I	4+11I	6+I
2	1	6+I	3+3I	4+I	8+I
	2	7+I	4+5I	3+6I	7+I

obtained from Tables 9 and 10. Ultimately, we develop Model 8 using NLPP to find the compromise solution for Model 2.

TABLE 8. The rate of carbon emission

m	$\tilde{e}_{m11}$	$\tilde{e}_{m12}$	$\tilde{e}_{m21}$	$\tilde{e}_{m22}$
1	2+4I	3+3I	3+2I	2+1I
2	2+3I	2+1I	2+2I	3+2I

TABLE 9. Pay off matrix for ${}^{IL}Z_i$

${}^{IL}Y$	${}^{IL}Z_1$	${}^{IL}Z_1$
${}^{IL}Y_1$	-99440.4	1947.75
${}^{IL}Y_1$	-31437.4	1578.64

TABLE 10. Pay off matrix for ${}^{IU}Z_i$

${}^{IU}Y$	${}^{IU}Z_1$	${}^{IU}Z_1$
${}^{IU}Y_1$	-80110.53	4334.3
${}^{IU}Y_1$	-261.27	2927.89

Model 8

$$
\begin{aligned}
&\text{Max}\, \alpha_T - \alpha_F + \alpha_I \\
&\text{subject to} \\
&{}^{IL}Z_i^*(y) + (U_i^L - L_i^L)\alpha_T \le U_i^L \\
&{}^{IL}Z_i^*(y) + (U_i^L - L_i^L)(\alpha_I - 1)s_1 i \le L_i^L \\
&{}^{IL}Z_I^*(y) - (U_i^L - L_i^L)\alpha_F(1 - r_1 i) \le U_i^L \\
&{}^{IU}Z_i^*(y) + (U_i^U - L_i^U)\alpha_T \le U_i^U \\
&{}^{IU}Z_i^*(y) + (U_i^U - L_i^U)(\alpha_I - 1)s_2 i \le L_i^U \\
&{}^{IU}Z_I^*(y) - (U_i^U - L_i^U)\alpha_F(1 - r_2 i) \le U_i^U \\
&\alpha_T \ge \alpha_I, \alpha_T \ge \alpha_F, \alpha_T + \alpha_I + \alpha_F \le 3, \\
&\alpha_T, \alpha_I, \alpha_F \in [0,1], i = 1,2.
\end{aligned}
\tag{11}
$$

Where U_i^L = PIS of ${}^{IL}Z_i$, L_i^L = NIS of ${}^{IL}Z_i$; U_i^U = PIS of ${}^{IU}Z_i$, L_i^U = NIS of ${}^{IU}Z_i$. The solutions for Model 8 are determined as follows;$\alpha_T = 0.7437, \alpha_I = 0.7437, \alpha_F = 0$,$x_{11211} = 63.23$, $x_{11221} = 2.379$, $x_{12112} = 119.4603$, $x_{12122} = 8.5396$, $x_{21112} = 139.495$, $x_{21122} = 0.5049$, $x_{21211} = 50.611$, $x_{21221} = 40.778$, $x_{22211} = 97$. ${}^{IL}Z_1 = 10379.72$, ${}^{IL}Z_2 = 1673.21$, ${}^{IU}Z_1 = 20332.83$ ${}^{IU}Z_2 = 3288.25$. Therefore the compromise solutions of proposed Model 1 is $Z_1 = [10379.72, 20332.83]$ and $Z_2 = [1673.21, 3288.25]$.

6.1. *Results and Discussion*

By employing NLPP to solve Model 2, the solutions for Model 1 are illustrated in Section 6. NLPP effectively addresses the MOTP. The analysis highlights the crucial need to incorporate additional constraints on CE and vehicle speed in our formulated problem. This approach not only safeguards the environment but also maximizes profitability in the transportation system by minimizing overall transportation costs and time.

7. **Sensitivity analysis**

To elucidate and interpret the impact of changes in the coefficients of objective functions, sensitivity analysis is a critical and meticulous procedure in optimization problems. Analyzing the range of all parametric values and their slight changes, while ensuring the optimal value remains constant, is challenging. There are several research papers on sensitivity analysis in TP using linear programming, such as Ebrahimnejad [40]. However, when there are significant changes in variables and constraints, complexity arises, leading to changes in the values of basic variables. To manage these complexities, we introduce sensitivity analysis for the NFCMO4DMITPCEBC with NLPP, ensuring that all basic variables remain fixed. The steps to determine the parameter ranges in NFCMO4DMITPCEBC are as follows:

(1) Keep all basic variables fixed for the optimal solution of NFCMO4DMITPCEBC.
(2) Vary each parameter's value one at a time, keeping the others fixed, and solve the NFCMO4DMITPCEBC using the LINGO 18 iterative scheme.
(3) Repeat Step 2 until either a basic variable changes or no feasible solution arises.
(4) Determine the range of each parameter from Step 3.

For supply, demand, and conveyance parameters, sensitivity analysis is conducted as follows:

- Let ${}^n\tilde{a}_{mg}$ change to ${}^n\tilde{a}^*_{mg}$ as ${}^n\tilde{a}^*_{mg} =^n \tilde{a}_{mg} + \theta_m g$
- Let ${}^n\tilde{b}_{tg}$ change to ${}^n\tilde{b}^*_{tg}$ as ${}^n\tilde{b}^*_{tg} =^n \tilde{b}_{tg} + \phi_t g$
- Let ${}^n\tilde{e}_v$ change to ${}^n\tilde{e}^*_v$ as ${}^n\tilde{e}^*_v =^n \tilde{e}_v + \omega_v$

Using this procedure, we derive the values of ${}^n\tilde{a}^*_m$,${}^n\tilde{b}^*_n$ and ${}^n\tilde{e}^*_v$, which are presented in Table 11.

8. **Conclusion and Future Work**

The transportation system is influenced by a variety of factors, including time, profit, budget, deterioration, CE, purchasing cost, selling price, and FC. Given that all data are sourced from real-world scenarios, complexities, restrictions, and uncertainties inevitably arise. Our model aims to address these complexities using neutrosophic data to find a compromise solution. By representing all data as NN and applying NLPP, we maximize the TMF and IMF while minimizing the FMF. Additionally, incorporating constraints on CE, vehicle speed, and

TABLE 11. Ranges of availability, demand and conveyance

Values of ${}^n\tilde{a}_{mg}, {}^n\tilde{b}_{ng} and {}^n\tilde{e}_v$	Changes values of ${}^n\tilde{a}_{mg}, {}^n\tilde{b}_{ng} and {}^n\tilde{e}_v$
${}^n\tilde{a}_{11}$= 380	100$\leq^n \tilde{a}^*_{11} \leq \infty$
${}^n\tilde{a}_{12}$= 290	125$\leq^n \tilde{a}^*_{12} \leq \infty$
${}^n\tilde{a}_{21}$= 250	90 $\leq^n \tilde{a}^*_{21} \leq \infty$
${}^n\tilde{a}_{22}$= 260	80.5$\leq^n \tilde{a}^*_{22} \leq \infty$
${}^n\tilde{b}_{11}$= 150	105$\leq^n \tilde{b}^*_{11} \leq 170$
${}^n\tilde{b}_{12}$= 130	60$\leq^n \tilde{b}^*_{12} \leq 180$
${}^n\tilde{b}_{21}$= 90	10$\leq^n \tilde{b}^*_{21} \leq 150$
${}^n\tilde{b}_{22}$= 120	35 $\leq^n \tilde{b}^*_{22} \leq 130$
${}^n\tilde{e}_1$= 425	230$\leq^n \tilde{e}^*_1 \leq \infty$
${}^n\tilde{e}_2$= 430	225$\leq^n \tilde{e}^*_2 \leq \infty$

disturbance rates improves profitability and reduces air pollution. NLPP has proven to be the most effective and straightforward method for multi-objective decision-making problems, yielding a compromise solution for our proposed model.

Unlike fuzzy sets and IFS, which only address uncertainty through membership and non-membership values, the neutrosophic system includes an indeterminacy membership value, providing a more reliable framework for handling incomplete and indeterminate information. The practicality of our approach has been demonstrated through a real-life example, with sensitivity analysis revealing stable parameter ranges. Furthermore, we have discussed decisions regarding budget and CE that can assist organizations in addressing economic and environmental concerns.

Future research could evolve the neutrosophic system into a bipolar-neutrosophic system for both linear and non-linear challenges. Additionally, the CE system could utilize a cap-and-trade policy with neutrosophic numbers. Neutrosophic programming using these numbers could also address fractional problems or multi-item transportation issues. Our model could be expanded to include various uncertainties, such as type-2 neutrosophic, type-2 uncertain variables, uncertain-random, and type-2 intuitionistic fuzzy variables. Furthermore, numerous heuristics, meta-heuristics, and hybrid approaches could be devised to solve larger instances of our proposed problem.

References

1. Hitchcock, F.L. The distribution of a product from several sources to numerous localities. Journal of Mathematics and Physics 1941, 20(1-4), 224–230.
2. Maity, G.; Roy, S.K.; Verdegay, J.L. Analyzing multimodal transportation problem and its application to artificial intelligence. Neural Computing and Applications 2019, 32(7), 2243–2256.

3. Ebrahimnejad, A. A simplified new approach for solving fuzzy transportation problems with generalized trapezoidal fuzzy numbers. Applied Soft Computing 2014, 19, 171–176.
4. Li, L.; Lai, K.K. A fuzzy approach to the multiobjective transportation problem. Computers and Operations Research 2000, 27(1), 43–57.
5. Maity, G.; Roy, S.K.; Verdegay, J.L. Multi-objective transportation problem with cost reliability under uncertain environment. International Journal of Computational Intelligence Systems 2016, 9(5), 839–849.
6. Hirsch, W.M.; Dantzig G.B. The Fixed Charge Problem. Naval Research Logistics 1968, 15(3), 413-424.
7. Midya, S.; Roy, S.K. Analysis of interval programming in different environments and its application to fixed charge transportation problem. Discrete Mathematics, Algorithms and Applications 2017, 9(03), 1750040.
8. Midya, S.; Roy, S.K. Solving single-sink, fixed charge, multi-objective, multi-index stochastic transportation problem. American Journal of Mathematical and Management Sciences 2004, 33(4), 300–314.
9. Roy, S.K.; Midya, S.; Yu, V.F. Multi-objective fixed charge transportation problem with random rough variables. International Journal Uncertainty Fuzziness and Knowledge-Based Systems 2008, 26(06), 971–996.
10. Haley, K. The solid transportation problem, Operations Research 1962, 10(4), 448–463.
11. Roy, S.K.; Midya, S. Multi-objective fixed-charge solid transportation problem with product blending under intuitionistic fuzzy environment. Applied Intelligence 2019, 49(10), 3524–3538.
12. Das, S.K.; Roy, S.K.; Weber, G.W. Application of type-2 fuzzy logic to a multi-objective green solid transportation– location problem with dwell time under carbon tax, cap, and offset policy: Fuzzy versus nonfuzzy techniques. IEEE Transactions on Fuzzy Systems 2020, 28(11), 2711–2725.
13. Ghosh, S.; Roy, S.K. Fuzzy-rough multi-objective product blending fixed-charge transportation problem with truck load constraints through transfer station. Rairo Operations Research 2021, 55, S2923–S29527.
14. Maity, G.; Roy, S.K.; Verdegay, J.L. Analyzing multi-modal transportation problem and its application to artificial intelligence. Neural Computing and Applications 2019, 32, 2243–2256.
15. Sifaouil, T.; Aider, M. Uncertain interval programming model for multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure. Soft Computing 2020, 24, 10123–10147.
16. Ding, J.; Jin, F.; Li, Y.; Wang, J. Analysis of transportation carbon emissions and its potential for reduction in china. Chinese Journal of Population Resources and Environment 2013, 11(1), 17–25.
17. Sengupta, D.; Das, A.; Bera, U.K. A gamma type- 2 defuzzification method for solving a solid transportation problem considering carbon emission. Applied Intelligence 2018, 48(11), 3995–4022.
18. Song, J.; Leng, M. Analysis of the single-period problem under carbon emissions policies. In Handbook of news vendor problems, Springer 2012, 297–313.
19. Tarulescu, S.; Tarulescu, R.; Soica, A.; Leahu, C.I. Smart transportation CO2 emission reduction strategies. In IOP Conference Series: Material Science and Engineering, IOP Publishing: 2017, 252, 012-051.
20. Midya, S.; Roy, S.K.; Vincent, F.Y. Intuitionistic fuzzy multi-stage multi-objective fixed-charge solid transportation problem in a green supply chain. International Journal of Machine Learning Cybernetics 2021, 12(3), 699–717.
21. M.Ali, A.; Muthuswamy, M. Neutrosophic Multi-Criteria Decision-Making Framework for Sustainable Evaluation of Power Production Systems in Renewable Energy Sources, Sustainable Machine Intelligence Journal 2023, 4(3), 1–10.
22. Thamaraiselvi, A.; Santhi, R. A new approach for optimization of real-life transportation problem in neutrosophic environment. Mathematical Problems in Engineering 2016, Article ID: 5950747. https://doi.org/10.1155/2016/5950747
23. Sengupta, D.; Das, A.; Dutta, A.; Bera, U.K. A carbon emission optimization model with reduction method of type-2 zigzag uncertain variable. Neural Computing and Applications 2018, 32, 10895–10914.

24. Roy, S.K.; Ebrahimnejad, A.; Verdegay, J.L.; Das, S. New approach for solving intuitionistic fuzzy multi-objective transportation problem. Sadhana 2018, 43(1), 3. https://doi.org/10.1007/s12046-017-0777-7
25. Gessesse, A.A.; Mishra, R.; Acharya, M.M.; Das, K.N. Genetic algorithm based fuzzy programming approach for multi-objective linear fractional stochastic transportation problem involving four parameter burr distribution. International Journal of Systems Assurance Engineering Management 2020, 11(1), 93–109.
26. Paul, N.; Sarma, D.; Bera, U.K. A neutrosophic solid transportation model with insufficient supply. In: 2019 IEEE region 10 symposium (TENSYMP). IEEE, 2019, 379–384. https://doi.org/10. 1109/TENSYMP46218.2019.8971130
27. Midya, S.; Roy, S.K.; Vincent, F.Y. Intuitionistic fuzzy multi-stage multi-objective fixed-charge solid transportation problem in a green supply chain. International Journal Machine Learning Cybernetics 2021, 12(3), 699–717.
28. Aktar, M.S.; De, M.; Maity, S.; Mazumder, S.K.; Maiti, M. (2020) Green 4D transportation problems with breakable incompatible items under type-2 fuzzy-random environment. Journal of Cleaner Production 2020, 275, 122376.
29. Adhami, A.Y.; Ahmad, F. Interactive Pythagorean-hesitant fuzzy computational algorithm for multi-objective transportation problem under uncertainty. International Journal of Management Science and Engineering Management 2020, 15(4), 288–297.
30. Malik, M.; Gupta, S. Goal programming technique for solving fully interval-valued intuitionistic fuzzy multiple objective transportation problems. Soft Computing 2020, 24(18), 13955–13977.
31. Samanta, S.; Jana, D.K.; Panigrahi, G.; Maiti, M. Novel multi-objective, multi-item and four-dimensional transportation problem with vehicle speed in LR-type intuitionistic fuzzy environment. Neural Computing and Applications 2020, 32, 11937–11955.
32. Mohammed, A.; Wang, Q. The fuzzy multi-objective distribution planner for a green meat supply chain. International Journal Production Economics 2017, 184, 47–58.
33. Rizk-Allah, R.M.; Hassanien, A.E.; Elhoseny, M. A multi-objective transportation model under neutrosophic environment. Computers and Electrical Engineering 2018, 69, 705–719.
34. Kumar, R.; Edalatpanah, S.A.; Jha, S.; Singh, R. A Pythagorean fuzzy approach to the transportation problem. Complex and Intelligence Systems 2019, 5, 255–263.
35. Pratihar, J.; Kumar, R.; Edalatpanah, S.A.; Dey, A. Modified Vogel's approximation method for transportation problem under uncertain environment. Complex and Intelligence Systems 2021, 7, 29–40.
36. Khalifa HAEW.; Kumar, P.; Mirjalili, S. A KKM approach for inverse capacitated transportation problem in neutrosophic environment. Sadhana 2021, 46, 166. https://doi.org/10.1007/s12046- 021-01682-5
37. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic set, Multi-space and multi-structure, 2010, 4, 410-413.
38. Smarandache, F. Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability, Education Publishing: Columbus, OH, USA, 2013;
39. Ye, J. Neutrosophic number linear programming method and its application under neutrosophic number environments. Soft computing 2018, 22(14), 4639–4646.
40. Ebrahimnejad, A. A simplified new approach for solving fuzzy transportation problems with generalized trapezoidal fuzzy numbers. Applied Soft Computing 2014, 19, 171–176.

Received: Sep 6, 2024. Accepted: Dec 4, 2024

Neutrosophic Sets and Systems, Vol. 78, 2025
University of New Mexico

On A Novel Topological Space Based On Partially Ordered Ring Of Weak Fuzzy Complex Numbers and Its Relation With The Partially Ordered Neutrosophic Ring of Real Numbers

Raed Hatamleh

Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan
Email: raed@jadara.edu.jo

Abstract: This work is dedicated to studying for the first time a novel topological space generated by weak fuzzy complex intervals based on the partial ordered ring structure of weak fuzzy complex numbers, where we build a topology structure over these intervals. Also, we present many novel results as mathematical theorems, mathematical proofs, and many related examples. On the other hand, we use the AH-isometry to determine the relationship between the weak fuzzy complex intervals, and neutrosophic intervals.

Keywords: weak fuzzy complex numbers, weak fuzzy interval, topological space, weak fuzzy complex topology.

Introduction

The algebraic extensions of real number field have attracted the attention of many different researchers from all over the world. In [9], Weak Fuzzy Complex numbers appeared for the first time, where they have been defined as a new generalization of classical real numbers through fuzzy operators. In [3, 5] authors studied weak fuzzy complex vector spaces and matrices. The weak fuzzy complex number theoretical concepts such as Diophantine equations and congruencies [2,7-8]. Also, in [1], authors have studied weak fuzzy complex linear Diophantine equations in two variables and presented an easy algorithm to solve them. In [10], the foundations of functions in weak fuzzy complex variables were studied. Actually, one of the important results of using this type of numbers, modeling the solutions of some vectorial equations defined by norms in 3-dimensional Euclidean space A-Curves [4]. However, Python introduced Weak Fuzzy Complex numbers in [11-20].

This has motivated us for the study of topological spaces based on the intervals defined over weak fuzzy complex numbers, where we study for the first time a novel topological space generated by weak fuzzy complex intervals based on the partial ordered ring structure of weak fuzzy complex numbers, where we build a topology structure over these intervals. Also, we present many novel results as mathematical theorems, mathematical proofs, and many related examples. On the other hand, we use the AH-isometry to determine the relationship between the weak fuzzy complex intervals, and neutrosophic intervals.

Main Discussion

Definition 1. ([9])

The set of Weak Fuzzy Complex numbers was defined as follows, where 'J' is the Weak Fuzzy Complex operator

($J \notin R$):

$$F_J = \{x_0 + x_1 J \; ; \; x_0 , x_1 \in R \, , J^2 = t \in]0,1[\}.$$

Properties of the Weak Fuzzy Complex numbers: ([9])

Let $X = x_0 + x_1 J$, $Y = y_0 + y_1 J \in F_J$, where $x_0 , x_1, y_0, y_1 \in R$

- ♦ Addition: $X + Y = (x_0 + y_0) + (x_1 + y_1)J$.
- ♦ Multiplication $X \times Y = (x_0 y_0 + x_1 y_1 t) + (x_0 y_1 + x_1 y_0)J$.
- ♦ The conjugate of X is: $\bar{X} = x_0 + x_1 J$.

Definition 2. ([9])

Let φ be *the transformation function* from F_J to $R \times R$, which we define as follows:

$$\varphi : F_J \mapsto R \times R.$$

$$\varphi(x_0 + x_1 J) = (x_0 + x_1 (-\sqrt{t}), x_0 + x_1(+\sqrt{t})) = (x_0 - x_1 \sqrt{t}, x_0 + x_1\sqrt{t}) .$$

Definition 3.

Let $\varphi : F_J \mapsto R \times R$ such that: $\varphi(X) = (a, b)$, the inverse function of φ is defined as follows:

$$\varphi^{-1} : R \times R \mapsto F_J$$

$$\varphi^{-1}(a, b) = \frac{1}{2}[a + b] + \frac{1}{2\sqrt{t}} J[b - a].$$

Definition 4.

The ring of weak fuzzy complex numbers is defined as follows:

$$\mathbb{C}_w = \{a + bJ \; ; \; J^2 = t \in]0,1[\; ; a, b \in \mathbb{R}\}.$$

Remark 1.

$\mathbb{C}_w$ is partially ordered set with $(\leq)$, with:

$$a + bJ \leq c + dJ \iff \begin{cases} a - b\sqrt{t} \leq c - d\sqrt{t} \\ a + b\sqrt{t} \leq c + d\sqrt{t} \end{cases}.$$

The relation $(<)$ can be defined as follows:

$$a + bJ < c + dJ \iff \begin{cases} a - b\sqrt{t} < c - d\sqrt{t} \\ a + b\sqrt{t} < c + d\sqrt{t} \end{cases}.$$

Definition 5.

We define the following types of weak fuzzy complex intervals:

$$I_1 = [a + bJ\,, c + dJ]$$

$$= \{x + yJ \in C_w \ ; a - b\sqrt{t} \leq x - y\sqrt{t} \leq c - d\sqrt{t}\,, a + b\sqrt{t} \leq x + y\sqrt{t} \leq c + d\sqrt{t}\}$$

$$I_2 =]a + bJ\,, c + dJ[$$

$$= \{x + yJ \in C_w \ ; a - b\sqrt{t} < x - y\sqrt{t} < c - d\sqrt{t}\,, a + b\sqrt{t} < x + y\sqrt{t} < c + d\sqrt{t}\}$$

$$I_3 = [a + bJ\,, c + dJ[$$

$$= \{x + yJ \in C_w \ ; a - b\sqrt{t} \leq x - y\sqrt{t} < c - d\sqrt{t}\,, a + b\sqrt{t} \leq x + y\sqrt{t} < c + d\sqrt{t}\}$$

$$I_4 =]a + bJ\,, c + dJ]$$

$$= \{x + yJ \in C_w \ ; a - b\sqrt{t} < x - y\sqrt{t} \leq c - d\sqrt{t}\,, a + b\sqrt{t} < x + y\sqrt{t} \leq c + d\sqrt{t}\}.$$

Remark 2.

1) The previous types of (WFC) intervals are defined under the condition $a + bJ \leq c + dJ$ in c_w.

2) (WFC) intervals with ∞ in any side can be recognized as:

$$]-\infty, a + bJ\,] = \{x + yJ \in C_w \ ; -\infty < x - y\sqrt{t} \leq a - b\sqrt{t}\,, -\infty < x + y\sqrt{t} \leq a + b\sqrt{t}\}.$$

$$[a + bJ\,, \infty[= \{x + yJ \in C_w \ ; a - b\sqrt{t} \leq x - y\sqrt{t} < \infty, a + b\sqrt{t} \leq x + y\sqrt{t} < \infty\}.$$

Example 1.

Take

$$I_1 =]1 + 2J, 3 + J[\ ;\ J^2 = t = \frac{1}{4}, \sqrt{t} = \frac{1}{2}.$$

$$\begin{cases} 1+2\sqrt{t}=2\leq 3+\sqrt{t}=\frac{7}{2} \\ 1-2\sqrt{t}=0\leq 3-\sqrt{t}=\frac{5}{2} \end{cases},$$

so that,

$$I_1=\left\{x+yJ\in C_w\ ;0<x-\frac{1}{2}y<\frac{5}{2}\ ,2<x+\frac{1}{2}y<\frac{7}{2}\right\}=\left\{(x,y)\in\mathbb{R}^2;\ 0<x-\frac{1}{2}y<\frac{5}{2}\ ,2<x+\frac{1}{2}y<\frac{7}{2}\right\}.$$

Take

$$I_5=\left]-\infty,-1+3J\right[\ ;\ J^2=\frac{1}{9}=t,\sqrt{t}=\frac{1}{3},$$

$$I_5=\left\{x+yJ\in C_w\ ;-\infty<x-\frac{1}{3}y<-2\ ,-\infty<x+\frac{1}{3}y<0\right\}=\left\{(x,y)\in\mathbb{R}^2;\ -\infty<x-\frac{1}{3}y<-2\ ,-\infty<x+\frac{1}{3}y<0\right\}.$$

Definition 6.

Let $a+bJ,c+dJ\in C_w$,

we define:

1) $\max(a+bJ,c+dJ)=f^{-1}\left[\left(\max\left(a-b\sqrt{t},c-d\sqrt{t}\right),\max\left(a+b\sqrt{t},c+d\sqrt{t}\right)\right)\right]$.

2) $\min(a+bJ,c+dJ)=f^{-1}\left[\left(\min\left(a-b\sqrt{t},c-d\sqrt{t}\right),\min\left(a+b\sqrt{t},c+d\sqrt{t}\right)\right)\right]$.

Where *(f)* is the isomorphism defined in [6] as follows:

$$f:C_w\to R\times R\ :f(a+bJ)=(a-b\sqrt{t},a+b\sqrt{t}),$$

$$f^{-1}:R\times R\to C_w,$$

such that,

$$f^{-1}(a,b)=\frac{b+a}{2}+\frac{1}{\sqrt{t}}J\left(\frac{b-a}{2}\right)\ ;J^2=t\in\left]0,1\right[.$$

Theorem 1.

In C_w, we have:

1) $\min(a+bJ,c+dJ)\leq a+bJ,c+dJ$,

2) $\max(a+bJ,c+dJ)\geq a+bJ,c+dJ$,

3) If $a+bJ\leq c+dJ$, then $\min(a+bJ,c+dJ)=a+bJ$, and $\max(a+bJ,c+dJ)=c+dJ$,

4) $\min(a+bJ,c+dJ)\leq\max(a+bJ,c+dJ)$.

Theorem 2.

In C_w, we have:

1) If $x + yJ \leq a + bJ, x + yJ \leq c + dJ$, then

$$x + yJ \leq \min(a + bJ, c + dJ).$$

2) If $x + yJ \geq a + bJ, x + yJ \geq c + dJ$, then

$$x + yJ \geq \max(a + bJ, c + dJ).$$

Proof of theorem 1.

1) $\min(a + bJ, c + dJ) = \frac{1}{2}\left[\min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) + \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right)\right] + \frac{1}{2\sqrt{t}} J \left[\min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) - \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right)\right] = k_1 + k_2 J.$

$k_1 + k_2\sqrt{t} = \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \leq a + b\sqrt{t}, c + d\sqrt{t}.$

$k_1 - k_2\sqrt{t} = \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \leq a - b\sqrt{t}, c - d\sqrt{t},$

Hence,

$$\min(a + bJ, c + dJ) \leq a + bJ, c + dJ.$$

2) $\max(a + bJ, c + dJ) = \frac{1}{2}\left[\max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) + \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right)\right] + \frac{1}{2\sqrt{t}} J \left[\max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) - \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right)\right] = k_1 + k_2 J.$

$k_1 + k_2\sqrt{t} = \max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \geq a + b\sqrt{t}, c + d\sqrt{t},$

$k_1 - k_2\sqrt{t} = \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \geq a - b\sqrt{t}, c - d\sqrt{t}.$

Hence,

$$\max(a + bJ, c + dJ) \geq a + bJ, c + dJ.$$

3) Assume that $a + bJ \leq c + dJ$,then

$$\begin{cases} a - b\sqrt{t} \leq c - d\sqrt{t} \\ a + b\sqrt{t} \leq c + d\sqrt{t} \end{cases}.$$

Hence,

$$\begin{cases} \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) = a - b\sqrt{t} \\ \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) = a + b\sqrt{t} \end{cases},$$

and

$$\begin{cases} \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right) = c - d\sqrt{t} \\ \max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) = c + d\sqrt{t} \end{cases}.$$

So that,

$$\min(a+bJ, c+dJ) = \frac{1}{2}\left(a - b\sqrt{t} + a + b\sqrt{t}\right) + \frac{1}{2\sqrt{t}}J\left(a + b\sqrt{t} - a + b\sqrt{t}\right) = a + bJ,$$

$$\max(a+bJ, c+dJ) = \frac{1}{2}\left(c - d\sqrt{t} + c + d\sqrt{t}\right) + \frac{1}{2\sqrt{t}}J\left(c + d\sqrt{t} - c + d\sqrt{t}\right) = c + dJ.$$

4) Since,

$$\min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \leq \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right),$$

$$\min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \leq \max\left(a + b\sqrt{t}, c + d\sqrt{t}\right),$$

thus,

$$\min(a+bJ, c+dJ) \leq \max(a+bJ, c+dJ).$$

Proof of theorem 2.

1) Assume that,

$$\begin{cases} x + yJ \leq a + bJ \\ x + yJ \leq c + dJ \end{cases},$$

then,

$$\begin{cases} x - y\sqrt{t} \leq a - b\sqrt{t} \\ x + y\sqrt{t} \leq a + b\sqrt{t} \end{cases} \text{and} \begin{cases} x - y\sqrt{t} \leq c - d\sqrt{t} \\ x + y\sqrt{t} \leq c + d\sqrt{t} \end{cases}.$$

Hence,

$$\begin{cases} x - y\sqrt{t} \leq \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \\ x + y\sqrt{t} \leq \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \end{cases},$$

and

$$x + yJ \leq \min(a+bJ, c+dJ).$$

2) Assume that:

$$\begin{cases} x + yJ \geq a + bJ \\ x + yJ \geq c + dJ \end{cases},$$

then,

$$\begin{cases} x - y\sqrt{t} \geq a - b\sqrt{t} \\ x + y\sqrt{t} \geq a + b\sqrt{t} \end{cases} \text{and} \begin{cases} x - y\sqrt{t} \geq c - d\sqrt{t} \\ x + y\sqrt{t} \geq c + d\sqrt{t} \end{cases}.$$

Hence,

$$\begin{cases} x - y\sqrt{t} \geq \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \\ x + y\sqrt{t} \geq \max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \end{cases},$$

and

$$x + yJ \geq \max(a+bJ, c+dJ).$$

Theorem 3.

In C_w, we have:

If $a+bJ<c+dJ, x+yJ<m+nJ$, then

1) $]a+bJ\,,c+dJ[\cap]x+yJ,m+nJ[=]\max(a+bJ,x+yJ)\,,\min(c+dJ,m+nJ)[$,

2) $[a+bJ\,,c+dJ]\cap[x+yJ,m+nJ]=[\max(a+bJ,x+yJ)\,,\min(c+dJ,m+nJ)]$.

Theorem 4.

In C_w, we have:

1) $]a+bJ\,,\infty[\cap]x+yJ,\infty[=]\max(a+bJ,x+yJ)\,,\infty[$.

2) $[a+bJ\,,\infty[\cap[x+yJ,\infty[=[\max(a+bJ,x+yJ)\,,\infty[$.

3) $]a+bJ\,,\infty[\cup]x+yJ,\infty[=]\min(a+bJ,x+yJ)\,,\infty[$.

4) $[a+bJ\,,\infty[\cup[x+yJ,\infty[=[\min(a+bJ,x+yJ)\,,\infty[$.

Theorem 5.

In C_w, we have:

1) $]-\infty,a+bJ\,[\cap]-\infty,x+yJ[=]-\infty,\min(a+bJ,x+yJ)[$.

2) $]-\infty,a+bJ\,]\cap]-\infty,x+yJ]=]-\infty,\min(a+bJ,x+yJ)]$.

3) $]-\infty,a+bJ\,[\cup]-\infty,x+yJ[=]-\infty,\max(a+bJ,x+yJ)[$.

4) $]-\infty,a+bJ\,]\cup]-\infty,x+yJ]=]-\infty,\max(a+bJ,x+yJ)]$.

Proof of theorem 3.

1) Let $k+hJ\in]a+bJ\,,c+dJ[\cap]x+yJ,m+nJ[$. Then

$$\begin{cases}k+hJ>a+bJ\\k+hJ>x+yJ\end{cases}\text{,}\begin{cases}k+hJ<c+dJ\\k+hJ<m+nJ\end{cases}.$$

Hence,

$$\begin{cases}k+hJ>\max(a+bJ,x+yJ)\\k+hJ<\min(c+dJ,m+nJ)\end{cases}\text{'}$$

so that

$$k+hJ\in]\max(a+bJ,x+yJ)\,,\min(c+dJ,m+nJ)[.$$

Now, let

$$k+hJ\in]\max(a+bJ,x+yJ)\,,\min(c+dJ,m+nJ)[.$$

Then,

$$\begin{cases} k+hJ > \max(a+bJ, x+yJ) \\ k+hJ < \min(c+dJ, m+nJ) \end{cases} \Longrightarrow \begin{cases} k+hJ > a+bJ, x+yJ \\ k+hJ < c+dJ, m+nJ \end{cases} \Longrightarrow k+hJ \in]a+bJ, c+dJ[\cap]x+yJ, m+nJ[.$$

2) The proof is exactly similar to 1 by replacing $(<)$ with $(\leq)$ and $(>)$ with $(\geq)$.

Proof of theorem 4.

1) $]a+bJ\,, \infty[\cap]x+yJ, \infty[=]\max(a+bJ, x+yJ)\,, \min(\infty, \infty)[=]\max(a+bJ, x+yJ)\,, \infty[,$

2) $[a+bJ\,, \infty[\cap [x+yJ, \infty[= [\max(a+bJ, x+yJ)\,, \min(\infty, \infty)[= [\max(a+bJ, x+yJ)\,, \infty[,$

3) Let $+hJ \in]a+bJ, \infty[\cup]x+yJ, \infty[$. Then

$$\begin{cases} k+hJ > a+bJ \\ \quad or \\ k+hJ > x+yJ \end{cases} \Longrightarrow k+hJ > \min(a+bJ, x+yJ) \Longrightarrow k+hJ \in]\min(a+bJ, x+yJ), \infty[.$$

Conversely, suppose that $k+hJ \in]\min(a+bJ, x+yJ), \infty[$, then

$$\begin{cases} k+hJ > a+bJ \\ \quad or \\ k+hJ > x+yJ \end{cases} \Longrightarrow k+hJ \in]a+bJ\,, \infty[\cup]x+yJ, \infty[.$$

4) It is similar to 3.

Proof of theorem 5.

1) $]-\infty, a+bJ\,[\cap]-\infty, x+yJ[=]\max(-\infty, -\infty)\,, \min(a+bJ, x+yJ)[=]-\infty, \min(a+bJ, x+yJ)[,$

2) $]-\infty, a+bJ\,] \cap]-\infty, x+yJ] =]\max(-\infty, -\infty)\,, \min(a+bJ, x+yJ)] =]-\infty, \min(a+bJ, x+yJ)],$

3) Let $k+hJ \in]-\infty, a+bJ[\cup]-\infty, x+yJ[$ .Then

$$\begin{cases} k+hJ < a+bJ \\ \quad or \\ k+hJ < x+yJ \end{cases} \Longrightarrow k+hJ < \max(a+bJ, x+yJ) \Longrightarrow k+hJ \in]-\infty, \max(a+bJ, x+yJ)[.$$

Conversely, assume that $k+hJ \in]-\infty, \max(a+bJ, x+yJ)[$, then

$$\begin{cases} k+hJ < a+bJ \\ \quad or \\ k+hJ < x+yJ \end{cases} \Longrightarrow k+hJ \in]-\infty, a+bJ\,[\cap]-\infty, x+yJ[.$$

4) It is similar to 3.

Definition7.

We define $\tau = \{\emptyset, C_w, S_i\}\,;\; S_i = \cup_{i\in I}]a_i+b_iJ, c_i+d_iJ[$.

Theorem 6.

(C_w, τ) is a topological space.

Proof.

We have:

$\emptyset, C_w \in \tau$ by a definition.

Let $S_i, S_j \in \tau$. Then

$$\begin{cases} S_i = \cup_{i\in I}]a_i + b_iJ, c_i + d_iJ[\\ S_j = \cup_{j\in L}]a_j + b_jJ, c_j + d_jJ[' \end{cases}$$

$$S_i \cap S_j = \cup_{l,k\in M}]\max(a_k + b_kJ, a_l + b_lJ), \min(c_k + d_kJ, c_l + d_lJ)[\in \tau.$$

Also, $\cup_{i=1}^{\infty} S_i$ is a union of open intervals in C_w, hence $\cup_{i=1}^{\infty} S_i \in \tau$, and τ is a topology on C_w, thus $(C_w, \tau)$is a topological space.

Remark 3.

1) Open sets in (C_w, τ) are the union of open intervals in C_w.

2) Closed sets in (C_w, τ) are the complements of open sets, i.e.

$$D_i = \{C_w / S_i \ ; i \in I, S_i \in \tau\}.$$

3) open intervals in C_w are open sets.

Example 2.

Take $S_1 =]1 + 2J, 3 + 5J[, S_2 =]2 + J, 3 + J[$; $J^2 = t = \frac{1}{4}$.

$S_1 = S_1 \cup \emptyset, S_2 = S_2 \cup \emptyset$, thus S_1, S_2 are open sets.

$S_1 \cap S_2 =]\max(1 + 2J, 2 + J), \min(3 + 5J, 3 + J)[;$

where,

$$\max(1 + 2J, 2 + J) = \frac{1}{2}\left[\max\left(1 + 2\left(\frac{1}{2}\right), 2 + (1)\left(\frac{1}{2}\right)\right) + \max\left(1 - 2\left(\frac{1}{2}\right), 2 - (1)\left(\frac{1}{2}\right)\right)\right] +$$

$$\frac{1}{2\left(\frac{1}{2}\right)}J\left[\max\left(1 + 2\left(\frac{1}{2}\right), 2 + (1)\left(\frac{1}{2}\right)\right) - \max\left(1 - 2\left(\frac{1}{2}\right), 2 - (1)\left(\frac{1}{2}\right)\right)\right] = \frac{1}{2}\left[\max\left(2, \frac{5}{2}\right) +\right.$$

$$\left.\max\left(0, \frac{3}{2}\right)\right] + J\left[\max\left(2, \frac{5}{2}\right) - \max\left(0, \frac{3}{2}\right)\right] = \frac{1}{2}\left(\frac{5}{2} + \frac{3}{2}\right) + J\left[\frac{5}{2} - \frac{3}{2}\right] = 2 + J.$$

$$\min(3 + 5J, 3 + J) = \frac{1}{2}\left[\min\left(3 + \frac{5}{2}, 3 + \frac{1}{2}\right) + \min\left(3 - \frac{5}{2}, 3 - \frac{1}{2}\right)\right] + \frac{1}{2\left(\frac{1}{2}\right)}J\left[\min\left(3 + \frac{5}{2}, 3 + \frac{1}{2}\right) -\right.$$

$$\left.\min\left(3 - \frac{5}{2}, 3 - \frac{1}{2}\right)\right] = \frac{1}{2}\left[\min\left(\frac{11}{2}, \frac{7}{2}\right) + \min\left(\frac{1}{2}, \frac{5}{2}\right)\right] + J\left[\min\left(\frac{11}{2}, \frac{7}{2}\right) - \min\left(\frac{1}{2}, \frac{5}{2}\right)\right] = \frac{1}{2}\left(\frac{7}{2} + \frac{1}{2}\right) +$$

$$J\left[\frac{7}{2} - \frac{1}{2}\right] = 2 + 3J.$$

$S_1 \cap S_2 =]2 + J, 2 + 3J[\in \tau$.

$S_1 \cup S_2 =]1 + 2J, 3 + 5J[\cup]2 + J, 3 + J[\in \tau$.

$\overline{S_1} = C_w/]1 + 2J, 3 + 5J[$, is a closed set.

$\overline{S_2} = C_w/]2 + J, 3 + J[$, is a closed set.

$\overline{S_1} \cap \overline{S_2} = (\overline{S_1 \cup S_2}), \overline{S_1} \cup \overline{S_2} = \overline{S_1 \cap S_2} \in \tau$.

Weak Fuzzy complex intervals representation as neutrosophic intervals:

Definition 8. [17]

Let $R(I) = \{a + bI \ ; a, b \in \mathbb{R}, I^2 = I\}$ be the neutrosophic ring of real numbers, then it is partially ordered with:

$$a + bI \leq c + dI \Leftrightarrow \begin{cases} a + b \leq c + d \\ a \leq c \end{cases}.$$

Remark. [17]

$h: R(I) \to R \times R: h(a + bI) = (a, a + b)$ is a ring isomorphism, and it is called the AH- isometry.

The inverse isometry is defined as:

$h^{-1}: R \times R \to R(I): h^{-1}(a, b) = a + (b - a)I$.

Definition 9.

The neutrosophic interval $[a + bJ, c + dJ]$ is defined under the condition $a + bI \leq c + dI$.

$$x + yI \in [a + bJ, c + dJ] \Leftrightarrow \begin{cases} a \leq x \leq c \\ a + b \leq x + y \leq c + d \end{cases}.$$

Remark 4.

The ring of weak fuzzy complex numbers is isomorphic to $R \times R$ with $f: C_w \to R \times R$ such that:

$f(a + bJ) = \left(a - b\sqrt{t}, a + b\sqrt{t}\right) \ ; J^2 = t$.

Theorem 7.

Every weak fuzzy complex number can be represented with one neutrosophic real number.

Proof.

Let $h: R(I) \to R \times R$ be the neutrosophic AH- isometry and $f: W_c \to R \times R$ be the (WFC) isomorphism, then we can define:

$$\varphi: C_w \to R(I) \ ; \quad \varphi = h^{-1} \circ f.$$

It is clear that (φ) is a ring isomorphism, with:

$$\varphi(a+bJ) = h^{-1} \circ f(a+bJ) = h^{-1}\left(a-b\sqrt{t}, a+b\sqrt{t}\right) = a-b\sqrt{t} + \left[\left(a+b\sqrt{t}\right) - \left(a-b\sqrt{t}\right)\right]I = \left(a-b\sqrt{t}\right) + 2b\sqrt{t}I$$

Hence, the proof is complete.

Result 1.

Every open interval in C_w $]a+bJ, c+dJ[$ can be represented with exactly one open interval in $R(I)$ as follows:

$$]a+bJ, c+dJ[\leftrightarrow]\varphi(a+bJ), \varphi(c+dJ)[= \left]\left(a-b\sqrt{t}\right) + 2b\sqrt{t}I, c-d\sqrt{t} + 2d\sqrt{t}I\right[.$$

Remark 5.

We call $\varphi: C_w \to R(I)$ the weak fuzzy neutrosophic AH- isometry.

Theorem 8.

Let $\varphi: C_w \to R(I)$ be the weak fuzzy neutrosophic AH- isometry, then:

If $a+bJ < c+dJ$, $\varphi(a+bJ) < \varphi(c+dJ)$.

Proof: Assume that $a+bJ < c+dJ$, then:

$$\begin{cases} a-b\sqrt{t} < c-d\sqrt{t} \\ a+b\sqrt{t} < c+d\sqrt{t} \end{cases} \Rightarrow \begin{cases} a-b\sqrt{t} < c-d\sqrt{t} \\ \left(a-b\sqrt{t}\right) + 2b\sqrt{t} < \left(c-d\sqrt{t}\right) + 2d\sqrt{t} \end{cases}$$

$$\Rightarrow a-b\sqrt{t} + 2b\sqrt{t}I < c-d\sqrt{t} + 2d\sqrt{t}I \Rightarrow \varphi(a+bJ) < \varphi(c+dJ).$$

Result 2.

The weak fuzzy complex interval topological space is isomorphic to the neutrosophic interval topology.

Proof.

Since every (WFC) interval can be linked with only one neutrosophic interval, then it is isomorphic to the neutrosophic topological space of intervals.

Result 3.

Every open set in $C_w, A = \bigcup_{i \in k}]a_i + b_iJ, c_i + d_iJ[$ can be linked with exactly one open set in $R(I)$ as follows:

$$\varphi(A) = \bigcup_{i \in k}]\varphi(a_i + b_i J), \varphi(c_i + d_i J)[.$$

Conclusion

In this work we studied for the first time a novel topological space generated by weak fuzzy complex intervals based on the partial ordered ring structure of weak fuzzy complex numbers, where we built a topology structure over these intervals. Also, we presented many novel results as mathematical theorems, mathematical proofs, and many related examples. On the other hand, we use the AH-isometry to determine the relationship between the weak fuzzy complex intervals, and neutrosophic intervals. For future studies, these mathematical tools can be used effectively with other tools and techniques that can be observed through [21-45].

References

[1] M.Abualhomos, W.M.M. Salameh, M. Bataineh, M.O. Al-Qadri, A. Alahmade, A.Al-Husban, An Effective Algorithm for Solving Weak Fuzzy Complex Diophantine Equations in Two Variables,(2024). (Doi:https://doi.org/10.54216/IJNS.230431).

[2] A.M.A. Alfahal, M. Abobala, Y.A. Alhasan, R.A. Abdulfatah, Generating Weak Fuzzy Complex and Anti Weak Fuzzy Complex Integer Solutions for Pythagoras Diophantine Equation $X^2 + Y^2 = Z^2$, International Journal of Neutrosophic Science 22 (2023) 8–14. (Doi: https://doi.org/10.54216/IJNS.220201).

[3] Y.A.Alhasan, A.M.A.Alfahal, R.A.Abdulfatah, G.Nordo, M.M.A.Zahra,On Some Novel Results About Weak Fuzzy Complex Matrices, International Journal of Neutrosophic Science 21 (2023) 134–140. (Doi:https://doi.org/10.54216/IJNS.210112).

[4] Y.A. Alhasan, L. Xu, R.A. Abdulfatah, A.M.A. Alfahal, The Geometrical Characterization for The Solutions of a Vectorial Equation By Using Weak Fuzzy Complex Numbers and Other Generalizations Of Real Numbers, International Journal of Neutrosophic Science 21 (2023) 155–159.(Doi: https://doi.org/10.54216/IJNS.210415).

[5] R. Ali, On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces, Neoma Journal of Mathematics and Computer Science 1 (2023). (Doi:https://doi.org/10.5281/zenodo.7953682).

[6] Sarkis, M. (2024). On The Diagonalization Problem of Weak Fuzzy Complex Matrices Based On a Special Isomorphism. Galoitica: Journal of Mathematical Structures and Applications, 22-27. (Doi: https://doi.org/10.54216/GJMSA.0110203).

[7] F.C. Galarza, M.L. Flores, D.P. Rivero, M. Abobala, On Weak Fuzzy Complex Pythagoras Quadruples, International Journal of Neutrosophic Science 22 (2023) 108–113. (Doi:https://doi.org/10.54216/IJNS.220209).

[8] Abobala, M. (2024). On Some Novel Results About Weak Fuzzy Complex Integers. Journal of Neutrosophic and Fuzzy Systems, 15-22**.** (Doi**:** https://doi.org/10.54216/JNFS.080202).

[9] A.Hatip, An Introduction to Weak Fuzzy Complex Numbers, Galoitica: Journal of Mathematical Structures and Applications 3 (2023) 08–13. (Doi:https://doi.org/10.54216/gjmsa.030101).

[10] L. Razouk, S. Mahmoud, M. Ali, On the Foundations of Weak Fuzzy Complex-Real Functions, 5 (2024) 116–140,(Doi: https://doi.org/10.22105/jfea.2024.435955.1369) .

[11] L. Razouk, S. Mahmoud, M. Ali, A Computer Program For The System Of Weak Fuzzy Complex Numbers And Their Arithmetic Operations Using Python, Galoitica: Journal of Mathematical Structures and Applications 8 (2023) 45–5.

https://doi.org/10.54216/gjmsa.080104.

[12] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Mowafaq Omar Al-Qadri, Abdallah Al-Husban,(2024), On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For 2≤n≤3, Neutrosophic Sets and Systems, Vol. 68,pp.8-25 . (Doi: 10.5281/zenodo.11406449).

[13] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems,vol.67, pp. 169-178.(DOI:10.5281/zenodo.11151930).

[14] Abubaker, Ahmad A, Hatamleh, Raed, Matarneh, Khaled, Al-Husban, Abdallah.(2024). On the Numerical Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, International Journal of Neutrosophic Science,25(2),197-205. (Doi: https://doi.org/10.54216/IJNS.250217).

[15] Raed Hatamleh , Ayman Hazaymeh,(2025),On The Topological Spaces of Neutrosophic Real Intervals, International Journal of Neutrosophic Science, Vol. 25 Issue. 1 PP. 130-136, (Doi: https://doi.org/10.54216/IJNS.250111.

[16] Raed Hatamleh , Ayman Hazaymeh.(2025),On Some Topological Spaces Based On Symbolic n-Plithogenic Intervals, International Journal of Neutrosophic Science,Vol.25Issue.1PP. 23-37.(Doi: https://doi.org/10.54216/IJNS.250102).

[17] Abobala, M., and Hatip, A., "An Algebraic Approach To Neutrosophic Euclidean Geometry", Neutrosophic Sets and Systems, Vol. 43, 2021.

[18] Hatamleh, R., Zolotarev,V. A. (2015). On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component, Journal of Mathematical Physics, Analysis, Geometry,11(2), 174–186. (Doi:https://doi.org/10.15407/mag11.02.174).

[19] Hatamleh, R., Zolotarev,V. A.(2014). On Two-Dimensional Model Representations of One Class of Commuting Operators, Ukrainian Mathematical Journal, 66(1), 122–144. https://doi.org/10.1007/s11253-014-0916-9.

[20] R.Hatamleh.(2024).On the Compactness and Continuity of Uryson's Operator in Orlicz Space, International Journal of Neutrosophic Science, 24 (3), 233-239.

[21] Raed Hatamleh, Ayman Hazaymeh.(2024). Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings, Neutrosophic Sets and Systems, Vol. 70, pp.1-16. (Doi: 10.5281/zenodo.13160839).

[22] Abubaker, Ahmad A, Hatamleh, Raed, Matarneh, Khaled, Al-Husban, Abdallah.(2024).On the Numerical Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, Inernational Journal of Neutrosophic Science,25(2),197-205.(Doi: https://doi.org/10.54216/IJNS.250217).

[23] Heilat, A.S.,Batiha,B,T.Qawasmeh ,Hatamleh R.(2023). Hybrid Cubic B-spline Method for Solving A Class of Singular Boundary Value Problems. European Journal of Pure and Applied Mathematics, 16(2),751-762.(Doi: https://doi.org/10.29020/nybg.ejpam.v16i2.4725).

[24] T.Qawasmeh, A.Qazza, R.Hatamleh, M.W. Alomari, R. Saadeh.(2023). Further accurate numerical radius inequalities.Axiom-MDPI, 12 (8), 801.

[25] T.Qawasmeh,R.Hatamleh,(2023).A new contraction based on H-simulation functions in the frame of extended b-metric spaces and application,International Journal of Electrical and Computer Engineering 13 (4),4212-4221.

[26] R.Hatamleh,V.A. Zolotarev.(2017).On the abstract inverse scattering problem for traces class pertubations. Journal of Mathematical Physics, Analysis, Geometry 13(1),1-32. https://doi.org/10.15407/mag13.01.003.

[27] R.Hatamleh, Abdallah Al-Husban,N.Sundarakannan4, M. S. Malchijah raj5.(2025). Complex cubic intuitionistic fuzzy set applied to subbisemirings of bisemirings using homomorphism.Communications on Applied Nonlinear Analysis,32(3),418-435.

[28] A.Rajalakshmi1, R.Hatamleh, Abdallah Al-Husban,K.Lenin Muthu Kumaran, M.S.Malchijah raj6.(2025). Various (ζ1; ζ2) neutrosophic ideals of an ordered ternary semigroups. Communications on Applied Nonlinear Analysis,32(3),400-417.

[29] R.Hatamleh. (2007).On a Class of Nonhomogeneous Fields in Hilbert Space.Journal of Mathematics and Statistics ,3 (4), 207-210.

[30] R.Hatamleh.(2024). On The Numerical Solutions of the Neutrosophic One-Dimensional Sine-Gordon System, International Journal of Neutrosophic Science, 25 (3), 25-36.

[31] Hamadneh,T., Alomari, M.W., Al-Shbeil, I., Alaqad,H.,Hatamleh, R., Heilat, A. S., & Al-Husban, A. (2023). Refinements of the Euclidean operator radius and Davis–Wielandt radius-type inequalities. Symmetry, 15(5), 1061.

[32] R.Hatamleh,V.A.Zolotarev.(2012).On the universal models of commutative systems of linear operators.Journal of Mathematical Physics, Analysis, Geometry,8(3),248-259.

[33] Abdallah Al-Husban & Abdul Razak Salleh 2015. Complex Fuzzy Hyperring Based on Complex Fuzzy Spaces. Proceedings of 2nd Innovation and Analytics Conference & Exhibition (IACE). Vol. 1691. AIP Publishing 2015. Pages 040009-040017.

[34] Al-Husban, A., & Salleh, A. R. (2016). Complex fuzzy hypergroups based on complex fuzzy spaces. International Journal of Pure and Applied Mathematics, *107*(4), 949-958.

[35] Alsarahead, M. O., & Al-Husban, A. (2022). Complex multi–fuzzy subgroups. Journal of discrete mathematical sciences and cryptography, 25(8), 2707-2716.

[36] Jaradat, A., & Al-Husban,A.(2021).The multi-fuzzy group spaces on multi-fuzzy space. J. Math. Comput. Sci.,11(6), 7535-7552.

[37] Al-Husban, A. (2021). Multi-fuzzy hypergroups. Italian Journal of Pure and Applied Mathematics, 46, 382-390.

[38] Al-Husban, A., Fuzzy Soft Groups based on Fuzzy Space. (2021) Wseas Transactions on Mathematics. 21, 53-57.

[39] Al-Husban,. Abdallah, Al-Sharoa, Doaa., Al-Kaseasbeh, Mohammad., & Mahmood, R.M.S. (2022). Structures of fibers of groups actions on graphs. Wseas Transactions on Mathematics, 7, 650-658.

[40] C. Sivakumar, Mowafaq Omar Al-Qadri, Abdallah shihadeh, Ahmed Atallah Alsaraireh, Abdallah Al-Husban, P. Maragatha Meenakshi, N. Rajesh, M. Palanikumar. (2024). q-rung square root interval-valued sets with respect to aggregated operators using multiple attribute decision making. Journal of, 23(3), pp. 154-174.

[41] Abu, Ibraheem.,T.,T.,Al-Husban, Abdallah.,Alahmade, Ayman.,Azhaguvelavan, S., Palanikumar, Murugan.(2024).Computer purchasing using new type neutrosophic sets and its extension based on aggregation operators. International Journal of Neutrosophic Science ,24(1), pp. 171-185.

[42] Raja, K., Meenakshi, P. M., Al-Husban, A., Al-Qadri, M. O., Rajesh, N., & Palanikumar, M. (2024). Multi-criteria group decision making approach based on a new type of neutrosophic vague approach is used to select the shares of the companies for purchase. International Journal of Neutrosophic Science, 23(3), 296-96.

[43] J., Lejo. , Damrah, Sadeq. , M., Mutaz. , Al-Husban, Abdallah. , Palanikumar, M.. (2024). Type-I extension Diophantine neutrosophic interval valued soft set in real life applications for a decision making. International Journal of Neutrosophic Science, 24(4), pp. 151-164.

[44] R.Hatamleh, V.A.Zolotarev.(2017). Jacobi operators and orthonormal matrix-valued polynomials.I. Ukrainian Mathematical Journal, 69(2), 228-239.

[45] R.Hatamleh, V.A.Zolotarev.(2017).Jacobi operators and orthonormal matrix-valued polynomials.II. Ukrainian Mathematical Journal, 69(6), 836-847.

Received: Sep 7, 2024. Accepted: Dec 6, 2024

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

It is an open access journal distributed under the Creative Commons Attribution License that permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, all NSS articles are indexed in Scopus.

NSS is also indexed by Google Scholar, Google Plus, Google Books, EBSCO, Cengage Thompson Gale (USA), Cengage Learning, ProQuest, Amazon Kindle, DOAJ (Sweden), University Grants Commission (UGC) - India, International Society for Research Activity (ISRA), Scientific Index Services (SIS), Academic Research Index (ResearchBib), Index Copernicus (European Union),CNKI (Tongfang Knowledge Network Technology Co., Beijing, China), etc.

Google Dictionary has translated the neologisms "neutrosophy" (1) and "neutrosophic" (2), coined in 1995 for the first time, into about 100 languages.

FOLDOC Dictionary of Computing (1, 2), Webster Dictionary (1, 2), Wordnik (1), Dictionary.com, The Free Dictionary (1),Wiktionary (2), YourDictionary (1, 2), OneLook Dictionary (1, 2), Dictionary / Thesaurus (1), Online Medical Dictionary (1, 2), and Encyclopedia (1, 2) have included these scientific neologisms.

DOI numbers are assigned to all published articles.

Registered by the Library of Congress, Washington DC, United States, https://lccn.loc.gov/2013203857.

Recently, NSS was also approved for Emerging Sources Citation Index (ESCI) available on the Web of Science platform, starting with Vol. 15, 2017.

Editors-in-Chief:

Prof. Dr. Florentin Smarandache, Postdoc, Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA, Email: smarand@unm.edu.
Dr. Mohamed Abdel-Basset, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamed.abdelbasset@fci.zu.edu.eg.
Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

23316055

$39.95